Concours Putnam

Atelier de Pratique
Le jeudi, 28 novembre 12h30-13h30
La salle 5448 Pav. André Aisenstad

Géométrie

1. Let A, B, C, D be four points in space forming a quadrilateral. Show that the midpoints of AB, BC, CD, DA form a parallellogram.

Solution: Consider A, B, C, D as vectors in $\mathbb{R}^n$. Then the midpoints are given by $P = (A+B)/2$, $Q = (B+C)/2$, $S = (A+D)/2$, $R = (D+C)/2$. Then $Q - P = (C-A)/2 = R - S$, $P - S = (B-D)/2 = Q - R$. So $PQRS$ is a parallelogram.

2. Is it possible to place a finite number of parabolas on a plane so that their interiors cover the entire plane?

Solution: It is easy to see that a parabola (with its interior) intersects a line that is not parallel to its axis by a segment, a point, or an empty set. Let us take a line that is not parallel to any of the axes of the parabolas. The parabolas will cut out a finite number of segments on the line, which means that they will not even cover this line.

3. Let A, B, C, D be four points on a plane. Prove that $|AB||CD| + |BC||AD| \geq |AC||BD|$.

Solution: Think of A, B, C, D as of four complex numbers. We know by algebraic manipulation with the complex multiplication that $(A-B)(C-D)+(B-C)(A-D) = (A-C)(B-D)$. Hence the inequality follows by taking norms and using their multiplicativity and the triangle inequality.

4. There are n red and n blue points on a plane, no three of which lie on the same line. Prove that it is possible to draw n segments with different-colored endpoints that have no common points.

Solution: Let us consider all partitions of the given points into pairs of different colors. There are a finite number of such partitions, so there is a partition (let us call it minimal) for which the sum of the lengths of the segments defined by the pairs of points of this partition is the smallest. Let us show that these segments will not intersect then.

Indeed, if two segments R_1B_1 and R_2B_2 with different colored ends intersected (we will denote the red points by the letters R, and the blue ones by B), then we could choose a partition with a smaller sum of the segment lengths by replacing the diagonals R_1B_1 and R_2B_2 of the convex quadrilateral $R_1B_2B_1R_2$ with its opposite sides R_1B_2 and R_2B_1. Then $R_1B_2 + R_2B_1 < R_1B_1 + R_2B_2$, which contradicts the choice of the minimal partition.

5. Let $P_1, \ldots, P_n$ be points on the unit sphere. Prove that $\sum_{i \leq j} |P_iP_j|^2 \leq n^2$.

Solution: Thinking of these points as vectors, recall that $|v|^2 = \langle v, v \rangle$ for all $v \in \mathbb{R}^k$, where $\langle v, w \rangle = v_1w_1 + \ldots + v_kw_k$ is the scalar product. Now we know that $|P_i| = 1$ for all i, and $|P_iP_j|^2 = \langle P_i - P_j, P_i - P_j \rangle$, hence

$$|P_iP_j|^2 = |P_i|^2 + |P_j|^2 - 2\langle P_i, P_j \rangle = 2 - 2\langle P_i, P_j \rangle.$$

Hence by summing over all $i < j$ (since $|P_iP_i| = 0$ for all i), we obtain

$$\sum_{i \leq j} |P_iP_j|^2 = 2n(n-1)/2 - 2\sum_{i<j} \langle P_i, P_j \rangle.$$

On the other hand we know that

$$0 \leq \langle \sum_i P_i, \sum_j P_j \rangle = \sum_i |P_i|^2 + 2\sum_{i<j} \langle P_i, P_j \rangle.$$

Hence $-2\sum_{i<j} \langle P_i, P_j \rangle \leq \sum_i |P_i|^2 = n$, so we conclude that

$$\sum_{i \leq j} |P_iP_j|^2 \leq n^2 - n + n = n^2$$

6. The plane is divided into equilateral triangles with side 1 by three infinite series of equally spaced parallel lines. M is the set of all their vertices. A and B are two vertices of one triangle. It is allowed to rotate the plane by 120° around any of the vertices of the set M. Is it possible to transform point A into point B using several such transformations?

Solution: It is easy to color all vertices in three colors so that each triangle contains vertices of all colors. Note that with an admissible rotation, each vertex transforms into a vertex of the same color. And points A and B are of different colors.

7. Does the circle $S^1 = \{x^2 + y^2 = 1\}$ in the plane contain a closed subset that contains exactly one of each pair of diametrically opposite points?

Solution: (This is Putnam 1975, B4) The map $\tau : v = (x, y) \mapsto -v = (-x, -y)$ is a homeomorphism from S^1 to S^1, a continuous mapping whose inverse is also continuous, and hence takes closed sets to closed sets. Hence K and $\tau(K)$ are disjoint closed sets whose union is S^1, but that is impossible since S^1 is connected. (For example, take a continuous arc $\gamma : [0, 1] \to \mathbb{R}$ joining $\gamma(0) = x \in K$ and $\gamma(1) = y \in \tau(K)$. Then for the point $t_0 = \inf\{t \mid \gamma(t) \in \tau(K)\}$ we have $\gamma(t_0) \in \tau(K)$ because $\tau(K)$ is closed and γ is continuous, but also $\gamma(t_0) \in K$ because we can find $t_j \to t_0$, $t_j \leq t_0$, $\gamma(t_j) \in K$, and K is closed and γ is contiuous. But then K, $\tau(K)$ are not disjoint, contradiction.)

8. Show that there are no 7 lines in the plane such that there are at least 6 points in the plane which lie on the intersection of just three of the lines and at least 4 points in the plane which lie on the intersection of just two of the lines.

Solution: (This is Putnam 1973, A6) From 7 lines we can choose just 21 pairs. An intersection of 3 lines accounts for 3 distinct pairs of lines, an intersection of 2 lines for 1 pair. Hence the configuration given would have at least $6 \cdot 3 + 4 \cdot 2 = 22$ distinct pairs of lines.