The Mahler measure of a family of polynomials with arbitrarily many variables

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Let

$$P_k(x_1, \ldots, x_k, y) = y + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_k}{1 + x_k}\right).$$

Theorem (Lalín, 2006)

$$\mathfrak{m}(P_{2n}) = \sum_{h=1}^{n} \frac{a_{n,h}}{\pi^{2h}} \zeta(2h+1),$$

and

$$\mathfrak{m}(P_{2n+1}) = \sum_{h=0}^{n} \frac{b_{n,h}}{\pi^{2h+1}} L(\chi_{-4}, 2h+2).$$

$a_{j,k}$ and $b_{j,k}$ are rational and can be described explicitly using coefficients of elementary symmetric polynomials.

Proof

$$P_k = y + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_k}{1+x_k}\right).$$

$$\Big\downarrow \text{ compare with}$$

$$Q_\gamma(y) = y + \gamma$$

$$\mathfrak{m}(P_k) = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \mathfrak{m}\left(Q_{\left(\frac{1-e^{i\theta_1}}{1+e^{i\theta_1}}\right) \cdots \left(\frac{1-e^{i\theta_k}}{1+e^{i\theta_k}}\right)}(y) \right) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_k$$

$$\Big\downarrow \text{ “clever” transformations}$$

$$= \frac{2^k}{\pi^k} \int_0^\infty \cdots \int_0^\infty \mathfrak{m}(Q_{y_k}) \frac{y_1 \mathrm{d}y_1}{(y_1^2+1)} \cdot \frac{y_2 \mathrm{d}y_2}{(y_2^2+y_1^2)} \cdots \frac{\mathrm{d}y_k}{(y_k^2+y_{k-1}^2)}.$$

We have

$$\int_0^\infty \cdots \int_0^\infty \mathfrak{m}(Q_{y_k}) \frac{y_1 \mathrm{d}y_1}{(y_1^2+1)} \cdot \frac{y_2 \mathrm{d}y_2}{(y_2^2+y_1^2)} \cdots \frac{\mathrm{d}y_k}{(y_k^2+y_{k-1}^2)}$$

which can be written as a linear combination of integrals of the form

$$\int_0^\infty \mathfrak{m}(Q_t) \log^j t \frac{\mathrm{d}t}{t^2 \pm 1},$$

and using

$$\int_0^1 \log^k t \, \frac{1}{t-a} \mathrm{d}t = (-1)^{k+1}(k!) \operatorname{Li}_{k+1}(1/a),$$

$\rightarrow$ gives zeta values and L-values

Extending these results

Lalín also looked at

$$S_{n,r} = (1+x)z + \left[\left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_n}{1+x_n}\right) \right]^r (1+y).$$

compare with

$$Q_\gamma(x,y,z) = (1+x)z + \gamma(1+y)$$

Theorem (Lalín, N., Roy; ϵ-close to submission!)

For $n \geq 1$,

$$\mathfrak{m}(S_{2n,r}) = \sum_{h=1}^{n} \frac{a'_{n,h}}{\pi^{2h}} \, \mathcal{C}_r(h),$$

and for $n \geq 0$,

$$\mathfrak{m}(S_{2n+1,r}) = \sum_{h=0}^{n} \frac{b'_{n,h}}{\pi^{2h+1}} \, \mathcal{D}_r(h)$$

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$$
\begin{aligned}
\mathcal{C}_r(h) :=& r(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\\
&+\frac{r^2(2h-1)!}{\pi^2}\times\\
&\left\{\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1}+(-1)^r2^{2h-1}+(-1)^{r+1}\right)\right.\\
&+(2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\
&-\sum_{\ell=0}^{2r-1}(-1)^\ell\left[\sum_{t=2}^{2h+2}\left(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^\ell)-\mathrm{Li}_t(-\xi_{2r}^\ell)\right)\right.\right.\\
&\left.\left.\left.-\binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\right)\times\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\right]\right\}.
\end{aligned}
$$

Examples

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\right]^2(1+y)z\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right]^2(1+y)z\right)=\frac{96}{\pi^3}L(\chi_{-4},4)-\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_3}{1+x_3}\right)\right]^2(1+y)z\right)=\frac{31}{2\pi^4}\zeta(5)-\frac{96}{\pi^3}L(\chi_{-4},4)+\frac{21}{2\pi^2}\zeta(3)$$

When $n=1$:

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=\frac{24}{\pi^3}L(\chi_{-4},4)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^2(1+y)z\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^3(1+y)z\right)=-\frac{8}{\pi^3}L(\chi_{-4},4)+\frac{12\sqrt{3}}{\pi^2}L(\chi_{12}(11,\cdot),3)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^4(1+y)z\right)=-\frac{105}{2\pi^2}\zeta(3)+\frac{64\sqrt{2}}{\pi^2}L(\chi_8(5,\cdot),3)$$

Why does this work – Möbius transformations?

The transformation

$$\phi(z) = \frac{1-z}{1+z}$$

sends the unit circle to the imaginary axis. For $z = e^{i\theta}$,

$$\frac{1-z}{1+z} = -2i \tan\left(\frac{\theta}{2}\right).$$

Some natural questions:

- Transformations that send unit circle to other lines?
- Those that preserve the unit circle?
  - These are

    $$\phi(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z},$$

    where $a \in \Delta$. Blaschke products?

A change of variables

Let $P(x, y_1, \ldots, y_n) \in \mathbb{C}[x, y_1, \ldots, y_n]$.
Choose $g(x) \in \mathbb{Z}[x]$ with all roots outside the unit disk.
Put $f(x) = \lambda \cdot x^k g(x^{-1})$, where $k > \deg(g)$ and $|\lambda| = 1$.
Let $\widetilde{P}$ the rational function obtained by replacing x by $f(x)/g(x)$ in P.

Theorem (Lalín & N., 2023)

$$\mathfrak{m}(P) = \mathfrak{m}(\widetilde{P}).$$

Take

$$\frac{f(x)}{g(x)} = \frac{x(2x+1)}{x+2},$$

$$P_k(x_1, \ldots, x_k, z) = z + \left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\cdots\left(\frac{1-x_k}{1+x_k}\right),$$

$$\hookrightarrow \widetilde{P}_k(x_1, \ldots, x_k, z) = z + \left(\frac{x_1^2-1}{x_1^2+x_1+1}\right)\left(\frac{1-x_2}{1+x_2}\right)\cdots\left(\frac{1-x_k}{1+x_k}\right).$$

Applications

$$\text{Condon}: \quad \mathfrak{m}(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3).$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X+1)}{X+2}} 2\,\frac{X^2+X+1+(X^2-1)(y+z)}{X+2}$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X^2-X+1)}{-(X^2-X+2)}} 2\,\frac{X^3-X^2+X-1+(X^3+1)(y+z)}{-(X^2-X+2)}$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X^3-X^2-X+1)}{-(X^3-X^2-X+2)}} 2\,\frac{X^4-X^3+X-1+(X^4-X^2+1)(y+z)}{-(X^3-X^2-X+2)}$$

Other results

Let

$$Q_k(z_1, \ldots, z_k, y) = y + \left(\frac{\overline{\alpha} z_1 + \alpha}{z_1 + 1}\right) \cdots \left(\frac{\overline{\alpha} z_k + \alpha}{z_k + 1}\right),$$

where $\alpha = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$.

Theorem (N., 2023++)

$$\mathfrak{m}(Q_{2n}) = \sum_{h=1}^{n} \frac{a_{n,h}}{\pi^{2h}} \zeta(2h+1) + \sum_{h=0}^{n-1} \frac{b_{n,h}}{\pi^{2h+1}} L(\chi_{-3}, 2h+2),$$

and

$$\mathfrak{m}(Q_{2n+1}) = \sum_{h=1}^{n} \frac{c_{n,h}}{\pi^{2h}} \zeta(2h+1) + \sum_{h=0}^{n} \frac{d_{n,h}}{\pi^{2h+1}} L(\chi_{-3}, 2h+2),$$

where the coefficients $a_{l,k}, b_{l,k}, c_{l,k}, d_{l,k} \in \mathbb{R}$ *are defined recursively.*

Examples

We have the first few examples in this family:

$$\mathfrak{m}(P_1) = \frac{5\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

$$\mathfrak{m}(P_2) = \frac{91}{18\pi^2}\zeta(3) + \frac{5}{4\sqrt{3}\pi} L(\chi_{-3}, 2)$$

$$\mathfrak{m}(P_3) = \frac{91}{36\pi^2}\zeta(3) + \frac{5}{4\sqrt{3}\pi} L(\chi_{-3}, 2) + \frac{153\sqrt{3}}{16\pi^3} L(\chi_{-3}, 4)$$

$$\mathfrak{m}(P_4) = \frac{91}{36\pi^2}\zeta(3) + \frac{3751}{108\pi^4}\zeta(5) + \frac{35}{36\sqrt{3}\pi} L(\chi_{-3}, 2) + \frac{51\sqrt{3}}{8\pi^3} L(\chi_{-3}, 4)$$

Proof

$$Q_n = \left(\frac{\overline{\alpha}z_1+\alpha}{z_1+1}\right)\cdots\left(\frac{\overline{\alpha}z_n+\alpha}{z_n+1}\right)+y.$$

compare with

$$P_\gamma(y) = \gamma + y$$

$$\mathfrak{m}(Q_n) = \frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathfrak{m}\left(P_{\left(\frac{\overline{\alpha}e^{i\theta_1}+\alpha}{e^{i\theta_1}+1}\right)\cdots\left(\frac{\overline{\alpha}e^{i\theta_n}+\alpha}{e^{i\theta_n}+1}\right)}(y)\right)\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n$$

"clever" transformations

$$\int\cdots\int\mathfrak{m}(P_{y_n})\frac{y_1\mathrm{d}y_1}{(y_1^2+y_1+1)}\cdot\frac{y_2\mathrm{d}y_2}{(y_2^2+y_2y_1+y_1^2)}\cdots\frac{\mathrm{d}y_n}{(y_n^2+y_ny_{n-1}+y_{n-1}^2)}.$$

$$\int_{y_1} \frac{y_1 \mathrm{d}y_1}{(y_1^2+y_1+1)(y_2^2+y_2y_1+y_1^2)} \rightsquigarrow \frac{2(y_2+1)(\log y_2)}{y_2^3-1} + \frac{2\pi}{3\sqrt{3}(y_2^2+y_2+1)}$$

$$\int_{y_2} \frac{(\ \cdots\cdots\ ) y_2 \mathrm{d}y_2}{(y_3^2+y_3y_2+y_2^2)} \rightsquigarrow \frac{4\pi^2}{9(y_3^2+y_3+1)} + \frac{2\pi(y_3+1)\log y_3}{\sqrt{3}(y_3^3-1)} + \frac{2\log^2 y_3}{y_3^2+y_3+1}$$

$$\int_{y_3} \frac{(\ \cdots\cdots\ ) y_3 \mathrm{d}y_3}{(y_4^2+y_4y_3+y_3^2)} \cdots$$

At the j^{th} stage, we have a linear combination of

$$\frac{y_j \log^k y_j}{(y_j^2 + y_j + 1)(y_{j+1}^2 + y_{j+1}y_j + y_j^2)},$$

and

$$\frac{y_j(y_j + 1) \log^k y_j}{(y_j^3 - 1)(y_{j+1}^2 + y_{j+1}y_j + y_j^2)},$$

for $k \le j$. Need to find

$$\int_0^\infty \frac{t \log^k t}{(t^2 + at + a^2)(t^2 + bt + b^2)} \mathrm{d}t,$$

and

$$\int_0^\infty \frac{t(t + a) \log^k t}{(t^3 - a^3)(t^2 + bt + b^2)} \mathrm{d}t$$

Contour integration!

$$\int \cdots \int \mathfrak{m}(P_{y_n}) \frac{y_1 \mathrm{d}y_1}{(y_1^2 + y_1 + 1)} \cdot \frac{y_2 \mathrm{d}y_2}{(y_2^2 + y_2 y_1 + y_1^2)} \cdots \frac{\mathrm{d}y_n}{(y_n^2 + y_n y_{n-1} + y_{n-1}^2)}$$

$$= \sum_{h=1}^{\lfloor n/2 \rfloor} a_h \int_0^1 \log^{2h} t \, \frac{1+t}{1-t^3} \mathrm{d}t \quad + \quad \sum_{h=0}^{\lfloor n/2 \rfloor} b_h \int_0^1 \frac{\log^{2h+1} t}{t^2+t+1} \mathrm{d}t$$

Again, we use

$$\int_0^1 \log^k t \, \frac{1}{t-a} \mathrm{d}t = (-1)^{k+1} (k!) \, \mathrm{Li}_{k+1}(1/a).$$

Relate combinations of polylogs to L-values

$$\mathrm{Li}_{2n}(\omega) - \mathrm{Li}_{2n}(\omega^2) = \sqrt{3} i \, L(\chi_{-3}, 2n)$$

$$\mathrm{Li}_{2n+1}(\omega) + \mathrm{Li}_{2n+1}(\omega^2) = -\left(1 - \frac{1}{3^{2n}}\right) \zeta(2n+1)$$

Finally!

We have

$$\mathfrak{m}(Q_{2n}) = \sum_{h=1}^{n} \frac{a_{n,h}}{\pi^{2h}} \zeta(2h+1) + \sum_{h=0}^{n-1} \frac{b_{n,h}}{\pi^{2h+1}} \, L(\chi_{-3}, 2h+2),$$

and

$$\mathfrak{m}(Q_{2n+1}) = \sum_{h=1}^{n} \frac{c_{n,h}}{\pi^{2h}} \zeta(2h+1) + \sum_{h=0}^{n} \frac{d_{n,h}}{\pi^{2h+1}} \, L(\chi_{-3}, 2h+2).$$

Further questions

Regarding the families of polynomials:

- Can we do this for other roots of unity? A general method?
- Do the coefficients have an elegant closed formula?

Regarding the $\frac{f}{g}$ transformations:

- Trying to understand what the transformations mean geometrically.
- Other such transformations that do not change the Mahler measure?