Concours Putnam

Atelier de Pratique
Le mercredi, 18 septembre 12h30-13h30
La salle 5448 Pav. André Aisenstad

1. Gryffindor fans tell the truth when Gryffindor wins and lie when it loses. Fans of Hufflepuff, Ravenclaw, and Slytherin behave similarly. After two matches of quidditch with the participation of these four teams (with no draws and each team playing exactly one game), among the wizards who watched the broadcast, 500 answered positively to the question “Do you support Gryffindor?”, 600 answered positively to the question “Do you support Hufflepuff?”, 300 answered positively to the question “Do you support Ravenclaw?”, and 200 answered positively to the question “Do you support Slytherin?”. How many wizards support each of the teams? Note: Each wizard is fan of exactly one of the teams.

Solution: Let A, B be the numbers of fans of the winning teams and C, D the numbers of fans of the losing teams. Then the numbers of positive answers to the question ”Do you support X” are $A + C + D$, $B + C + D$, D, and C, respectively. Note that each of the latter two numbers is always less or equal to each of the former two. So we conclude that Ravenclaw and Slytherin lost and had, respectively, 200 and 300 fans. This means that there were $500 - 200 - 300 = 0$ Gryffindor fans and there were $600 - 200 - 300 = 100$ Hufflepuff fans.

2. Into how many regions do n lines divide the plane, assuming no two lines are parallel and no three lines intersect in the same point?

Solution: The answer is $\frac{n(n+1)}{2} + 1$. We give two solutions.

#1. One observes that the number a_n of regions satisfies the recursive relation $a_n = a_{n-1} + n$, $n \geq 1$, since focusing on a given line l, we observe that it intersects exactly $n - 1$ other lines, and hence itersects exactly n regions, splitting each one of them into two. Together with $a_0 = 1$, we have $a_n = a_0 + \sum_{j=1}^{n} j = \frac{n(n+1)}{2} + 1$.

#2. One considers the numbers V, E, F of, respectively, vertices given by intersection points of the lines, edges into which the lines are subdivided by the intersection points, and regions in the complement of the lines. If we focus only on the numbers V, E', F' of vertices, bounded edges, and bounded regions, Euler's formula for planar graphs implies $V - E' + F' = 1$. We observe that $-E + F = -E' + F'$, which gives $V - E + F = 1$. Now $V = \binom{n}{2} = \frac{n(n-1)}{2}$ since each two lines intersect at precisely one point, and $E = n^2$ since each line is subdivided into precisely n edges. Hence $F = 1 - V + E = 1 - \frac{n^2-n}{2} + n^2 = \frac{n(n+1)}{2} + 1$.

Note: both the relations $V - E' + F' = 1$ and $V - E + F = 1$ can be obtained by adding one unbounded face in the first case, and one vertex at infinity in the second case, and using Euler's formula

$$V - E + F = 2$$

for graphs in the (two-dimensional) sphere.

3. Find at least one positive integer that is at least 3 times smaller than the sum of its positive proper divisors.

Solution: Observe $1/(2^n + 1) + 1/(2^n + 2) + \ldots + 1/2^{n+1} > 1/2$, so we have $1/2 + 1/3 + \ldots + 1/64 > 3$. Next, $64!$ is divisible by every integer between 2 and 64, so the sum of its proper divisors is at least $(1/2 + 1/3 + \ldots + 1/64) \cdot 64! > 3 \cdot 64!$.

4. A certain country has finitely many cities. Any pair of these cities is connected by a road. However, all roads in this country are one-way roads, and it is therefore not always possible to travel from one city to another city. Show that the country has a city ("capital") that can be reached from every other city either directly or via exactly one intermediate city.

Solution: We give three solutions: first relying on mathematical induction, second relying on the technique of minimization, and third relying on both.

#1. We prove it by induction on n the number of cities. For two cities it is clear. The induction step from n to $n + 1$ is shown as follows. Consider a city c' in our country C'. Let C be the sub-country of C' obtained by forgetting the city c' and all roads ending or starting at it. By induction hypothesis, C has a capital c. We claim that either c or c' is a capital for C. If the road between c and c' enters c, then c is a capital. Similarly, if there is a road from a city c_1, in C, to c, and the road between c' and c_1 (in C') enters c_1, then c is a capital. Hence if c is not a capital, then there is a road from c to c', and hence c' can be reached from c or from any city in C with that has a road to c, by a sequence of at most two roads. Moreover, for each sequence of two roads c_1 to c_2, c_2 to c (in C), which always exists as c is a capital of C, there must be a road from c_2 to c', and hence a sequence of two roads c_1 to c_2, c_2 to c'. Hence c' is a capital.

#2. (Inspired by solution of Julien Codsi) Let c be a city for which the number k of roads entering it is maximal among the n cities. We claim that c is a capital. Denote by $c_1, \ldots, c_k$ the cities from which there are roads entering c. For these cities the capital condition (i.e. being able to reach c by a sequence of at most two roads) is clearly satisfied. Now note that the number $l = n - k$ of roads exiting c is minimal among the n cities. Hence for each city d other than $c, c_1, \ldots, c_k$ there are at least l roads exiting it. However, since $l + k + 1 = n + 1 > n$, there must be at least one road that exits d and enters one of the cities $c, c_1, \ldots, c_k$, and therefore the capital condition is satisfied for d. This proves our claim.

#3. We prove it by induction on n the number of cities. For two cities it is clear. Denote the cities by $c_1, \ldots, c_n$, and for each c_i let A_i those cities c_j for which the road between c_i and c_j is from c_j to c_i, and B_i those where the road goes from c_i to c_j. Assume the statement is true for $n-1$. Pick a city c_i for which the size of A_i is minimal. Without loss of generality, assume it is c_n. The induction hypothesis gives a capital among $c_1, \ldots, c_{n-1}$, say c_{n-1}. If there is a road from c_n to c_{n-1}, there is nothing to prove. If not, the road goes from c_{n-1} to c_n and c_{n-1} belongs to A_n and c_n does NOT belong to A_{n-1}. Since A_n is of minimal size, there is an element of c_i of A_{n-1} which does not belong to A_n. That means that there is a road from c_n to c_i and a road from c_i to c_{n-1}. Thus c_{n-1} is the capital.

5. Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two points are separated by a distance of at least $r^{1/3}$.

Solution: First notice that any triangle (which is assumed above to be non-degenerate) with integer coordinates has area at least $\frac{1}{2}$. There are various ways to show this. For example, this is trivial if one of the sides is parallel to one of the axes, and then, for general triangles, one can show that it is always decompose them into such triangles. In another way, one observes that area S of the triangle with vertices $u, v, w \in \mathbb{R}^2$ is given by

$$S = \frac{1}{2} |\det(v - u, w - u)|,$$

where $v - u, w - u$ are considered as column vectors. Now if u, v, w have integer coordinates, so do $v - u, w - u$, and hence $2S$ is a positive integer, whence $2S \geq 1$.

On the other hand, the area of a triangle of sides a, b, and c which is inscribed into a circle of radius r is $\frac{abc}{4r}$. To see this, start from the formula $\frac{ab\sin\gamma}{2}$ where γ is the angle between a and b, and use that $\frac{c}{\sin\gamma} = 2r$. Then we have

$$\frac{abc}{4r} \geq \frac{1}{2}$$

If we assume that a, b, c are less or equal to $r^{1/3}$, we obtain $\frac{1}{4} \geq \frac{1}{2}$, a contradition.

Note: the fact that any triangle with integer coordinates has area at least $\frac{1}{2}$ is a particular case of Pick's theorem: a lattice polygon whose boundary consists of a sequence of connected nonintersecting straight-line segments has area $i + \frac{b}{2} - 1$, where i is the number of interior points, and b is the number of boundary points.

6. Find the digit before the decimal point and the first two digits after the decimal point in the decimal notation of $(1+\sqrt{2})^{2024}$.

Solution: First, observe that $(1+\sqrt{2})^{2024}+(1-\sqrt{2})^{2024}$ is an integer and $-1/2 < (1-\sqrt{2}) < 0$. So $(1-\sqrt{2})^{2024}$ is positive and smaller than 0.01, hence the first two digits of $(1+\sqrt{2})^{2024}$ after the decimal point are 99.

Next, the sequence of integers $x_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$ satisfies $x_{n+1} = 2x_n + x_{n-1}$, so it is periodic modulo 10. The direct computation shows that the period length is 12. So the 2024th term of this sequence is the same as the 8th, and it is 4 modulo 10. Thus the last digit before the decimal point is 3.