THE MAHLER MEASURE OF AN n-VARIABLE FAMILY WITH NON-LINEAR DEGREE

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Abstract. We investigate the Mahler measure of a particular family of rational functions with arbitrary number of variables and arbitrary degree in one of the variables, generalizing previous results for families of arbitrary number of variables but linear dependence in each variable obtained in [Lal06].

1. Introduction

The (logarithmic) Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)^\times$ is defined as

$$\mathrm{m}(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where the integration is taken with respect to the Haar measure on the n-dimensional unit torus $\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\}$.

When P is a single variable polynomial, Jensen's formula implies that $\mathrm{m}(P)$ can be expressed in terms of the roots of P. While in the multivariable case there is no general formula for $\mathrm{m}(P)$, various examples are known where $\mathrm{m}(P)$ is related to special values of functions that are arithmetically significant, such as the Riemann zeta function, L-functions, etc. The first formula of this type was given by Smyth [Smy81, Boy81]:

$$\mathrm{m}(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2),$$

where $L(\chi_{-3}, s)$ is the Dirichlet L-function associated to the primitive character χ_{-3} of conductor 3. The appearance of these special values has been explained in terms of evaluations of regulators and Beilinson's conjectures by Deninger [Den97], Boyd [Boy98], and Rodriguez-Villegas [RV99] (see also the book of Brunault and Zudilin [BZ20] for a more detailed exposition).

Very few examples are known with more than three variables. Such examples represent important evidence towards understanding the relationship between Mahler measure and regulators. In [Lal03, Lal06] Lalín considered the Mahler measures of the following families of rational functions:

$$\begin{aligned} R_n(x_1, \dots, x_n, z) &:= z + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_n}{1 + x_n}\right), \\ S_n(x_1, \dots, x_n, x, y, z) &:= (1 + x)z + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_n}{1 + x_n}\right)(1 + y), \\ T_n(x_1, \dots, x_n, x, y) &:= 1 + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_n}{1 + x_n}\right) x + \left(1 - \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_n}{1 + x_n}\right)\right) y. \end{aligned}$$

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Notice that multiplication by $(1+x_1)\cdots(1+x_n)$ turns the above functions into polynomials, without changing the Mahler measure. They are written as rational functions for convenience.

For $a_1,\dots a_n\in\mathbb{C}$, define the symmetric functions as the coefficients of the polynomial $(x+a_1)\cdots(x+a_n)$, namely,

$$s_\ell(a_1,\dots,a_n)=\begin{cases}1 & \text{if } \ell=0,\\ \sum_{i_1<\cdots<i_\ell} a_{i_1}\cdots a_{i_\ell} & \text{if } 0<\ell\leq n,\\ 0 & \text{if } n<\ell.\end{cases} \tag{1}$$

We also set $s_0=1$ when $n=0$.

Recall that the Bernoulli numbers B_k are given by

$$\frac{x}{e^x-1}=\sum_{k=0}^\infty\frac{B_kx^k}{k!}.$$

Let for $n\geq 1$,

$$L(\chi_{-4},\ell)=\sum_{k=1}^\infty\frac{\chi_{-4}(k)}{k^\ell},\qquad \chi_{-4}(k)=\left(\frac{-4}{k}\right).$$

The Mahler measures of the polynomials R_n,S_n,T_n are then given by the following formulas [Lal06, LL16]. For $k\geq 1$,

$$\mathrm{m}\left(R_{2k}\right)=\sum_{h=1}^k\frac{s_{k-h}(2^2,4^2,\dots,(2k-2)^2)}{(2k-1)!}\left(\frac{2}{\pi}\right)^{2h}\mathcal{A}(h),$$

where

$$\mathcal{A}(h):=(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1).$$

For $k\geq 0$,

$$\mathrm{m}\left(R_{2k+1}\right)=\sum_{h=0}^k\frac{s_{k-h}(1^2,3^2,\dots,(2k-1)^2)}{(2k)!}\left(\frac{2}{\pi}\right)^{2h+1}\mathcal{B}(h),$$

where

$$\mathcal{B}(h):=(2h+1)!L(\chi_{-4},2h+2).$$

For $k\geq 1$,

$$\mathrm{m}\left(S_{2k}\right)=\sum_{h=1}^k\frac{s_{k-h}(2^2,4^2,\dots,(2k-2)^2)}{(2k-1)!}\left(\frac{2}{\pi}\right)^{2h+2}\mathcal{C}(h),$$

where

$$\mathcal{C}(h):=\sum_{\ell=1}^h\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{4h}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3).$$

For $k\geq 0$,

$$\mathrm{m}\left(S_{2k+1}\right)=\sum_{h=0}^k\frac{s_{k-h}(1^2,3^2\dots,(2k-1)^2)}{(2k)!}\left(\frac{2}{\pi}\right)^{2h+3}\mathcal{D}(h),$$

where

$$\mathcal{D}(h):=\sum_{\ell=0}^h\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2(2h+1)}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+3)!L(\chi_{-4},2\ell+4).$$

For $k \geq 1$,

$$\mathrm{m}\left(T_{2k}\right) = \frac{\log 2}{2} + \sum_{h=1}^{k} \frac{s_{k-h}(2^2, 4^2, \dots, (2k-2)^2)}{(2k-1)!} \left(\frac{2}{\pi}\right)^{2h} \mathcal{E}(h),$$

where

$$\mathcal{E}(h) := \frac{(2h)!}{2}\left(1 - \frac{1}{2^{2h+1}}\right)\zeta(2h+1) + \sum_{\ell=1}^{h} (2^{2(h-\ell)-1} - 1)\binom{2h}{2\ell}\frac{(-1)^{h-\ell+1}}{2h} \times B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell)!\left(1 - \frac{1}{2^{2\ell+1}}\right)\zeta(2\ell+1).$$

For $k \geq 0$,

$$\mathrm{m}\left(T_{2k+1}\right) = \frac{\log 2}{2} + \sum_{h=1}^{k} \frac{s_{k-h}(2^2, 4^2, \dots, (2k-2)^2)}{(2k+1)!} \left(\frac{2}{\pi}\right)^{2h+2} \mathcal{F}(h),$$

where

$$\mathcal{F}(h) := \frac{(2h+2)!}{2}\left(1 - \frac{1}{2^{2h+3}}\right)\zeta(2h+3) + \frac{\pi^2 k^2}{2}(2h)!\left(1 - \frac{1}{2^{2h+1}}\right)\zeta(2h+1) + k(2k+1)\sum_{\ell=1}^{h} (2^{2(h-\ell)-1} - 1)\binom{2h}{2\ell}\frac{(-1)^{h-\ell+1}}{4h} B_{2(h-\ell)}\pi^{2h+2-2\ell}(2\ell)!\left(1 - \frac{1}{2^{2\ell+1}}\right)\zeta(2\ell+1).$$

The above formulas are quite miraculous. Their computations are possible because the Möbius transformation $\frac{1-x}{1+x}$ has a particular elegant effect mapping the unit circle to the imaginary axis. The resulting differential in the change of variables also has very special properties, allowing for certain recurrences relating the case $n+2$ to the case n, which explains why the above formulas depend on the parity of n.

A similar phenomenon was recently explored by Nair [Nai23] who considered the family

$$Q_n(x_1, \dots, x_n, z) := z + \left(\frac{\omega + \overline{\omega} x_1}{1 + x_1}\right) \cdots \left(\frac{\omega + \overline{\omega} x_n}{1 + x_n}\right),$$

where

$$\omega = \frac{-1 + \sqrt{3}i}{2},$$

and proved similar formulas involving linear combinations of values of $\frac{\zeta(k)}{\pi^{k-1}}$ and $\frac{L(\chi_{-3},k)}{\pi^{k-1}}$ with certain rational coefficients.

In [Boy06], Boyd proposed the study of polynomials of the form $a(x) + b(x)y + c(x)z$, where $a(x), b(x), c(x)$ are products of cyclotomic polynomials. The reason for studying this particular class of polynomials comes from the Cassaigne–Maillot formula for the Mahler measure of $a + by + cz$ [Mai00], which has an expression that is particularly convenient for numerical integration. The investigation of such polynomials led to the discovery of several interesting numerical formulas involving L-functions of elliptic curves. Recently Brunault further pursued these computations with higher degree cyclotomic polynomials. This led to the discovery of certain formulas with arbitrary degree such as

$$\mathrm{m}\left(1 + (x^2 - x + 1)y + (x+1)^r z\right) = r\frac{3\sqrt{3}}{4\pi} L\left(\chi_{-3}, 2\right), \tag{2}$$

where r is an arbitrary positive integer.

In this work our aim is to combine both ideas. More precisely, we generalize the family S_n to

$$S_{n,r}(x_1,\dots,x_n,x,y,z) := (1+x)z + \left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r(1+y)$$

and we prove the following result.

Theorem 1. *Let $r\geq 1$. For $k\geq 1$, we have*

$$\mathrm{m}(S_{2k,r}) = \sum_{h=1}^{k}\frac{s_{k-h}(2^2,4^2,\dots,(2k-2)^2)}{(2k-1)!}\left(\frac{2}{\pi}\right)^{2h}\mathcal{C}_r(h),$$

where

$$\begin{aligned}\mathcal{C}_r(h) :=& r(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\\ &+\frac{r^2(2h-1)!}{\pi^2}\left\{\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1}+(-1)^r2^{2h-1}+(-1)^{r+1}\right)\right.\\ &+(2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\ &-\sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=2}^{2h+2}\left(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^\ell)-\mathrm{Li}_t(-\xi_{2r}^\ell)\right)-\binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\right)\right.\\ &\left.\left.\times\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\right]\right\}.\end{aligned}$$

For $k\geq 0$, we have

$$\mathrm{m}(S_{2k+1,r}) = \sum_{h=0}^{k}\frac{s_{k-h}(1^2,3^2,\dots,(2k-1)^2)}{(2k)!}\left(\frac{2}{\pi}\right)^{2h+1}\mathcal{D}_r(h),$$

where

$$\begin{aligned}\mathcal{D}_r(h) :=& r(2h+1)!L(\chi_{-4},2h+2)\\ &+\frac{2ir^2(2h)!}{\pi^2}\left\{\frac{(-1)^{h+1}(2^{2h+4}-1)B_{2h+4}\pi^{2h+4}}{r^{2h+3}(2h+4)!}-i\frac{(-1)^hE_{2h}\pi^{2h+1}}{r^22^{2h}(2h)!}\left(\mathrm{Li}_3((-i)^r)-\frac{1}{8}\mathrm{Li}_3((-1)^r)\right)\right.\\ &+(2h+3)(2h+2)\frac{1}{r^{2h+3}}\left(\mathrm{Li}_{2h+4}((-i)^r)-\frac{1}{2^{2h+4}}\mathrm{Li}_{2h+4}((-1)^r)\right)\\ &+\sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=1}^{2h+3}\left(\frac{(t-1)(t-2)}{2}(-1)^t\mathrm{Li}_t(-i\xi_{2r}^\ell)+\binom{t-1}{2h}\mathrm{Li}_t(-i)\right)\right.\\ &\left.\left.\times\frac{(2\pi i)^{2h+4-t}}{(2h+4-t)!}B_{2h+4-t}\left(\frac{\ell}{2r}\right)\right]\right\}.\end{aligned}$$

In the above formulas, ξ_{2r} denotes a primitive $2r$-root of unity, $\mathrm{Li}_\ell(z)$ denotes the polylogarithm function (see Definition 5), and $B_n(t)$ denotes de Bernoulli polynomial given by

$$\frac{xe^{xt}}{e^x-1}=\sum_{k=0}^{\infty}\frac{B_k(t)x^k}{k!}.$$

The importance of Theorem 1 is that it provides formulas for the Mahler measure of a family with arbitrarily many variables and arbitrarily large degree. In contrast, the previous results involve the families R_n, S_n and T_n that have arbitrarily many variables, but are linear in those variables. Moreover, the degree r plays a non-crucial role in the Mahler measure of $S_{n,r}$, as varying r fundamentally changes $\mathrm{m}(S_{n,r})$, as opposed to formula (2), where r is merely a factor in the final formula.

We remark that in the case $r = 1$, Theorem 1 reduces to the cases previously known for S_n, namely,

$$\mathcal{C}_1(h) = \frac{4}{\pi^2}\mathcal{C}(h) \qquad \text{and} \qquad \mathcal{D}_1(h) = \frac{4}{\pi^2}\mathcal{D}(h).$$

The case $r = 2$ also admits an interesting simplification as follows.

$$\begin{aligned}\mathcal{C}_2(h) =&(-1)^{h+1}\frac{7}{4h}B_{2h}\pi^{2h-2}\left(2^{2h}-1\right)\zeta(3)\\ &+4\sum_{\ell=0}^{h-1}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{h}\left(2^{2h-2\ell}-1\right)B_{2(h-\ell)}\pi^{2h-2\ell-2}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3)\\ &+\sum_{\ell=1}^{h}\binom{2h-1}{2\ell-1}\frac{(-1)^{h-\ell}}{2^{2h-2\ell-2}}E_{2(h-\ell)}\pi^{2h-2\ell-1}(2\ell+1)!L(\chi_{-4},2\ell+2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_2(h) =&(-1)^{h}\frac{21}{2^{2h+2}}E_{2h}\pi^{2h-1}\zeta(3)\\ &+8\sum_{\ell=0}^{h-1}\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2h+1}B_{2(h-\ell)}\pi^{2h-2\ell-2}(2\ell+3)!\left(2^{2h-2\ell}-1\right)L(\chi_{-4},2\ell+4)\\ &+\sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{2^{2h+1}}E_{2(h-\ell)}\pi^{2h-2\ell-1}(2\ell+2)!\left(2^{2\ell+3}-1\right)\zeta(2\ell+3),\end{aligned}$$

where the E_k are the Euler numbers given by

$$\frac{2e^x}{1+e^{2x}} = \sum_{k=0}^{\infty}\frac{E_k x^k}{k!}.$$

Tables 1 and 2 record the formulas for the Mahler measures of $S_{n,1}$ and $S_{n,2}$ respectively for the first few values of n. We have included the case $n = 0$, not covered in Theorem 1, for comparison purposes. We see that, although there is a clear distinction between the cases n even and odd for $\mathrm{m}(S_{n,1})$ in the sense that the formulas for n even only contain special values of the Riemann zeta function, and the formulas for n odd only contain special values of the Dirichlet L-function, for $\mathrm{m}(S_{n,2})$ the formulas are mixed.

When $r > 2$ it is more difficult to evaluate $\mathcal{C}_r(h)$ and $\mathcal{D}_r(h)$ in terms of special values of the Riemann zeta function and Dirichlet L-functions, due to the difficulty relating polylogarithms evaluated at roots of unity of higher order to special values of L-functions. We illustrate the formulas for the Mahler measures of $S_{1,r}$ for the first few values of r in Table 3. We remark the appearance of Dirichlet L-functions in the characters $\chi_{12}(11,\cdot) := \left(\frac{12}{\cdot}\right)$ of conductor 12 and $\chi_8(5,\cdot) := \left(\frac{8}{\cdot}\right)$ of conductor 8. This is a key distinction from the previous results for the families R_n, S_n and T_n.

$\pi^2 \mathrm{m}\left(1+x+(1+y)z\right)$	$\frac{7}{2}\zeta(3)$
$\pi^4 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)$	$93\zeta(5)$
$\pi^6 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)(1+y)z\right)$	$\frac{1905}{2}\zeta(7)+31\pi^2\zeta(5)$
$\pi^8 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_6}{1+x_6}\right)(1+y)z\right)$	$7154\zeta(9)+635\pi^2\zeta(7)+\frac{248\pi^4}{15}\zeta(5)$
$\pi^3 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)$	$24L(\chi_{-4},4)$
$\pi^5 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)(1+y)z\right)$	$320L(\chi_{-4},6)+4\pi^2L(\chi_{-4},4)$
$\pi^7 \mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_5}{1+x_5}\right)(1+y)z\right)$	$2688L(\chi_{-4},8)+160\pi^2L(\chi_{-4},6)+\frac{9\pi^4}{5}L(\chi_{-4},4)$

TABLE 1. Mahler measure of $S_{n,1}$ for $n \leq 6$.

$\pi^2 \mathrm{m}\left(1+x+(1+y)z\right)$	$\frac{7}{2}\zeta(3)$
$\pi^4 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right]^2(1+y)z\right)$	$96\pi L(\chi_{-4},4)-\frac{21\pi^2}{2}\zeta(3)$
$\pi^6 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)\right]^2(1+y)z\right)$	$1280\pi L(\chi_{-4},6)-372\pi^2\zeta(5)+$ $112\pi^3L(\chi_{-4},4)-\frac{21\pi^4}{2}\zeta(3)$
$\pi^8 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_6}{1+x_6}\right)\right]^2(1+y)z\right)$	$10752\pi L(\chi_{-4},8)-3810\pi^2\zeta(7)+$ $1920\pi^3L(\chi_{-4},6)-496\pi^4\zeta(5)+$ $\frac{596\pi^5}{5}L(\chi_{-4},4)-\frac{21\pi^6}{2}\zeta(3)$
$\pi^3 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\right]^2(1+y)z\right)$	$\frac{21\pi}{2}\zeta(3)$
$\pi^5 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_3}{1+x_3}\right)\right]^2(1+y)z\right)$	$\frac{31\pi}{2}\zeta(5)-96\pi^2L(\chi_{-4},4)+\frac{21\pi^3}{2}\zeta(3)$
$\pi^7 \mathrm{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_5}{1+x_5}\right)\right]^2(1+y)z\right)$	$\frac{127\pi}{24}\zeta(7)-1280\pi^2L(\chi_{-4},6)+\frac{62\pi^3}{3}\zeta(5)-$ $112\pi^4L(\chi_{-4},4)+\frac{21\pi^5}{2}\zeta(3)$

TABLE 2. Mahler measure of $S_{n,2}$ for $n \leq 6$.

$\pi^3\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)$	$24L(\chi_{-4},4)$
$\pi^3\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^2(1+y)z\right)$	$\frac{21\pi}{2}\zeta(3)$
$\pi^3\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^3(1+y)z\right)$	$-8L(\chi_{-4},4)+12\sqrt{3}\pi L(\chi_{12}(11,\cdot),3)$
$\pi^3\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^4(1+y)z\right)$	$-\frac{105\pi}{2}\zeta(3)+64\sqrt{2}\pi L(\chi_8(5,\cdot),3)$

Table 3. Mahler measure of $S_{1,r}$ for $r \leq 4$.

The proof of Theorem 1 relies on similar recursive strategies as used in the proofs of the previous results from [Lal06, Nai23] discussed above. For Theorem 1 we introduce a clever application of partial fractions that allows us to write the Mahler measure in terms of hyperlogarithms evaluated at roots of unity. This new idea allows us to make the important transition from the previous results at $r=1$ to the more general case of arbitrary r. These hyperlogarithms give rise to multiple polylogarithms that can then be reduced to length-one polylogarithms.

This article is organized as follows. Section 2 presents some preliminary results on evaluating certain necessary integrals that where proven in previous work ([Lal03, Lal06, LL16]). An introduction to the general theory of polylogarithms and hyperlogarithms is given in Section 3. The proof of Theorem 1 is given in Sections 4 and 5. More precisely, Section 4 describes the iterative process that leads to the Mahler measure being expressed in terms of integrals that can be related to hyperlogarithms, while these integrals are evaluated in Section 5. Discussions of the case $r=2$ and of the cases $n=1$ and $r=3,4$ are included in Section 6.

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2. Some preliminary results

The goal of this section is to state some results concerning the evaluation of certain integrals that were proven in [Lal03, Lal06, LL16] and that are necessary for the proof of Theorem 1.

Let $P_\alpha(y,w,z) = 1+y+\alpha(1+w)z$. The Mahler measure of this polynomial was initially computed by Smyth [Boy81, Smy02]. We state here a version given in [Lal03, Theorem 17].

Theorem 2.

$$\pi^2\mathrm{m}(1+y+\alpha(1+w)z) = \begin{cases} 2\mathcal{L}_3(|\alpha|) & \text{for} \quad |\alpha|\leq 1, \\ \pi^2\log|\alpha| + 2\mathcal{L}_3(|\alpha|^{-1}) & \text{for} \quad |\alpha|>1, \end{cases}$$

where, for $\beta > 0$,

$$\mathcal{L}_3(\beta) = -\frac{2}{\beta}\int_0^1 \frac{dt}{t^2-\frac{1}{\beta^2}} \circ \frac{dt}{t} \circ \frac{dt}{t} := -\frac{2}{\beta}\int_{0\le t_1\le t_2\le t_3\le 1} \frac{dt_1}{t_1^2-\frac{1}{\beta^2}}\frac{dt_2}{t_2}\frac{dt_3}{t_3}.$$

The following proposition allows us to compute an integral that will be key for the iterative process leading to Theorem 1.

Proposition 3. [Lal06, Proposition 5], [LL16, Proposition 5.5] *Let* $a, b > 0$ *and* $k \in \mathbb{Z}_{\ge 0}$. *We have*

$$\int_0^\infty \frac{x\log^k x dx}{(x^2+a^2)(x^2+b^2)} = \left(\frac{\pi}{2}\right)^{k+1} \frac{A_k\left(\frac{2\log a}{\pi}\right) - A_k\left(\frac{2\log b}{\pi}\right)}{a^2-b^2},$$

where the $A_k(x)$ *are polynomials in* $\mathbb{Q}[x]$ *given by*

$$R(T;x) = \frac{e^{xT}-1}{\sin T} = \sum_{k\ge 0} A_k(x)\frac{T^k}{k!}.$$

Remark 4. *The polynomials* $A_k(x)$ *satisfy the following recurrence.*

$$A_k(x) = \frac{x^{k+1}}{k+1} + \frac{1}{k+1}\sum_{\substack{j>1\\ \text{odd}}}^{k+1} (-1)^{\frac{j+1}{2}}\binom{k+1}{j} A_{k+1-j}(x),$$

and can be explicitly given by

$$A_k(x) = -\frac{2}{k+1}\sum_{h=0}^{k} B_h \binom{k+1}{h}(2^{h-1}-1)i^h x^{k+1-h},$$

where the B_n *are the Bernoulli numbers. (See the Appendix to* [Lal06] *and* [LL16, Lemma 5.2]*.)*

3. Integrals and polylogarithms

In order to understand how special values of zeta functions and L-series arise in our formulas, we need the definition of polylogarithms. Here we follow the notation of Goncharov [Gon95, Gon96].

Definition 5. *Multiple polylogarithms are defined as the power series*

$$\mathrm{Li}_{n_1,\dots,n_m}(x_1,\dots,x_m) := \sum_{0<k_1<k_2<\cdots<k_m} \frac{x_1^{k_1}x_2^{k_2}\dots x_m^{k_m}}{k_1^{n_1}k_2^{n_2}\dots k_m^{n_m}},$$

which are convergent for $|x_i| \le 1$ *and* $|x_m| < 1$ *if* $n_m = 1$. *The length of a polylogarithm function is the number* m *and its weight is the number* $w = n_1 + \dots + n_m$.

Definition 6. *Hyperlogarithms are defined as the iterated integrals*

$$\mathrm{I}_{n_1,\dots,n_m}(a_1 : \dots : a_m : a_{m+1}) :=$$
$$\int_0^{a_{m+1}} \underbrace{\frac{dt}{t-a_1}\circ\frac{dt}{t}\circ\dots\circ\frac{dt}{t}}_{n_1} \circ \underbrace{\frac{dt}{t-a_2}\circ\frac{dt}{t}\circ\dots\circ\frac{dt}{t}}_{n_2} \circ\dots\circ \underbrace{\frac{dt}{t-a_m}\circ\frac{dt}{t}\circ\dots\circ\frac{dt}{t}}_{n_m},$$

where n_i *are integers,* a_i *are complex numbers, and*

$$\int_0^{b_{k+1}} \frac{dt}{t-b_1}\circ\dots\circ\frac{dt}{t-b_k} = \int_{0\le t_1\le\dots\le t_k\le b_{k+1}} \frac{dt_1}{t_1-b_1}\cdots\frac{dt_k}{t_k-b_k}.$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C}\setminus\{a_1,\dots,a_m\}$.

It is easy to see (for instance, in [Gon96]) that,

$$\begin{aligned}
\mathrm{I}_{n_1,\dots,n_m}(a_1:\dots:a_m:a_{m+1}) &= (-1)^m \mathrm{Li}_{n_1,\dots,n_m}\left(\frac{a_2}{a_1},\frac{a_3}{a_2},\dots,\frac{a_m}{a_{m-1}},\frac{a_{m+1}}{a_m}\right),\\
\mathrm{Li}_{n_1,\dots,n_m}(x_1,\dots,x_m) &= (-1)^m \mathrm{I}_{n_1,\dots,n_m}((x_1\dots x_m)^{-1}:\dots:x_m^{-1}:1),
\end{aligned}$$

which gives an analytic continuation of multiple polylogarithms.

We remark that we recover the special value of the Riemann zeta function $\zeta(n)$ for $n\geq 2$ as

$$\mathrm{Li}_n(1)=\zeta(n),\qquad \mathrm{Li}_n(-1)=-\left(1-\frac{1}{2^{n-1}}\right)\zeta(n).$$

The evaluations at $x=i$ also give the Riemann zeta function as well as a Dirichlet L-function:

$$\mathrm{Re}(\mathrm{Li}_n(i))=-\frac{1}{2^n}\left(1-\frac{1}{2^{n-1}}\right)\zeta(n),\qquad \mathrm{Im}(\mathrm{Li}_n(i))=L(\chi_{-4},n).$$

We also have the following useful identity due to Jonquière [Jon89]

$$\mathrm{Li}_n(e^{2\pi ix})+(-1)^n\mathrm{Li}_n(e^{-2\pi ix})=-\frac{(2\pi i)^n}{n!}B_n(x),$$

where $B_n(x)$ is the Bernoulli polynomial, and $0\leq \mathrm{Re}(x)<1$ if $\mathrm{Im}(x)\geq 0$ and $0<\mathrm{Re}(x)\leq 1$ if $\mathrm{Im}(x)<0$. Notice that in particular we have for $0<\ell<2r$,

$$\mathrm{Li}_1(\xi_{2r}^{\ell})-\mathrm{Li}_1(\xi_{2r}^{-\ell})=\frac{(r-\ell)\pi i}{r},$$

where ξ_{2r} is a primitive $2r$-root of the unity.

We recall a technical result that will help us recognize special values of the Riemann zeta function and Dirichlet L-functions from certain integrals.

Lemma 7. [Lal06, Lemma 9] *We have the following length-one identities:*

$$\begin{aligned}
\int_0^1 \log^j x\frac{dx}{x^2-1} &= (-1)^{j+1}j!\left(1-\frac{1}{2^{j+1}}\right)\zeta(j+1),\\
\int_0^1 \log^j x\frac{dx}{x^2+1} &= (-1)^j j! L(\chi_{-4},j+1).
\end{aligned}$$

Some combinations of length 2 polylogarithms can be written in terms of length 1 polylogarithms. We use some results due to Nakamura [Nak12] and Panzer [Pan17]. Here we state the formulation of [LL18].

Theorem 8. [LL18, Theorem 3] *Let r,s be positive integers, $k=r+s$, and let u,v be complex numbers such that $|u|=|v|=1$. In addition, we assume that $v\neq 1$ if $s=1$. Let*

$$\mathrm{Re}_k=\begin{cases}\mathrm{Re} & k \text{ odd},\\ i\,\mathrm{Im} & k \text{ even}.\end{cases}$$

Then,

$$\begin{aligned}2\mathrm{Re}_k(\mathrm{Li}_{r,s}(u,v)) =&(-1)^k\mathrm{Li}_k(\overline{uv})+(-1)^{k+1}\mathrm{Li}_r(\overline{u})\mathrm{Li}_s(\overline{v})+(-1)^{r-1}\mathrm{Li}_r(\overline{u})\mathrm{Li}_s(v)\\&+(-1)^{r-1}\left(\binom{k-1}{r-1}\mathrm{Li}_k(\overline{u})+\binom{k-1}{s-1}\mathrm{Li}_k(v)\right)\\&+\sum_{t=1}^{k-1}\left(\binom{t-1}{r-1}\mathrm{Li}_t(\overline{u})+\binom{t-1}{s-1}(-1)^{k+t}\mathrm{Li}_t(v)\right)\\&\times((-1)^r\mathrm{Li}_{k-t}(uv)+(-1)^{s+t}\mathrm{Li}_{k-t}(\overline{uv})).\end{aligned}$$

The following statement is a direct application of the above result.

Corollary 9. *Let ξ_{2r} denote a primitive $2r$-root of unity. If h is a nonnegative integer, we have*

$$\begin{aligned}2i\,\mathrm{Im}\left(\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell},-i)\right) =&\mathrm{Li}_{2h+4}(\xi_{2r}^{\ell})-\mathrm{Li}_3(-i\xi_{2r}^{\ell})\mathrm{Li}_{2h+1}(i)+\mathrm{Li}_3(-i\xi_{2r}^{\ell})\mathrm{Li}_{2h+1}(-i)\\&+\left(\binom{2h+3}{2}\mathrm{Li}_{2h+4}(-i\xi_{2r}^{\ell})+\binom{2h+3}{2h}\mathrm{Li}_{2h+4}(-i)\right)\\&+\sum_{t=1}^{2h+3}\left(\binom{t-1}{2}\mathrm{Li}_t(-i\xi_{2r}^{\ell})+\binom{t-1}{2h}(-1)^t\mathrm{Li}_t(-i)\right)\\&\times(-\mathrm{Li}_{2h+4-t}(\xi_{2r}^{-\ell})-(-1)^t\mathrm{Li}_{2h+4-t}(\xi_{2r}^{\ell})).\end{aligned}\tag{3}$$

If h is a positive integer, we have

$$\begin{aligned}2\,\mathrm{Re}(\mathrm{Li}_{3,2h}\left(\pm\xi_{2r}^{-\ell},\pm1\right)) =&-\mathrm{Li}_{2h+3}(\xi_{2r}^{\ell})+2\mathrm{Li}_3(\pm\xi_{2r}^{\ell})\mathrm{Li}_{2h}(\pm1)\\&+\left(\binom{2h+2}{2}\mathrm{Li}_{2h+3}(\pm\xi_{2r}^{\ell})+\binom{2h+2}{2h-1}\mathrm{Li}_{2h+3}(\pm1)\right)\\&+\sum_{t=1}^{2h+2}\left(\binom{t-1}{2}\mathrm{Li}_t(\pm\xi_{2r}^{\ell})-\binom{t-1}{2h-1}(-1)^t\mathrm{Li}_t(\pm1)\right)\\&\times(-\mathrm{Li}_{2h+3-t}(\xi_{2r}^{-\ell})+(-1)^t\mathrm{Li}_{2h+3-t}(\xi_{2r}^{\ell})).\end{aligned}\tag{4}$$

Lemma 10. *We have*

$$\sum_{\ell=0}^{2r-1}(-1)^\ell\mathrm{Li}_h(\xi_{2r}^{\ell})=\frac{2-2^{1-h}}{r^{h-1}}\zeta(h),$$

$$\sum_{\ell=0}^{2r-1}(-1)^\ell\mathrm{Li}_h(-\xi_{2r}^{\ell})=(-1)^r\frac{2-2^{1-h}}{r^{h-1}}\zeta(h),\tag{5}$$

and

$$\sum_{\ell=0}^{2r-1}(-1)^\ell\mathrm{Li}_h(-i\xi_{2r}^{\ell})=\frac{2}{r^{h-1}}\left(\mathrm{Li}_h((-i)^r)-2^{-h}\mathrm{Li}_h((-1)^r)\right).$$

Proof. Indeed, we have

$$\begin{aligned}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_h(\xi_{2r}^\ell) =&\sum_{n=1}^\infty\sum_{\ell=0}^{2r-1}\frac{(-1)^\ell\xi_{2r}^{\ell n}}{n^h} = \sum_{n=1}^\infty\sum_{\ell=0}^{2r-1}\frac{(\xi_{2r}^{n+r})^\ell}{n^h}\\ =&2r\sum_{\substack{n=1\\ n\equiv r \bmod 2r}}^\infty \frac{1}{n^h} = \frac{2r}{r^h}\sum_{j=0}^\infty\frac{1}{(2j+1)^h}\\ =&\frac{2(1-2^{-h})}{r^{h-1}}\zeta(h).\end{aligned}$$

The proof of (5) is similar. We also have

$$\begin{aligned}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_h(-i\xi_{2r}^\ell) =&\sum_{n=1}^\infty\sum_{\ell=0}^{2r-1}\frac{(-1)^\ell(-i\xi_{2r}^{\ell})^n}{n^h} = \sum_{n=1}^\infty\frac{(-i)^n}{n^h}\sum_{\ell=0}^{2r-1}(\xi_{2r}^{n+r})^\ell\\ =&2r\sum_{\substack{n=1\\ n\equiv r \bmod 2r}}^\infty \frac{(-i)^n}{n^h} = \frac{2r}{r^h}\sum_{j=0}^\infty\frac{(-i)^{(2j+1)r}}{(2j+1)^h}\\ =&\frac{2}{r^{h-1}}\left(\mathrm{Li}_h((-i)^r)-2^{-h}\mathrm{Li}_h((-1)^r)\right).\end{aligned}$$

□

We finish this section by recalling some particular formulas for special values of $\zeta(s)$ and $L(\chi_{-4},2)$:

$$\zeta(2n)=\frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}\qquad\text{and}\qquad L(\chi_{-4},2n+1)=\frac{(-1)^nE_{2n}\pi^{2n+1}}{2^{2n+2}(2n)!},$$

where B_n and E_n denote the n^{th} Bernoulli and Euler numbers respectively.

4. General set-up

We start by first describing a general setting that could be applied to various rational functions. Then we will specialize this setting in the particular polynomial from the statement.

Let $P_\alpha \in \mathbb{C}(\mathbf{x})$ be a non-zero rational function such that its coefficients depend (as rational functions) on a parameter $\alpha\in\mathbb{C}$. We replace α by $\left[\left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_n-1}{x_n+1}\right)\right]^r$ and obtain a new rational function $\tilde{P}\in\mathbb{C}(\mathbf{x},x_1,\dots,x_n)$. By definition of the Mahler measure, one can see that

$$\mathrm{m}(\tilde{P})=\frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\mathrm{m}\left(P_{\left[\left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_n-1}{x_n+1}\right)\right]^r}\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

We first apply a change of variables to polar coordinates, $x_j=e^{i\theta_j}$:

$$=\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathrm{m}\left(P_{\left[i^n\tan\left(\frac{\theta_1}{2}\right)\cdots\tan\left(\frac{\theta_n}{2}\right)\right]^r}\right)d\theta_1\dots d\theta_n.$$

Now let $y_i = \tan\left(\frac{\theta_i}{2}\right)$. We get,

$$\begin{aligned}
=&\frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{m}\left(P_{(i^n y_1\cdots y_n)^r}\right)\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}\\
=&\frac{2^{n-1}}{\pi^n}\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(i^n y_1\cdots y_n)^r}\right)\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}\\
&+\frac{2^{n-1}}{\pi^n}\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(-i^n y_1\cdots y_n)^r}\right)\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}.
\end{aligned}$$

By making one more change of variables, $\hat{x}_1 = y_1, \dots, \hat{x}_{n-1} = y_1\cdots y_{n-1}, \hat{x}_n = y_1\cdots y_n$, we finally obtain

$$\begin{aligned}
=&\frac{2^{n-1}}{\pi^n}\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(i^n \hat{x}_n)^r}\right)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1} d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}\\
&+\frac{2^{n-1}}{\pi^n}\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(-i^n \hat{x}_n)^r}\right)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1} d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}.
\end{aligned}$$

Thus, to obtain our final formula, we need to compute this integral.

By iterating Proposition 3, the above integral can be written as a linear combination, with coefficients that are rational numbers and powers of π in such a way that the weights are homogeneous, of integrals of the form

$$\int_0^\infty \mathrm{m}\left(P_{(i^n x)^r}\right)\log^j x\frac{dx}{x^2\pm 1}+\int_0^\infty \mathrm{m}\left(P_{(-i^n x)^r}\right)\log^j x\frac{dx}{x^2\pm 1}. \tag{6}$$

One can see that j is even if and only if n is odd and the corresponding sign in that case is "+". This leads to the following construction.

Definition 11. [Lal06, Definition15] *Let $a_{k,j}\in\mathbb{Q}$ be defined for $k\geq 1$, $n=2k$ and $j=0,\dots,k-1$ by*

$$\begin{aligned}
&\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(\pm i^n \hat{x}_n)^r}\right)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1} d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}\\
=&\sum_{h=1}^{k} a_{k,h-1}\left(\frac{\pi}{2}\right)^{2k-2h}\int_0^\infty \mathrm{m}\left(P_{(\pm i^n x)^r}\right)\log^{2h-1}x\frac{dx}{x^2-1}.
\end{aligned}$$

Let $b_{k,j}\in\mathbb{Q}$ be defined for $k\geq 0$, $n=2k+1$ and $j=0,\dots,k$ by

$$\begin{aligned}
&\int_{0}^{\infty}\cdots\int_{0}^{\infty} \mathrm{m}\left(P_{(\pm i^n \hat{x}_n)^r}\right)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1} d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}\\
=&\sum_{h=0}^{k} b_{k,h}\left(\frac{\pi}{2}\right)^{2k-2h}\int_0^\infty \mathrm{m}\left(P_{(\pm i^n x)^r}\right)\log^{2h}x\frac{dx}{x^2+1}.
\end{aligned}$$

The following result is proven in [Lal06].

Theorem 12. [Lal06, Theorem 17] *For $k\geq 1$ and $h=0,\dots,k-1$, we have*

$$a_{k,h}=\frac{s_{k-1-h}(2^2,\dots,(2k-2)^2)}{(2k-1)!}.$$

For $k \geq 0$ and $h = 0, \ldots, k$, we have

$$b_{k,h} = \frac{s_{k-h}(1^2, \ldots, (2k-1)^2)}{(2k)!},$$

where we recall that the symmetric polynomials are given by (1).

It remains to evaluate the integrals of the type (6).

5. Integral reduction

In this section, we focus on evaluating the integral

$$\mathcal{I}_{r,j} := \int_0^\infty \mathrm{m}\left(P_{(i^n x)^r}\right) \log^j x \frac{dx}{x^2 + (-1)^j}$$

for the polynomial $P_\alpha = 1 + y + \alpha(1+w)z$ and we deduce our main result. Notice that in this case the Mahler measure is independent of the complex argument of α, and it therefore suffices to evaluate $\mathrm{m}\left(P_{x^r}\right)$. We have the following result.

Proposition 13. *Let $P_\alpha = 1 + y + \alpha(1+w)z$. When $h \geq 0$ we have*

$$\begin{aligned}\mathcal{I}_{r,2h} =& \frac{2ir^2(2h)!}{\pi^2}\left\{\frac{(-1)^{h+1}(2^{2h+4}-1)B_{2h+4}\pi^{2h+4}}{r^{2h+3}(2h+4)!} - i\frac{(-1)^h E_{2h}\pi^{2h+1}}{r^2 2^{2h}(2h)!}\left(\mathrm{Li}_3((-i)^r) - \frac{1}{8}\mathrm{Li}_3((-1)^r)\right)\right.\\ &+ (2h+3)(2h+2)\frac{1}{r^{2h+3}}\left(\mathrm{Li}_{2h+4}((-i)^r) - \frac{1}{2^{2h+4}}\mathrm{Li}_{2h+4}((-1)^r)\right)\\ &+ \sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=1}^{2h+3}\left(\frac{(t-1)(t-2)}{2}(-1)^t\mathrm{Li}_t(-i\xi_{2r}^\ell) + \binom{t-1}{2h}\mathrm{Li}_t(-i)\right)\right.\\ &\left.\left.\times \frac{(2\pi i)^{2h+4-t}}{(2h+4-t)!}B_{2h+4-t}\left(\frac{\ell}{2r}\right)\right]\right\} + r(2h+1)!L(\chi_{-4}, 2h+2).\end{aligned}$$

When $h \geq 1$ we have

$$\begin{aligned}\mathcal{I}_{r,2h-1} =& \frac{r^2(2h-1)!}{\pi^2}\left\{\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1} + (-1)^r 2^{2h-1} + (-1)^{r+1}\right)\right.\\ &+ (2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\ &- \sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=2}^{2h+2}\left(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^\ell) - \mathrm{Li}_t(-\xi_{2r}^\ell)\right) - \binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\right)\right.\\ &\left.\left.\times\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\right]\right\} + r(2h)!\left(1 - \frac{1}{2^{2h+1}}\right)\zeta(2h+1).\end{aligned}$$

Proof. We start by splitting the integral according to $0 \leq x \leq 1$ and $1 \leq x$.

$$\mathcal{I}_{r,j} = \int_0^1 \mathrm{m}\left(P_{x^r}\right)\log^j x\frac{dx}{x^2+(-1)^j} + \int_1^\infty \mathrm{m}\left(P_{x^r}\right)\log^j x\frac{dx}{x^2+(-1)^j}.$$

By applying Theorem 2, we obtain

$$\mathcal{I}_{r,j} = \int_0^1 \left(-\frac{4}{x^r\pi^2}\right)\int_0^1 \frac{dt}{t^2-\frac{1}{x^{2r}}}\circ\frac{dt}{t}\circ\frac{dt}{t}\frac{\log^j x dx}{x^2+(-1)^j}$$
$$+\int_1^\infty \left(\log\left(x^r\right)+\left(-\frac{4x^r}{\pi^2}\right)\int_0^1 \frac{dt}{t^2-x^{2r}}\circ\frac{dt}{t}\circ\frac{dt}{t}\right)\frac{\log^j x dx}{x^2+(-1)^j}.$$

Denoting the t-variables by $0\le t_1\le t_2\le t_3\le 1$, we consider the following changes of variables. For the first term above, we let

$$t_1=\frac{s_1^r}{x^r},\quad t_2=\frac{s_2^r}{x^r},\quad t_3=\frac{s_3^r}{x^r},$$

and for the second term we let

$$t_1=\frac{x^r}{s_1^r},\quad t_2=\frac{x^r}{s_2^r},\quad t_3=\frac{x^r}{s_3^r}.$$

This leads to

$$\mathcal{I}_{r,j} = -\frac{4}{\pi^2}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1}\circ\frac{rds}{s}\circ\frac{rds}{s}\circ\frac{\log^j x dx}{x^2+(-1)^j}$$
$$+r\int_1^\infty \frac{\log^{j+1} x dx}{x^2+(-1)^j}-\frac{4}{\pi^2}\int_1^\infty \frac{\log^j x dx}{x^2+(-1)^j}\circ\frac{(-r)ds}{s}\circ\frac{(-r)ds}{s}\circ\frac{(-r)s^{r-1}ds}{1-s^{2r}}.$$

In the last two integrals, we reverse $s\to\frac{1}{s}$ and $x\to\frac{1}{x}$ to get

$$\mathcal{I}_{r,j} = -\frac{4}{\pi^2}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1}\circ\frac{rds}{s}\circ\frac{rds}{s}\circ\frac{\log^j x dx}{x^2+(-1)^j}$$
$$-r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j}-\frac{4}{\pi^2}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1}\circ\frac{rds}{s}\circ\frac{rds}{s}\circ\frac{\log^j x dx}{x^2+(-1)^j}$$
$$=-\frac{8r^2}{\pi^2}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1}\circ\frac{ds}{s}\circ\frac{ds}{s}\circ\frac{\log^j x dx}{x^2+(-1)^j}-r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j}$$
$$=-\frac{4r^2j!(-1)^j}{\pi^2 i^{j+1}}\int_0^1 \frac{rs^{r-1}ds}{s^{2r}-1}\circ\frac{ds}{s}\circ\frac{ds}{s}\circ\left(\frac{1}{x-i^{j+1}}-\frac{1}{x+i^{j+1}}\right)dx\circ\underbrace{\frac{du}{u}\circ\cdots\circ\frac{du}{u}}_{j\text{ times}}$$
$$-r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j}.$$

Let ξ_{2r} be a primitive $(2r)^{\text{th}}$ root of unity. We can then write

$$s^r-1=\prod_{\ell=0}^{r-1}(s-\xi_{2r}^{2\ell})\quad\text{and}\quad s^r+1=\prod_{\ell=0}^{r-1}(s-\xi_{2r}^{2\ell+1}).$$

By applying logarithmic derivatives above, we get

$$\frac{rs^{r-1}}{s^r-1}=\sum_{\ell=0}^{r-1}\frac{1}{s-\xi_{2r}^{2\ell}}\quad\text{and}\quad\frac{rs^{r-1}}{s^r+1}=\sum_{\ell=0}^{r-1}\frac{1}{s-\xi_{2r}^{2\ell+1}}.$$

This gives

$$\frac{rs^{r-1}}{s^{2r}-1}=\frac{rs^{r-1}}{2(s^r-1)}-\frac{rs^{r-1}}{2(s^r+1)}=\frac{1}{2}\sum_{\ell=0}^{2r-1}\frac{(-1)^\ell}{s-\xi_{2r}^{\ell}}.$$

Finally we have

$$\begin{aligned}\mathcal{I}_{r,j} =& -\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^\ell I_{3,j+1}\left(\xi_{2r}^\ell : i^{j+1} : 1\right) + \frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^\ell I_{3,j+1}\left(\xi_{2r}^\ell : -i^{j+1} : 1\right)\\ & - r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j}\\ =& -\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_{3,j+1}(i^{j+1}\xi_{2r}^{-\ell}, i^{-j-1}) + \frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_{3,j+1}(-i^{j+1}\xi_{2r}^{-\ell}, -i^{-j-1})\\ & - r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j}. \end{aligned} \tag{7}$$

By Lemma 7, we have

$$-r\int_0^1 \frac{\log^{j+1} x dx}{x^2+(-1)^j} = \begin{cases} r(j+1)!\left(1-\dfrac{1}{2^{j+2}}\right)\zeta(j+2) & j \text{ odd},\\ \\ r(j+1)!L(\chi_{-4}, j+2) & j \text{ even}.\end{cases} \tag{8}$$

When $j = 2h$ is even, we have that

$$\begin{aligned} & -\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,j+1}(i^{j+1}\xi_{2r}^{-\ell}, i^{-j-1}) - \mathrm{Li}_{3,j+1}(-i^{j+1}\xi_{2r}^{-\ell}, -i^{-j-1})\right)\\ &= \frac{2r^2 i(-1)^h(2h)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,2h+1}(i(-1)^h\xi_{2r}^{-\ell}, i(-1)^{h+1}) - \mathrm{Li}_{3,2h+1}(i(-1)^{h+1}\xi_{2r}^{-\ell}, i(-1)^h)\right)\\ &= \frac{2r^2 i(2h)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell}, -i) - \mathrm{Li}_{3,2h+1}(-i\xi_{2r}^{-\ell}, i)\right)\\ &= \frac{2r^2 i(2h)!}{\pi^2}\sum_{\ell=0}^{2r-1}\left((-1)^\ell\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell}, -i) - (-1)^{2r-\ell}\mathrm{Li}_{3,2h+1}(-i\xi_{2r}^{2r-\ell}, i)\right)\\ &= -\frac{4r^2(2h)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell\,\mathrm{Im}\left(\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell}, -i)\right). \end{aligned} \tag{9}$$

When $j = 2h-1$ is odd, we have that

$$\begin{aligned} & -\frac{2r^2 j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,j+1}(i^{j+1}\xi_{2r}^{-\ell}, i^{-j-1}) - \mathrm{Li}_{3,j+1}(-i^{j+1}\xi_{2r}^{-\ell}, -i^{-j-1})\right)\\ &= \frac{2r^2(-1)^h(2h-1)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,2h}((-1)^h\xi_{2r}^{-\ell}, (-1)^h) - \mathrm{Li}_{3,2h}(-(-1)^h\xi_{2r}^{-\ell}, -(-1)^h)\right)\\ &= \frac{2r^2(2h-1)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Li}_{3,2h}(\xi_{2r}^{-\ell}, 1) - \mathrm{Li}_{3,2h}(-\xi_{2r}^{-\ell}, -1)\right). \end{aligned} \tag{10}$$

Notice that one can combine

$$\begin{aligned}(-1)^{\ell}\mathrm{Li}_{3,2h}\left(\xi_{2r}^{-\ell},1\right)+(-1)^{2r-\ell}\mathrm{Li}_{3,2h}\left(\xi_{2r}^{-2r+\ell},1\right)=&(-1)^{\ell}\mathrm{Li}_{3,2h}\left(\xi_{2r}^{-\ell},1\right)+(-1)^{\ell}\mathrm{Li}_{3,2h}\left(\xi_{2r}^{\ell},1\right)\\ =&(-1)^{\ell}2\,\mathrm{Re}(\mathrm{Li}_{3,2h}\left(\xi_{2r}^{-\ell},1\right))\end{aligned}$$

and similarly with

$$(-1)^{\ell}\mathrm{Li}_{3,2h}\left(-\xi_{2r}^{-\ell},-1\right)+(-1)^{2r-\ell}\mathrm{Li}_{3,2h}\left(-\xi_{2r}^{-2r+\ell},-1\right)=(-1)^{\ell}2\,\mathrm{Re}(\mathrm{Li}_{3,2h}\left(-\xi_{2r}^{-\ell},-1\right)).$$

By combining the above with (10), we finally have that, when $j=2h-1$ is odd,

$$\begin{aligned}&-\frac{2r^2j!(-1)^j}{\pi^2 i^{j+1}}\sum_{\ell=0}^{2r-1}(-1)^{\ell}\left(\mathrm{Li}_{3,j+1}(i^{j+1}\xi_{2r}^{-\ell},i^{-j-1})-\mathrm{Li}_{3,j+1}(-i^{j+1}\xi_{2r}^{-\ell},-i^{-j-1})\right)\\ &=\frac{2r^2(2h-1)!}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^{\ell}\left(\mathrm{Re}\left(\mathrm{Li}_{3,j+1}(\xi_{2r}^{-\ell},1)\right)-\mathrm{Re}\left(\mathrm{Li}_{3,j+1}(-\xi_{2r}^{-\ell},-1)\right)\right).\end{aligned}\tag{11}$$

In order to continue the simplification, we apply Corollary 9. Equation (3) gives, for $j=2h$,

$$\begin{aligned}&2i\sum_{\ell=0}^{2r-1}(-1)^{\ell}\,\mathrm{Im}\left(\mathrm{Li}_{3,2h+1}(i\xi_{2r}^{-\ell},-i)\right)\\ &=\sum_{\ell=0}^{2r-1}(-1)^{\ell}\Bigg[\mathrm{Li}_{2h+4}(\xi_{2r}^{\ell})-\mathrm{Li}_3(-i\xi_{2r}^{\ell})\mathrm{Li}_{2h+1}(i)+\mathrm{Li}_3(-i\xi_{2r}^{\ell})\mathrm{Li}_{2h+1}(-i)+\binom{2h+3}{2}\mathrm{Li}_{2h+4}(-i\xi_{2r}^{\ell})\\ &+\sum_{t=1}^{2h+3}\left(\binom{t-1}{2}\mathrm{Li}_t(-i\xi_{2r}^{\ell})+\binom{t-1}{2h}(-1)^t\mathrm{Li}_t(-i)\right)\left(-\mathrm{Li}_{2h+4-t}(\xi_{2r}^{-\ell})-(-1)^t\mathrm{Li}_{2h+4-t}(\xi_{2r}^{\ell})\right)\Bigg].\end{aligned}$$

We now apply part of Lemma 10 and other identities from Section 3 to see that the above equals

$$\begin{aligned}=&\frac{(-1)^{h+1}(2^{2h+4}-1)B_{2h+4}\pi^{2h+4}}{r^{2h+3}(2h+4)!}-i\frac{(-1)^hE_{2h}\pi^{2h+1}}{r^2 2^{2h}(2h)!}\left(\mathrm{Li}_3((-i)^r)-\frac{1}{8}\mathrm{Li}_3((-1)^r)\right)\\ &+(2h+3)(2h+2)\frac{1}{r^{2h+3}}\left(\mathrm{Li}_{2h+4}((-i)^r)-\frac{1}{2^{2h+4}}\mathrm{Li}_{2h+4}((-1)^r)\right)\\ &+\sum_{\ell=0}^{2r-1}(-1)^{\ell}\left[\sum_{t=1}^{2h+3}\left(\frac{(t-1)(t-2)}{2}(-1)^t\mathrm{Li}_t(-i\xi_{2r}^{\ell})+\binom{t-1}{2h}\mathrm{Li}_t(-i)\right)\frac{(2\pi i)^{2h+4-t}}{(2h+4-t)!}B_{2h+4-t}\left(\frac{\ell}{2r}\right)\right].\end{aligned}\tag{12}$$

Equation (4) gives for $j=2h-1$,

$$
\begin{aligned}
&2\sum_{\ell=0}^{2r-1}(-1)^\ell\left(\mathrm{Re}\left(\mathrm{Li}_{3,j+1}(\xi_{2r}^{-\ell},1)\right)-\mathrm{Re}\left(\mathrm{Li}_{3,j+1}(-\xi_{2r}^{-\ell},-1)\right)\right)\\
&=\sum_{\ell=0}^{2r-1}(-1)^\ell\Bigg[2\mathrm{Li}_3(\xi_{2r}^\ell)\mathrm{Li}_{2h}(1)-2\mathrm{Li}_3(-\xi_{2r}^\ell)\mathrm{Li}_{2h}(-1)+\binom{2h+2}{2}\left(\mathrm{Li}_{2h+3}(\xi_{2r}^\ell)-\mathrm{Li}_{2h+3}(-\xi_{2r}^\ell)\right)\\
&+\sum_{t=1}^{2h+2}\left(\binom{t-1}{2}\left(\mathrm{Li}_t(\xi_{2r}^\ell)-\mathrm{Li}_t(-\xi_{2r}^\ell)\right)-\binom{t-1}{2h-1}(-1)^t\left(\mathrm{Li}_t(1)-\mathrm{Li}_t(-1)\right)\right)\\
&\times\left(-\mathrm{Li}_{2h+3-t}(\xi_{2r}^{-\ell})+(-1)^t\mathrm{Li}_{2h+3-t}(\xi_{2r}^\ell)\right)\Bigg].
\end{aligned}
$$

Again, we apply part of Lemma 10 and other identities from Section 3 to see that the above equals

$$
\begin{aligned}
=&\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1}+(-1)^r2^{2h-1}+(-1)^{r+1}\right)+(2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\
&-\sum_{\ell=0}^{2r-1}(-1)^\ell\Bigg[\sum_{t=2}^{2h+2}\left(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^\ell)-\mathrm{Li}_t(-\xi_{2r}^\ell)\right)-\binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\right)\\
&\times\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\Bigg].
\end{aligned}
\tag{13}
$$

Combining equations (12) and (13) with (9), (11), and (8) in (7) concludes the proof of the statement. □

Proof of Theorem 1. By Definition 11, we have that

$$
\mathrm{m}(S_{2k,r})=\sum_{h=1}^{k}a_{k,h-1}\left(\frac{2}{\pi}\right)^{2h}\mathcal{I}_{r,2h-1}
$$

and

$$
\mathrm{m}(S_{2k+1,r})=\sum_{h=0}^{k}b_{k,h}\left(\frac{2}{\pi}\right)^{2h+1}\mathcal{I}_{r,2h}.
$$

The result the follows from Theorem 12 and Proposition 13, by setting $\mathcal{C}_r(h):=\mathcal{I}_{r,2h-1}$ and $\mathcal{D}_r(h):=\mathcal{I}_{r,2h}$. □

6. Some particular cases

In this section we focus on the simplest cases, for low values of r or n.

For the case $r=1$, and $j=2h$, we have, from (12),

$$\begin{aligned}&\sum_{\ell=0}^{1}(-1)^{\ell}\operatorname{Im}\left(\mathrm{Li}_{3,2h+1}(i(-1)^{-\ell},-i)\right)\\&=-(2h+3)(h+1)L(\chi_{-4},2h+4)+(2h+1)L(\chi_{-4},2h+2)\frac{\pi^2}{4}\\&+\sum_{s=2}^{h+1}(2s-1)(s-1)L(\chi_{-4},2s)\frac{(-1)^{h+1-s}B_{2h+4-2s}\pi^{2h+4-2s}}{(2h+4-2s)!}.\end{aligned}$$

This gives, for $r=1$,

$$\mathcal{I}_{1,2h}=\frac{2}{\pi^2}\sum_{\ell=0}^{h}\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2h+1}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+3)!L(\chi_{-4},2\ell+4)=\frac{4}{\pi^2}\mathcal{D}(h),$$

where we have set $s=\ell+2$.

For the case $r=1$, and $j=2h-1$, we have, from (13),

$$\begin{aligned}&\sum_{\ell=0}^{1}(-1)^{\ell}\left(\operatorname{Re}\left(\mathrm{Li}_{3,j+1}((-1)^{-\ell},1)\right)-\operatorname{Re}\left(\mathrm{Li}_{3,j+1}(-(-1)^{-\ell},-1)\right)\right)\\&=(h+1)(2h+1)\left(2-\frac{1}{2^{2h+2}}\right)\zeta(2h+3)-h\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\pi^2\\&-\sum_{s=2}^{h}s(2s-1)\left(2-\frac{1}{2^{2s}}\right)\zeta(2s+1)\frac{(-1)^{h-s}B_{2h+2-2s}\pi^{2h+2-2s}}{(2h+2-2s)!}.\end{aligned}$$

This gives, for $r=1$,

$$\mathcal{I}_{1,2h-1}=\frac{1}{\pi^2}\sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{h}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3)=\frac{4}{\pi^2}\mathcal{C}(h),$$

where we have set $s=\ell+1$.

For the case $r=2$ and $j=2h$, we have, from (12),

$$\begin{aligned}&\sum_{\ell=0}^{3}(-1)^{\ell}\operatorname{Im}\left(\mathrm{Li}_{3,2h+1}(i^{1-\ell},-i)\right)\\&=\frac{(-1)^{h+1}21\pi^{2h+1}}{2^{2h+6}(2h)!}E_{2h}\zeta(3)+(2h+1)L(\chi_{-4},2h+2)\frac{\pi^2}{8}\\&-\sum_{s=2}^{h+1}\binom{2s-1}{2}(-1)^{h-s}L(\chi_{-4},2s)\left(2^{2h+4-2s}-1\right)\frac{\pi^{2h+4-2s}}{(2h+4-2s)!}B_{2h+4-2s}\\&+\sum_{s=2}^{h+1}\binom{2s}{2}(-1)^{h-s}\left(2^{2s+1}-1\right)\zeta(2s+1)\frac{\pi^{2h+3-2s}}{2^{2h+4}(2h+2-2s)!}E_{2h+2-2s}.\end{aligned}$$

This gives, for $r=2$,

$$\begin{aligned}\mathcal{I}_{2,2h} =&\frac{(-1)^h 21}{2^{2h+2}}E_{2h}\pi^{2h-1}\zeta(3)\\ &+8\sum_{\ell=0}^{h-1}\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2h+1}B_{2(h-\ell)}\pi^{2h-2\ell-2}(2\ell+3)!\left(2^{2h-2\ell}-1\right)L(\chi_{-4},2\ell+4)\\ &+\sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{2^{2h+1}}E_{2(h-\ell)}\pi^{2h-2\ell-1}(2\ell+2)!\left(2^{2\ell+3}-1\right)\zeta(2\ell+3)\\ =&\mathcal{D}_2(h).\end{aligned}$$

For the case $r=2$, and $j=2h-1$, we have, from (13),

$$\begin{aligned}&\sum_{\ell=0}^{3}(-1)^\ell\left(\mathrm{Re}\left(\mathrm{Li}_{3,2h}(i^{-\ell},1)\right)-\mathrm{Re}\left(\mathrm{Li}_{3,2h}(-i^{-\ell},-1)\right)\right)\\ =&\frac{(-1)^{h+1}7\pi^{2h}}{16(2h)!}B_{2h}\left(2^{2h}-1\right)\zeta(3)-h\left(2-\frac{1}{2^{2h}}\right)\zeta(2h+1)\frac{\pi^2}{4}\\ &-\sum_{s=1}^{h}\binom{2s}{2}(-1)^{h-s}\left(2-\frac{1}{2^{2s}}\right)\zeta(2s+1)\left(2^{2h+2-2s}-1\right)\frac{\pi^{2h+2-2s}}{(2h+2-2s)!}B_{2h+2-2s}\\ &-\sum_{s=1}^{h+1}\binom{2s-1}{2}(-1)^{h-s}L(\chi_{-4},2s)\frac{\pi^{2h+3-2s}}{2^{2h+2-2s}(2h+2-2s)!}E_{2h+2-2s}.\end{aligned}$$

This gives, for $r=2$,

$$\begin{aligned}\mathcal{I}_{2,2h-1} =&\frac{(-1)^{h+1}7}{4h}B_{2h}\pi^{2h-2}\left(2^{2h}-1\right)\zeta(3)\\ &+4\sum_{\ell=0}^{h-1}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{h}\left(2^{2h-2\ell}-1\right)B_{2(h-\ell)}\pi^{2h-2\ell-2}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3)\\ &+\sum_{\ell=1}^{h}\binom{2h-1}{2\ell-1}\frac{(-1)^{h-\ell}}{2^{2h-2\ell-2}}E_{2(h-\ell)}\pi^{2h-2\ell-1}(2\ell+1)!L(\chi_{-4},2\ell+2)\\ =&\mathcal{C}_2(h).\end{aligned}$$

The evaluation of $\mathcal{I}_{r,j}$ and $\mathrm{m}(S_{n,r})$ for $r>2$ quickly becomes computationally involved. We will focus on the case $n=1$. This corresponds to the case $k=h=0$ and $\mathcal{I}_{r,0}$. We remark that for $j=0$ we have

$$\begin{aligned}\mathcal{I}_{r,0} =&\mathrm{Re}\left[\frac{12i}{\pi^2 r}\mathrm{Li}_4((-i)^r)-\frac{2}{\pi}\mathrm{Li}_3((-i)^r)+\frac{2r}{\pi}\sum_{\ell=0}^{2r-1}(-1)^\ell\mathrm{Li}_3(-i\xi_{2r}^\ell)\ell\right]\\ &-\frac{3r^2}{16\pi}\zeta(3)+\frac{1}{4\pi}\mathrm{Li}_3((-1)^r)\end{aligned}$$

and

$$\mathrm{m}(S_{1,r}) = \mathrm{Re}\left[\frac{24i}{\pi^3 r}\mathrm{Li}_4((-i)^r) - \frac{4}{\pi^2}\mathrm{Li}_3((-i)^r) + \frac{4r}{\pi^2}\sum_{\ell=0}^{2r-1}(-1)^\ell \mathrm{Li}_3(-i\xi_{2r}^\ell)\ell\right]$$
$$- \frac{3r^2}{8\pi^2}\zeta(3) + \frac{1}{2\pi^2}\mathrm{Li}_3((-1)^r).$$

We get different cases according to the class of $r \bmod 4$.

For $r = 2s+1$, we have

$$\mathrm{m}(S_{1,2s+1}) = \frac{24(-1)^s}{(2s+1)\pi^3}L(\chi_{-4},4) - \frac{3(2s+1)^2}{8\pi^2}\zeta(3) + \frac{4(2s+1)}{\pi^2}\sum_{\ell=0}^{4s+1}(-1)^\ell\,\mathrm{Re}(\mathrm{Li}_3(-i\xi_{4s+2}^\ell))\ell.$$

For $r = 4s$, we have

$$\mathrm{m}(S_{1,4s}) = -\frac{12s^2+7}{2\pi^2}\zeta(3) + \frac{16s}{\pi^2}\sum_{\ell=0}^{8s-1}(-1)^\ell\,\mathrm{Re}(\mathrm{Li}_3(-i\xi_{8s}^\ell))\ell.$$

For $r = 4s+2$, we have

$$\mathrm{m}(S_{1,4s+2}) = -\frac{6s^2+6s-2}{\pi^2}\zeta(3) + \frac{16s+8}{\pi^2}\sum_{\ell=0}^{8s+3}(-1)^\ell\,\mathrm{Re}(\mathrm{Li}_3(-i\xi_{8s+4}^\ell))\ell.$$

Specializing in $r = 1, 2$ we recover the formulas for the Mahler measures of $S_{1,1}$ and $S_{1,2}$. We now provide additional details for the cases $r = 3, 4$.

For $r = 3$, we must find

$$\sum_{\ell=0}^{5}(-1)^\ell\,\mathrm{Re}(\mathrm{Li}_3(-i\xi_6^\ell))\ell = -\mathrm{Re}(\mathrm{Li}_3(e^{\frac{11\pi i}{6}})) + 2\,\mathrm{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{6}})) - 3\,\mathrm{Re}(\mathrm{Li}_3(i))$$
$$+ 4\,\mathrm{Re}(\mathrm{Li}_3(e^{\frac{5\pi i}{6}})) - 5\,\mathrm{Re}(\mathrm{Li}_3(e^{\frac{7\pi i}{6}}))$$
$$= \mathrm{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{6}})) - \mathrm{Re}(\mathrm{Li}_3(e^{\frac{5\pi i}{6}})) - 3\,\mathrm{Re}(\mathrm{Li}_3(i)),$$

since $\mathrm{Li}(\overline{z}) = \overline{\mathrm{Li}(z)}$. Now consider

$$\mathrm{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{6}})) = \sum_{k=1}^{\infty}\frac{\cos\frac{k\pi}{6}}{k^3}$$
$$= \frac{\sqrt{3}}{2}\left(\frac{1}{1^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{11^3} + \cdots\right) + \frac{1}{2}\left(\frac{1}{2^3} - \frac{1}{4^3} - \frac{2}{6^3} - \frac{1}{8^3} + \frac{1}{10^3} + \frac{2}{12^3} + \cdots\right).$$

This sum is absolutely convergent and we may rearrange the terms as desired. Let $\chi_{12}(11,n)$ be the Dirichlet character of conductor 12 given by $\left(\frac{12}{n}\right)$. This corresponds to the character $\chi_{12,4}$ according to Mathematica. Its values are given by

n	1	5	7	11
$\chi_{12}(11,n)$	1	-1	-1	1

so that

$$\frac{\sqrt{3}}{2}\left(\frac{1}{1^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{11^3} + \cdots\right) = \frac{\sqrt{3}}{2}\cdot L(\chi_{12}(11,\cdot),3).$$

We can also write

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{2^3}-\frac{1}{4^3}-\frac{2}{6^3}-\frac{1}{8^3}+\frac{1}{10^3}+\frac{2}{12^3}+\cdots\right) &= \frac{1}{2}\left(\frac{1}{2^3}-\frac{1}{4^3}+\frac{1}{6^3}-\frac{1}{8^3}+\frac{1}{10^3}-\frac{1}{12^3}+\cdots\right)\\ &\quad -\frac{3}{2}\left(\frac{1}{6^3}-\frac{1}{12^3}+\frac{1}{18^3}+\cdots\right)\\ &= -\left(\frac{1}{2\cdot 2^3}-\frac{3}{2\cdot 6^3}\right)\mathrm{Li}_3(-1)\\ &= \frac{\zeta(3)}{24}.\end{aligned}$$

Therefore,

$$\mathrm{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{6}})) = \frac{\zeta(3)}{24}+\frac{\sqrt{3}}{2}L(\chi_{12}(11,\cdot),3).$$

Similarly, we can show that

$$\mathrm{Re}(\mathrm{Li}_3(e^{\frac{5\pi i}{6}})) = \frac{\zeta(3)}{24}-\frac{\sqrt{3}}{2}L(\chi_{12}(11,\cdot),3),$$

and using that $\mathrm{Re}(\mathrm{Li}_3(i)) = -\frac{3}{32}\zeta(3)$, we obtain

$$\sum_{\ell=0}^{5}(-1)^\ell\,\mathrm{Re}(\mathrm{Li}_3(-i\xi_6^\ell))\ell = \frac{9}{32}\zeta(3)+\sqrt{3}L(\chi_{12}(11,\cdot),3),$$

which gives

$$\mathrm{m}(S_{1,3}) = \frac{12\sqrt{3}}{\pi^2}L(\chi_{12}(11,\cdot),3)-\frac{8}{\pi^3}L(\chi_{-4},4).$$

When $m=4$, using similar manipulations, we can also show

$$\begin{aligned}\mathrm{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{4}})) &= -\frac{3}{4^4}\zeta(3)+\frac{1}{\sqrt{2}}L(\chi_8(5,\cdot),3),\\ \mathrm{Re}(\mathrm{Li}_3(e^{\frac{3\pi i}{4}})) &= -\frac{3}{4^4}\zeta(3)-\frac{1}{\sqrt{2}}L(\chi_8(5,\cdot),3),\end{aligned}$$

where $\chi_8(5,n)$ is the Dirichlet character of conductor 8 given by $\left(\frac{8}{n}\right)$. This corresponds to the character $\chi_{8,2}$ according to Mathematica. Its values are given by

n	1	3	5	7
$\chi_8(5,n)$	1	-1	-1	1

.

Thus,

$$\begin{aligned}\sum_{\ell=0}^{7}(-1)^\ell\,\mathrm{Re}(\mathrm{Li}_3(-i\xi_8^\ell))\ell &= -4\,\mathrm{Re}(\mathrm{Li}_3(e^{\frac{\pi i}{4}}))-12\,\mathrm{Re}(\mathrm{Li}_3(e^{\frac{3\pi i}{4}}))-\frac{23}{8}\zeta(3)\\ &= -\frac{43}{16}\zeta(3)+4\sqrt{2}L(\chi_8(5,\cdot),3),\end{aligned}$$

and

$$\mathrm{m}(S_{1,4}) = -\frac{105}{2\pi^2}\zeta(3)+\frac{64\sqrt{2}}{\pi^2}L(\chi_8(5,\cdot),3).$$

7. Conclusion

Our results show that the Mahler measure of the family $S_{n,r}$ is even richer and more interesting than the previously known Mahler measure of $S_{n,1}$. It is clear from the case $n=1$ that we can not expect a formula of the form (2). Such formula is certainly true if we consider an analogous construction for the R_n family, namely, if we let

$$R_{n,r}(x_1,\dots,x_n,z) := z + \left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r,$$

Then, we trivially have that

$$\begin{aligned}\mathrm{m}(R_{n,r}(x_1,\dots,x_n,z)) =& \mathrm{m}(R_{n,r}(x_1,\dots,x_n,-z^r)) = \sum_{j=0}^{r}\mathrm{m}\left(z-\xi_r^j\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right)\\ =& r\mathrm{m}(R_{n,1}(x_1,\dots,x_n,z)).\end{aligned}$$

Thus, the case of $R_{n,1}$ is trivial. Similar considerations apply to the family $Q_{n,r}$ given by

$$Q_{n,r}(x_1,\dots,x_n,z) := z + \left[\left(\frac{\omega+\overline{\omega}x_1}{1+x_1}\right)\cdots\left(\frac{\omega+\overline{\omega}x_n}{1+x_n}\right)\right]^r,$$

An interesting project would be to consider the construction of this article for the family T_n:

$$T_{n,r}(x_1,\dots,x_n,x,y) := 1 + \left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r x + \left(1-\left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)\right]^r\right)y.$$

As we remarked in the introduction, there is a clear distinction between the cases n even and odd for $\mathrm{m}(S_n)$, namely, the formulas for n even only contain special values of the Riemann zeta function, and the formulas for n odd only contain special values of the Dirichlet L-function at χ_{-4}. However, for $\mathrm{m}(S_{n,2})$, the formulas are mixed. The case of $\mathrm{m}(R_n)$ also shows an alternation of formulas involving special values of the Riemann zeta function or special values of the Dirichlet L-function, and by the discussion above, since $\mathrm{m}(R_{n,r}) = r\mathrm{m}(R_n)$, the same is true for $\mathrm{m}(R_{n,r})$ independently of r. Finally, all the formulas involving $\mathrm{m}(T_n)$ are given in terms of $\log 2$ and special values of the Riemann zeta function. It would be interesting to see how this extends to $\mathrm{m}(T_{n,r})$.

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