Concours Putnam

Atelier de Pratique
Le jeudi, 14 novembre 12h30-13h30
La salle 5448 Pav. André Aisenstad

Jeus et invariants

1. Given an 8×8 grid representing a chessboard. At each step, any two columns or any two rows may be swapped. Is it possible to make the top half of the grid white and the bottom half black in a few steps?

> **Solution:** Both permitted operations do not change the number of black and white cells in any row, therefore, the required coloring of the grid cannot be obtained.

2. Consider the following two-player game. Initially, there is a large number of tokens on the table. Each player in their turn can remove between 1 and 10 tokens. The player who must take the last token loses. Give a strategy for the first player to win if we start with 99 tokens; and a strategy for the second player to win if we start with 100 tokens.

> **Solution:** La clé est de travailler mod 11. Supposons que le nombre de jetons soit N et que le joueur A soit au tour. Si $N \equiv 1 \bmod 11$ et que le joueur A prend m jetons avec $1 \leq m \leq 10$, alors le joueur B prend $11 - m$ jetons de sorte qu'il reste $N - 11$ jetons, ce qui est à nouveau $\equiv 1 \bmod 11$. On continue à faire cela jusqu'à ce qu'il reste 1 jeton. Ainsi, si nous commencons avec 100 jetons, le premier joueur peut être contraint de perdre. Si nous commencons avec 99 jetons, le premier joueur prend 10 pour quitter $89 \equiv 1 \bmod 11$, et ainsi le deuxième joueur peut être forcé de perdre par la même stratégie.

3. The numbers $1, \frac{1}{2}, ..., \frac{1}{n}$ are written on the board. It is allowed to erase any two numbers a and b and replace them with the number $ab + a + b$. What number remains after n–1 such operations?

> **Solution:** Let the numbers $a, b, \ldots, z$ be written on the board at some point. Consider the product $P = (a + 1)(b + 1)...(z + 1)$. After replacing the pair of numbers a, b with $ab + a + b$, the factors $(a + 1)(b + 1)$ in the product will be replaced by $ab + a + b + 1 = (a + 1)(b + 1)$. Thus, the operation does not change the value of P. It follows that the number that remains at the end does not depend on the order of operations and is equal to $(1+1)(\frac{1}{2}+1)\ldots(\frac{1}{n}+1)–1 = \frac{2}{1}\frac{3}{2}\ldots\frac{n+1}{n}–1 = (n+1)–1 = n$.

4. Borgov places white bishops on a chess board, and Beth Harmon places black bishops in her turn, starting with Borgov, in such a way that a new bishop can only be placed on a square that **can** be "taken" by a bishop of the other color already on the board. A player loses if they cannot place a bishop during their turn. Give a strategy for Beth Harmon to win.

Solution: Symétrie. Supposons que les carrés de l'échiquier soient étiquetés en coordonnées $\{1,...,8\} \times \{1,...,8\}$. Si Borgov joue un évêque à (m,n) alors Harmon répond à $(9-m,n)$. Vous devriez réfléchir aux raisons pour lesquelles cette stratégie fonctionne.

5. Four grasshoppers are sitting at the vertices of a square. Every minute one of them jumps to a point symmetrical to it relative to some other grasshopper. Can the grasshoppers at some point end up at the vertices of a larger square?

Solution: Let's imagine that the square with the grasshoppers at its vertices is a square of checkered paper (the size of the square is 1×1). Note that the grasshoppers always jump along the vertices of the cells: if we place the grasshoppers at the vertices of the cells on the checkered paper, then after each jump each grasshopper will again end up at some node of the square grid.

Suppose that the grasshoppers managed to end up at the vertices of the larger square, then, jumping in the opposite order, they should end up at the vertices of the smaller one. But, starting to jump from the vertices of the larger square, they will always end up at the nodes of the grid consisting of large squares. In other words, the distance between them cannot be less than the side of the large square, which is a contradiction.

6. Given 11 red chips, 30 white chips and 19 blue chips, we can replace any two chips of two different colours, by two chips of the third colour. (For example, we may replace a white chip and a blue chip by two red chips.) Can we ever have the same number of chips of two different colours?

Solution: In our example we replace (r, w, b) by $(r-1, w+2, b-1)$. Therefore $b - w \bmod 3$ and $r - b \bmod 3$ are invariants, and are always $\equiv 1 \pmod 3$, and so r, b and w are always distinct mod 3, so no two can be equal.

7. In a 10×10 square table, nine 1×1 cells are shaded. After that, you can sequentially shade the cells that have at least two adjacent cells (i.e., that have a common side) already shaded. Prove that it is impossible to shade all the cells.

Solution: Let's consider the boundary of the shaded area. Initially, the length of the boundary was no more than $9 \times 4 = 36$, since only 9 cells were shaded. It is easy to see that in the process of shading, the length of the boundary cannot increase. But if the entire 10×10 table were shaded at some point, the length of the boundary would be equal to $10 \times 4 = 40$. This is a contradiction.

8. There are two three-liter vessels. One contains 1 liter of water, the other contains 1 liter of a two-percent solution of table salt. It is allowed to pour any part of the liquid from one vessel to the other, and then mix. Is it possible to obtain a 1.5% solution in the vessel that initially contained water after several such pourings?

Solution: Until all the liquid ends up in one vessel (then you get a 1% solution and nothing will change), the salt concentration in the first vessel (where the water was) is lower than in the second. Let the first vessel contain a 1.5% solution at the end. Then it should not be lower in the second. By pouring everything into one vessel, we get 2 liters with a salt concentration higher than 1%, which is impossible.