Concours Putnam
Atelier de Pratique
Le lundi, 29 octobre 12h30-13h30 (Salle: Pavillon André-Aisenstadt 5448)
Suites et Séries

Geometric series
\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x| < 1 \]

Finite geometric series
\[ \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \quad n = 1, 2, \ldots, x \neq 1 \]

Exponential series
\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \]

Logarithmic series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = \log(1+x) \quad -1 < x \leq 1 \]

1. Compute the following
   1. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)
   2. \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \)
   3. \( \sum_{n=k}^{\infty} x^n \)
   4. \( \sum_{n=0}^{\infty} \frac{n^n}{2^n} \)
   5. \( \sum_{n=1}^{\infty} \frac{1}{n2^n} \)
   6. \( \sum_{n=1}^{\infty} \frac{1}{n(n+3)} \)
   7. \( \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} \)
   8. \( \sum_{n=0}^{\infty} \binom{n+k}{k} x^n \) for \( k = 0, 1, 2, \ldots |x| < 1 \).

Solution:
   1. \(-\log 2\). Put \( x = 1 \) in the logarithmic series.
   2. 1. Write \( \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \).
   3. \( \frac{x^k}{1-x} \). Multiply both sides of the geometric series by \( x^k \).
4. We have \( \sum_{n=0}^{\infty} \frac{n}{2^n} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{2^n} = \sum_{k=1}^{\infty} \frac{2}{2^k} = 2. \)

5. \( \log 2. \) Put \( x = -\frac{1}{2} \) in the logarithmic series.

6. \( \frac{11}{18} = \frac{1}{3} (1 + \frac{1}{2} + \frac{1}{3}). \) Write \( \frac{1}{n(n+1)} = \frac{1}{3} \left( \frac{1}{n} - \frac{1}{n+3} \right). \)

7. \( (n+1)^2 = n(n-1) + 3n + 1 \) and use the exponential series.

8. \( \frac{1}{(1-x)^{k+1}}. \) Write \( \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \sum_{n=0}^{\infty} \frac{(-k-1)(-k-2)\ldots(-k-n)}{n!}(-x)^n = (1-x)^{-k-1}. \)

2. Evaluate \( \sum_{n=1}^{\infty} \frac{(-1)^{[2^nx]}}{2^n} \quad 0 < x < 1, \) where \([t]\) denotes the greatest integer \( \leq t.\)

**Solution:** The numerator of the \( n^{th} \) term is \( 1 - 2k \) where \( k \in \{0,1\} \) is the \( n^{th} \) digit in the binary expansion of \( x. \) The answer is, then, \( 1 - 2x. \)

3. Evaluate \( \sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)}, \)

where \( s(n) \) is the number of 1’s in the binary expansion of \( n.\)

**Solution:** This is (Putnam ’81, B5). Let \( S = \sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)}. \) Since \( s(2n) = s(n) \) and \( s(2n+1) = s(2n) + 1, \) it follows that
\[
S - \frac{s(1)}{2} = \sum_{n=1}^{\infty} \frac{s(2n)}{2n(2n+1)} + \frac{s(2n+1)}{(2n+1)(2n+2)} = \sum_{n=1}^{\infty} \frac{s(n)}{2n(n+1)} + \frac{1}{(2n+1)(2n+2)}.
\]

Thus, \( S - \frac{1}{2} = \frac{S}{2} + \frac{1}{3} - \frac{1}{4} + \ldots. \) Then \( S = 2 \log 2. \)

4. Let \( a \) and \( d \) be positive integers. Show that the arithmetic progression \( a, a+d, a+2d, \ldots \) either contains no perfect square or contains infinitely many perfect squares.

**Solution:** Let \( n^2 \) be an element of the sequence. Then \( (n+kd)^2 = n^2 + (2kn+k^2d)d \) is also an element of the sequence, for any positive integer \( k.\)
5. Solve: \(x_{n+1} = 2x_n(1 - x_n)\), with \(x_1 = -1\).

**Solution:** We write \(x_n = \frac{1-y_n}{2}\). Thus the equation becomes
\[
\frac{1 - y_{n+1}}{2} = (1 - y_n) \frac{1 + y_n}{2}
\]
Thus \(y_n = 3^{2^{n-1}}\) and \(x_n = \frac{1-3^{2^{n-1}}}{2}\).

6. Let \(\{x_n\}\) be a sequence of real numbers satisfying \(x_n = \frac{x_{n-1} + x_{n-2}}{2}\). Show that the sequence converges, and find the limit in terms of \(x_0\) and \(x_1\).

**Solution:** The characteristic equation is given by
\[
2x^2 = x + 1
\]
The roots are 1 and \(-\frac{1}{2}\). Thus, \(x_n = \frac{x_0+2x_1}{3} + \left(-\frac{1}{2}\right)^n \frac{2(x_0-x_1)}{3}\). The limit is clearly \(x_0 + \frac{2x_1}{3}\).

7. Let \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) be infinite sequences of positive integers. Show that there exist distinct indices \(p\) and \(q\) such that \(x_p \geq x_q\), \(y_p \geq y_q\), and \(z_p \geq z_q\).

**Solution:** First, we claim that there is a (fixed) index \(q\) for which there are infinitely many \(p\) such that \(x_p \geq x_q\) and \(y_p \geq y_q\). First take \(x_q\) such that the sequence \(\{x_n\}\) reaches a minimal value in \(x_q\). Then all the other elements in the sequence satisfy \(x_p \geq x_q\). Now we look at \(y_q\). If there are infinitely many \(y_p\) such that \(y_p \geq y_q\), then the claim is true. If not, then there are infinitely many elements in the sequence \(\{y_n\}\) taking finitely many values between 1 and \(y_q-1\). Therefore, there are infinitely many indexes that take the same value. We take such a subsequence and look at the situation in \(\{x_n\}\). There is a minimal element there and we rename that element \(x_q\). This proves the claim.

Now we look at \(z_q\). If there is a \(z_p \geq z_q\), we are done. If not, we proceed as before and take infinitely many indexes in the sequence \(\{z_n\}\) that take the same value. Now we take those subsequences in \(\{x_n\}\) and \(\{y_n\}\) and apply the previous claim.

8. Let \(a_1 = a_2 = 1\) and \(a_n = \frac{a_{n-1} + 2}{a_{n-2}}\) for \(n \geq 3\). Show that \(a_n\) is an integer for all \(n\).
Solution: We write \( a_n a_{n-2} - a_{n-1}^2 = 2 = a_{n-1} a_{n-3} - a_{n-2}^2 \). Thus
\[
\frac{a_n + a_{n-2}}{a_{n-1}} = \frac{a_{n-1} + a_{n-3}}{a_{n-2}},
\]
and therefore, \( b_n = \frac{a_n + a_{n-2}}{a_{n-1}} \) is a constant sequence. Placing \( n = 3 \) we obtain \( b_n = 4 \).

Thus, \( a_n = 4a_{n-1} - a_{n-2} \) and since \( a_1, a_2 \) are integers, the same is true for \( a_n \).

9. For each integer \( n \geq 0 \), let \( d(n) = n - m^2 \), where \( m \) is the largest integer with \( m^2 \leq n \). Define a sequence \( \{b_k\} \) by \( b_0 = B; b_{k+1} = b_k + d(b_k) \). For what positive integers \( B \) is \( \{b_k\} \) eventually constant?

Solution: This is (Putnam '91, B1). If \( b_k \) is a square, then \( d(b_k) = 0 \), so \( b_{k+1} = b_k \) and the sequence is constant from that point on.

If \( b_k \) is not a square, then for some \( m \) it lies between \( m^2 \) and \( (m + 1)^2 \), so we can write it as \( m^2 + r \), where \( 1 \leq r \leq 2m \). So \( b_{k+1} = m^2 + 2r \). But \( m^2 < b_{k+1} < (m + 2)^2 \) and \( b_{k+1} \neq (m + 1)^2 \), so \( b_{k+1} \) is not a square (and is > \( b_k \)). Thus \( \{b_k\} \) is eventually constant iff \( B \) is a square.