Concours Putnam  
Atelier de Pratique  
Le lundi, 15 octobre 12h30-13h30 (Salle: Pavillon André-Aisenstadt 5448)  
Suites et Séries

Geometric series
\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad |x| < 1 \]

Finite geometric series
\[ \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \quad n = 1, 2, \ldots, x \neq 1 \]

Exponential series
\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \]

Logarithmic series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = \log(1+x) \quad -1 < x \leq 1 \]

1. Compute the following
   1. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)
   2. \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \)
   3. \( \sum_{n=k}^{\infty} x^n \)
   4. \( \sum_{n=0}^{\infty} \frac{n}{2^n} \)
   5. \( \sum_{n=1}^{\infty} \frac{1}{n2^n} \)
   6. \( \sum_{n=1}^{\infty} \frac{1}{n(n+3)} \)
   7. \( \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} \)
   8. \( \sum_{n=0}^{\infty} \binom{n+k}{k} x^n \) for \( k = 0, 1, 2, \ldots |x| < 1. \)

2. Evaluate
\[ \sum_{n=1}^{\infty} \frac{(-1)^{\left\lfloor 2^n x \right\rfloor}}{2^n} \quad 0 < x < 1, \]
where \( \left\lfloor t \right\rfloor \) denotes the greatest integer \( \leq t. \)

3. Evaluate
\[ \sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)}, \]
where \( s(n) \) is the number of 1’s in the binary expansion of \( n. \)
4. Let $a$ and $d$ be positive integers. Show that the arithmetic progression $a, a+d, a+2d,\ldots$ either contains no perfect square or contains infinitely many perfect squares.

5. Solve: $x_{n+1} = 2x_n(1 - x_n)$, with $x_1 = -1$.

6. Let $\{x_n\}$ be a sequence of real numbers satisfying $x_n = (x_{n-1} + x_{n-2})/2$. Show that the sequence converges, and find the limit in terms of $x_0$ and $x_1$.

7. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be infinite sequences of positive integers. Show that there exist distinct indices $p$ and $q$ such that $x_p \geq x_q$, $y_p \geq y_q$, and $z_p \geq z_q$.

8. Let $a_1 = a_2 = 1$ and $a_n = \frac{a_{n-1}^2 + 2}{a_{n-2}}$ for $n \geq 3$. Show that $a_n$ is an integer for all $n$.

9. For each integer $n \geq 0$, let $d(n) = n - m^2$, where $m$ is the largest integer with $m^2 \leq n$. Define a sequence $\{b_k\}$ by $b_0 = B; b_{k+1} = b_k + d(b_k)$. For what positive integers $B$ is $\{b_k\}$ eventually constant?