Concours Putnam

Atelier de Pratique
Le jeudi, 31 octobre 12h30-13h30
La salle 5448 Pav. André Aisenstad

Le principe du pigeonnier/de Dirichlet

If $n+1$ objects ("pigeons") are distributed among n boxes ("pigeon holes"), at least one of the boxes contains more than one object. More generally, if $kn+1$ objects are distributed among n boxes, at least one of the boxes contains more than k objects.

1. Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most $1/2$.
2. Given a set of 7 integers, show that there exist two of them whose difference or sum is divisible by 10.
3. Prove that from a set of ten distinct two-digit integers it is possible to select two disjoint non-empty subsets whose members have the same sum.
4. Show that any set $A \subset \{1, 2, ..., 2n\}$ with at least $n+1$ elements contains two elements, one of which divides the other.
5. Several chess players are playing a single round robin tournament. Prove that at any moment of the tournament there are two chess players who have played the same number of matches by that moment.
6. Let S be the set of real numbers of the form $a + b\sqrt{2}$, where a and b are integers. Show that S is dense on the real line, in the sense that, given any $\epsilon > 0$ and any real number x there exists an element $s \in S$ with $|s - x| < \epsilon$.
7. Each point in the plane with integer coordinates is colored in one of n colors. Prove that there is a rectangle with vertices at points of the same color.
8. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that, given any positive integer k, there exists a Fibonacci number F_n ending in at least k zeros.
9. Suppose $\mathcal{A}$ is a collection of subsets of $\{1, 2, \ldots, n\}$ with the property that any two sets in $\mathcal{A}$ have a non-empty intersection. Show that $\mathcal{A}$ has at most 2^{n-1} elements. Can the bound 2^{n-1} be lowered?
10. Given any 5 distinct points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them.
11. A partition of a set S is a collection of disjoint non-empty subsets (parts) whose union is S. For a partition π of $\{1, 2, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there exist $x, y \in \{1, 2, \ldots, 9\}$, $x \neq y$, such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.