Concours Putnam

Atelier de Pratique
Le mercredi, 18 septembre 12h30-13h30
La salle 5448 Pav. André Aisenstad

Inequalities

Cauchy's Inequality

$$\left(\sum a_i^2\right)\left(\sum b_i^2\right) \geq \left(\sum a_ib_i\right)^2$$

$$\left(\int f(x)^2dx\right)\left(\int g(x)^2dxv\right) \geq \left(\int f(x)g(x)dx\right)^2$$

Arithmetic-Geometric Mean Inequality If $a_i \geq 0$,

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$$

Jensen's inequality A real-valued function $f(x)$ is called convex if

$$\frac{f(x_1)+f(x_2)}{2} \geq f\left(\frac{x_1+x_2}{2}\right)$$

for all real x_1, x_2. If $f(x)$ is convex, and $p_i \geq 0$, $\sum p_i = 1$, then for any real x_i,

$$\sum p_i f(x_i) \geq f\left(\sum p_i x_i\right).$$

1. Given n positive real numbers with sum 1, show that the sum of the squares of these numbers is at least $\frac{1}{n}$.

Solution: Apply Cauchy-Schwarz to the sum $1 = \sum_{i=1}^{n} 1 \cdot a_i$.

2. Given n positive real numbers $a_1, \ldots, a_n$, define

$$H = n\left(\frac{1}{a_1} + \ldots + \frac{1}{a_n}\right)^{-1}.$$

(This number H is called the **harmonic mean** of the numbers a_i .) Show that $H \leq G$, where $G = (a_1 \ldots a_n)^{\frac{1}{n}}$ is the geometric mean of the a_i's.

Solution: Note that H is the reciprocal of the arithmetic mean of the numbers $b_i = \frac{1}{a_i}$, while G is the reciprocal of the geometric mean of these numbers. Then apply the Arithmetic-Geometric Mean (AGM) inequality.

Remark. In fact the following generalization holds. Consider the function $F_0 : \mathbb{R} \setminus \{0\} \to \mathbb{R}_{>0}$ given by $F_0(t) = \left(\frac{1}{n}\sum_{i=1}^{n}(a_i)^t\right)^{\frac{1}{t}}$. Then

1. F_0 extends to a monotone increasing continuous function $F : \mathbb{R} \to \mathbb{R}_{>0}$, by setting $F(0) := (a_1 \cdot \ldots a_n)^{\frac{1}{n}}$.
2. $\lim_{t\to+\infty} F(t) = \max\{a_i\}$, $\lim_{t\to-\infty} F(t) = \min\{a_i\}$.

3. Let $a_1, \ldots, a_n$ be positive integers, and let $b_1, \ldots, b_n$ be a permutation of the a_i's. Show that

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n.$$

Solution: Notice that $\prod_{i=1}^{n} \frac{a_i}{b_i} = 1$. Apply AGM to the numbers $\frac{a_i}{b_i}$.

4. Suppose f is a nonnegative function defined on the interval $[0, 1]$ and satisfying $\int_0^1 f(x)^2 dx = 1$. What is the maximum value of $\int_0^1 f(x)x^{2024}dx$?

Solution: Let I denote the integral $\int_0^1 f(x)x^{2024}dx$. Apply the integral version of Cauchy-Schwarz with the functions $f(x)$ and x^{2024} to get

$$I^2 \leq \int_0^1 f(x)^2 dx \int_0^1 x^{4048} dx = \frac{1}{4049}$$

therefore, $I \leq \frac{1}{\sqrt{4049}}$. To show that this upper bound can be achieved, take f to be proportional to x^{2024} , i.e., $f(x) = cx^{2024}$ with $c = \sqrt{4049}$.

5. Let $x_1, \ldots, x_n$ be real numbers with $0 < x_i < 1$, and let $x = (1/n)\sum_{i=1}^{n} x_i$ be the arithmetic mean of these numbers. Show that

$$\left(\frac{\sin x}{x}\right)^n \geq \prod_{i=1}^{n} \left(\frac{\sin x_i}{x_i}\right)$$

Solution: Taking logarithms the inequality to be shown is equivalent to

$$n \log\left(\frac{\sin x}{x}\right) \geq \sum_{i=1}^{n} \log\left(\frac{\sin x_i}{x_i}\right).$$

Note that $f(x) = \log((\sin x)/x) = \log(\sin x) - \log x$ is concave on $(0, 1)$, as

$$f''(x) = \frac{1}{x^2} - \frac{1}{\sin^2 x} < 0$$

The result now follows from Jensen's inequality.

6. Let u, v, w be real numbers. Show that

$$\log\left(\frac{e^u + e^v + e^w}{3}\right) \geq \frac{u + v + w}{3}.$$

When does equality hold?

Solution: Exponentiate both sides, then apply Jensen's inequality with the function $f(x) = e^x$ to get

$$(1/3)(f(u) + f(v) + f(w)) \geq f((1/3)u + (1/3)v + (1/3)w).$$

7. Suppose $x_1, \ldots, x_n$ are positive real numbers with $\sum_{i=1}^{n} x_i = 1$. Show that

$$\log \sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i \log x_i$$

Solution: Apply Jensen's inequality with $f(x) = -\log x$ (which is convex since $\log x$ is concave) and $p_i = x_i$.

8. Let k be a positive constant. The sequence x_i of positive reals has sum k. What are the possible values for the sum of x_i^2?

Solution: This is (Putnam '00, A1). The answer is any value in the open interval $(0, k^2]$.

Since the terms are positive we have,

$$0 < \sum x_i^2 \leq (\sum x_i)^2 = k^2$$

So certainly all sums must lie in the interval $(0, k^2]$.

We show how to get any value in the interval by explicit construction. We may represent the value as hk^2 where $0 < h \leq 1$. Take $c = \frac{1-h}{1+h}$. Take the geometric sequence with first term $k(1-c)$ and ratio c, i.e., $(k(1-c), kc(1-c), kc^2(1-c), ...)$. We have $\sum x_i = k$ as required, and $\sum x_i^2 = k^2(1-c)^2/(1-c^2) = hk^2$ as required.

9. Prove the following inequality for all $a_i > 0$

$$\prod_{i=1}^{n} a_i^{a_i} \geq (\frac{1}{n}\sum_{i=1}^{n} a_i)^{\sum\limits_{i=1}^{n} a_i}$$

Solution: Taking logarithms of both parts and dividing by n, we get to Jensen inequality for the function $x \log x$. The latter is convex since its second derivative is $\frac{1}{x} > 0$.