Concours Putnam

Atelier de Pratique
Le mercredi, 18 septembre 12h30-13h30
La salle 5448 Pav. André Aisenstad

Inequalities

Cauchy's Inequality

$$\left(\sum a_i^2\right)\left(\sum b_i^2\right) \geq \left(\sum a_i b_i\right)^2$$

$$\left(\int f(x)^2 dx\right)\left(\int g(x)^2 dxv\right) \geq \left(\int f(x)g(x)dx\right)^2$$

Arithmetic-Geometric Mean Inequality If $a_i \geq 0$,

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$$

Jensen's inequality A real-valued function $f(x)$ is called convex if

$$\frac{f(x_1)+f(x_2)}{2} \geq f\left(\frac{x_1+x_2}{2}\right)$$

for all real x_1, x_2. If $f(x)$ is convex, and $p_i \geq 0$, $\sum p_i = 1$, then for any real x_i,

$$\sum p_i f(x_i) \geq f\left(\sum p_i x_i\right).$$

1. Given n positive real numbers with sum 1, show that the sum of the squares of these numbers is at least $\frac{1}{n}$.

2. Given n positive real numbers $a_1, \ldots, a_n$, define

$$H = n\left(\frac{1}{a_1} + \ldots + \frac{1}{a_n}\right)^{-1}.$$

(This number H is called the **harmonic mean** of the numbers a_i .) Show that $H \leq G$, where $G = (a_1 \ldots a_n)^{\frac{1}{n}}$ is the geometric mean of the a_i's.

Remark. In fact the following generalization holds. Consider the function $F_0 : \mathbb{R} \setminus \{0\} \to \mathbb{R}_{>0}$ given by $F_0(t) = \left(\frac{1}{n}\sum_{i=1}^{n}(a_i)^t\right)^{\frac{1}{t}}$. Then

1. F_0 extends to a monotone increasing continuous function $F : \mathbb{R} \to \mathbb{R}_{>0}$, by setting $F(0) := (a_1 \cdot \ldots a_n)^{\frac{1}{n}}$.
2. $\lim_{t\to+\infty} F(t) = \max\{a_i\}$, $\lim_{t\to-\infty} F(t) = \min\{a_i\}$.

3. Let $a_1, \dots, a_n$ be positive integers, and let $b_1, \dots, b_n$ be a permutation of the a_i's. Show that
$$\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n.$$

4. Suppose f is a nonnegative function defined on the interval $[0,1]$ and satisfying $\int_0^1 f(x)^2 dx = 1$. What is the maximum value of $\int_0^1 f(x) x^{2024} dx$?

5. Let $x_1, \dots, x_n$ be real numbers with $0 < x_i < 1$, and let $x = (1/n)\sum_{i=1}^{n} x_i$ be the arithmetic mean of these numbers. Show that
$$\left(\frac{\sin x}{x}\right)^n \geq \prod_{i=1}^{n} \left(\frac{\sin x_i}{x_i}\right)$$

6. Let u, v, w be real numbers. Show that
$$\log\left(\frac{e^u + e^v + e^w}{3}\right) \geq \frac{u+v+w}{3}.$$
When does equality hold?

7. Suppose $x_1, \dots, x_n$ are positive real numbers with $\sum_{i=1}^{n} x_i = 1$. Show that
$$\log \sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i \log x_i$$

8. Let k be a positive constant. The sequence x_i of positive reals has sum k. What are the possible values for the sum of x_i^2?

9. Prove the following inequality for all $a_i > 0$
$$\prod_{i=1}^{n} a_i^{a_i} \geq (\frac{1}{n}\sum_{i=1}^{n} a_i)^{\sum_{i=1}^{n} a_i}$$