A Family of Polynomials with Mahler Measure $\frac{28}{5\pi^2}\zeta(3)$

Siva Sankar Nair

Université de Montréal

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Definition of the Mahler Measure

For a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, we define the (logarithmic) Mahler measure of P to be

$$\begin{aligned} m(P) &:= \int_{[0,1]^n} \log \left| P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}) \right| \, d\theta_1 \cdots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \, \frac{dx_1}{x_1} \cdots \frac{dx_1}{x_n}, \end{aligned}$$

where $\mathbb{T}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : \; |x_i| = 1\}$. It is the average value of $\log |P|$ over the unit n-torus.

The one-variable case

If P is a univariate polynomial of the form $P(x) = A\prod_{j=1}^{d}(x-\alpha_j)$, then using Jensen's formula, it follows that

$$m(P) = \int_0^1 \log|P(e^{2\pi i\theta})|d\theta = \log|A| + \sum_{\substack{j \\ |\alpha_j|>1}} \log|\alpha_j|.$$

This means that polynomials with integer coefficients have Mahler measure greater than or equal to zero.

Some Properties

- Kronecker's Lemma: $P \in \mathbb{Z}[x]$, $P \neq 0$,
$$m(P) = 0 \text{ if and only if } P(x) = x^n \prod_i \Phi_i(x),$$
where $\Phi_i(x)$ are cyclotomic polynomials.
- Lehmer's Question (1933, still open): Does there exist a constant $\delta > 0$ such that for every polynomial $P \in \mathbb{Z}[x]$ having non-zero Mahler measure, we must also have $m(P) > \delta$?
So far the polynomial with the smallest Mahler measure is
$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \approx 0.162357612\ldots$$
- The Mahler measure of $P(x)$ is related to heights. For an algebraic integer α, with minimal polynomial f_α and logarithmic Weil height $h(\alpha)$
$$m(f_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]h(\alpha).$$

More variables, more problems

In general, calculating the Mahler measure of multi-variable polynomials is much more difficult than the univariate case. However, there are more intriguing results concerning such polynomials that suggest that something deeper is in play.

We have the Boyd-Lawton formula for any rational function $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$,

$$\lim_{k_2 \to \infty} \cdots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots, x^{k_n})) = m(P(x_1, x_2, \ldots, x_n)),$$

where the k_i's vary independently.

Examples

Turns out that for certain polynomials, the Mahler measure is not just any random real number, but in fact a special value of an L-function.
Smyth, 1981:

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) = -14\zeta'(-2)$$

Examples

Condon, 2004:

- $$m(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3) = -\frac{112}{5}\zeta'(-2)$$

Lalín, 2003:

- $$m\left(1+x+\left(\frac{1-v}{1+v}\right)\left(\frac{1-w}{1+w}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5) = 124\zeta'(-4)$$

Rogers and Zudilin, 2010:

- $$m(x+\frac{1}{x}+y+\frac{1}{y}+8) = \frac{24}{\pi^2}L(E_{24a3},2) = 4L'(E_{24a3},0)$$

Coming up with such Identities

- In general, Mahler measures are arbitrary real values. Only polynomials with a certain structure end up giving interesting values.
- Oftentimes, such identities are obtained after a numerical experiment on the computer of certain special polynomials. For example Boyd conducted many numerical experiments on polynomials of the type

$$A(x) + B(x)y + C(x)z,$$

where A, B and C are products of cyclotomic polynomials.

Calculations by Brunault and Zudilin

Brunault and Zudilin carried out extensive numerical calculations investigating the Mahler measure of polynomials of the form $A(x) + B(x)(y+z)$. These led to several conjectural identities that mysteriously evaluated to the same Mahler measure

$$\left.\begin{array}{l} m(x^2+x+1+(x^2-1)(y+z)) \\ m(x^3-x^2+x-1+(x^3+1)(y+z)) \\ m(x^4-x^3+x-1+(x^4-x^2+1)(y+z)) \\ m(x^4-x^3+x-1+(x^4-x^3+x^2-x+1)(y+z)) \\ m(x^4-x^3+x^2-x+1+(x^4-1)(y+z)) \\ m(x^4-x^3+x-1+(x^4+1)(y+z)) \\ m(x^5-x^4+x-1+(x^5+1)(y+z)) \end{array}\right\} \stackrel{?}{=} \frac{28}{5\pi^2}\zeta(3).$$

Is there some connection between these polynomials?

John Condon proved the following relation in his doctoral dissertation

$$m(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3).$$

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$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X+1)}{X+2}} 2\frac{X^2+X+1+(X^2-1)(y+z)}{X+2}$$

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 & 2\,\frac{X^2+X+1+(X^2-1)(y+z)}{X+2} \\
x+1+(x-1)(y+z) \arrow[ur, "x=\frac{X(2X+1)}{X+2}"] \arrow[r, "x=\frac{X(2X^2-X+1)}{-(X^2-X+2)}"] & 2\,\frac{X^3-X^2+X-1+(X^3+1)(y+z)}{-(X^2-X+2)}
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 & 2\frac{X^4-X^3+X-1+(X^4-X^2+1)(y+z)}{-(X^3-X^2-X+2)}
\end{tikzcd}$$

We notice that the transformations are of the form

$$x = \frac{f(X)}{g(X)},$$

where

1. If α is a root of $g(X)$, then $|\alpha| > 1$.
2. $f(X)$ is of the form

$$f(X) = \pm X^k \overline{g}(X^{-1}),$$

where k is an integer that is **greater** than the degree of $g(X)$.

Notation

If $p(X) = \sum_{i=1}^{d} p_i X^i$, then define

$$\overline{p}(X) := \sum_{i=1}^{d} \overline{p_i} X^i.$$

$$\oint \oint \oint \log |x + 1 + (x-1)(y+z)| \frac{dx}{x} \frac{dy}{y} \frac{dz}{z}$$

$$\oint \oint \oint \log \left| \frac{2(X^2+X+1+(X^2-1)(y+z))}{X+2} \right| \left(\frac{dX}{X} - \frac{dX}{2(X+2)} - \frac{dX}{2X^2(X^{-1}+2)} \right) \frac{dy}{y} \frac{dz}{z}.$$

$$\oint\oint\oint \log\left|x+1+(x-1)(y+z)\right| \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}$$

$$\Big\downarrow {\scriptstyle x=\frac{X(2X+1)}{X+2}}$$

$$\oint\oint\oint \log\left|\frac{2(X^2+X+1+(X^2-1)(y+z))}{X+2}\right| \left(\frac{dX}{X}-\frac{dX}{2(X+2)}-\frac{dX}{2X^2(X^{-1}+2)}\right)\frac{dy}{y}\frac{dz}{z}.$$

The Main Result

Let $g(X) \in \mathbb{C}[X]$ with all roots outside the unit disc, and let $f(X) = \pm X^k \overline{g}(X^{-1})$, for any integer $k > \mathrm{degree}\ g$. Then

$$m(f + g + (f - g)(y + z)) = \frac{28}{5\pi^2}\zeta(3) + m(g).$$

- Take $g(X) = X + 2$ and $f(X) = X(2X + 1)$ so that

$$m\left(2(X^2 + X + 1 + (X^2 - 1)(y + z))\right) = \frac{28}{5\pi^2}\zeta(3) + m(X + 2)$$
$$\implies m\left(X^2 + X + 1 + (X^2 - 1)(y + z)\right) = \frac{28}{5\pi^2}\zeta(3).$$

Ongoing Work

- Filling in the gaps in the proof
- More general results involving polynomials with more variables
- Apply methods to other cases involving, say, L-functions of elliptic curves.

  Boyd conjectured

$$m(1+(x-1)y+(x+1)z) \stackrel{?}{=} \frac{5}{4}L'(E_{21a1},-1).$$

  The above methods can be extended to come up with identities such as

$$m\left(\frac{x+2}{2}+(x^2+x+1)y+(x^2-1)z\right) \stackrel{?}{=} m(1+(x-1)y+(x+1)z).$$