The Mahler measure of some polynomial families

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The definition

For a non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)^\times$, we define the (logarithmic) **Mahler measure** of P to be

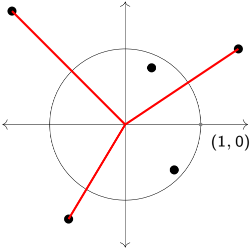
$$\mathfrak{m}(P) := \int_{[0,1]^n} \log \left| P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}) \right| d\theta_1 \cdots d\theta_n.$$

- Average value of $\log |P|$ over the unit n-torus.
- Introduced as a height function

The one-variable case

If $P(x) = A\prod_{j=1}^{d}(x-\alpha_j)$, then Jensen's formula implies

$$\mathfrak{m}(P) = \int_0^1 \log|P(e^{2\pi i\theta})|d\theta = \log|A| + \sum_{\substack{j\\|\alpha_j|>1}} \log|\alpha_j|.$$



- Thus, if $P(x) \in \mathbb{Z}[x] \Longrightarrow \mathfrak{m}(P) \geq 0$

Some Properties

- Kronecker's Lemma: $P \in \mathbb{Z}[x]$, $P \neq 0$,

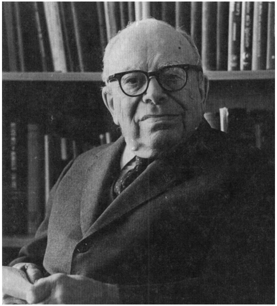
$$\mathfrak{m}(P) = 0 \text{ if and only if } P(x) = x^n \prod_i \Phi_i(x),$$

  where $\Phi_i(x)$ are cyclotomic polynomials.
- Lehmer's Question (1933, still open):
  Do we have a constant $\delta > 0$ such that for any $P \in \mathbb{Z}[x]$ with non-zero Mahler measure, we must also have $\mathfrak{m}(P) > \delta$?

$$\mathfrak{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \approx 0.162357612\ldots$$

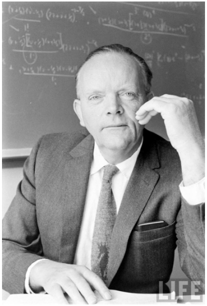
- Related to heights. For an algebraic integer α with logarithmic Weil height $h(\alpha)$,

$$\mathfrak{m}(f_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}] h(\alpha).$$



Kurt Mahler

Johan Jensen



Derrick Lehmer

More variables, more problems (more fun?)

Calculating the Mahler measure of multi-variable polynomials is very difficult.
For certain polynomials, the Mahler measure comes up as a value of an L-function!
Smyth, 1981:

- $$\mathfrak{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

- $$\mathfrak{m}(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) = -14\zeta'(-2)$$

More examples

Condon, 2004:

- $$\mathfrak{m}(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3) = -\frac{112}{5}\zeta'(-2)$$

Lalín, 2006:

- $$\mathfrak{m}\left(1+x+\left(\frac{1-v}{1+v}\right)\left(\frac{1-w}{1+w}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5) = 124\zeta'(-4)$$

Rogers and Zudilin, 2010:

- $$\mathfrak{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right) = \frac{24}{\pi^2}L(E_{24a3},2) = 4L'(E_{24a3},0)$$

Matilde Lalín

Chris Smyth

Wadim Zudilin

Coming up with such Identities

- In general, Mahler measures are arbitrary real values.
- Polynomials with a certain structure may give interesting values.
- Use the computer to compare with known L-values.
- Commonly associated to evaluating certain *polylogarithms*.

Families of polynomials with arbitrarily many variables

Let

$$P_k = y + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_k}{1+x_k}\right).$$

Theorem (Lalín, 2006)

$$\mathfrak{m}(P_{2n}) = \sum_{h=1}^{n} \frac{a_{n,h}}{\pi^{2h}}\zeta(2h+1),$$

and

$$\mathfrak{m}(P_{2n+1}) = \sum_{h=0}^{n} \frac{b_{n,h}}{\pi^{2h+1}}L(\chi_{-4}, 2h+2).$$

$a_{j,k}, b_{j,k} \in \mathbb{Q}$ related to coefficients of elementary symmetric polynomials.

Proof

$$P_k = y + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_k}{1+x_k}\right).$$

$\rightsquigarrow$ compare with

$$Q_\gamma(y) = y + \gamma$$

$$\mathfrak{m}(P_k) = \frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathfrak{m}\left(Q_{\left(\frac{1-e^{i\theta_1}}{1+e^{i\theta_1}}\right)\cdots\left(\frac{1-e^{i\theta_k}}{1+e^{i\theta_k}}\right)}(y)\right)\mathrm{d}\theta_1\cdots\mathrm{d}\theta_k$$

$\rightsquigarrow$ "clever" transformations

$$= \frac{2^k}{\pi^k}\int_0^{\infty}\cdots\int_0^{\infty}\mathfrak{m}(Q_{y_k})\frac{y_1\mathrm{d}y_1}{(y_1^2+1)}\cdot\frac{y_2\mathrm{d}y_2}{(y_2^2+y_1^2)}\cdots\frac{\mathrm{d}y_k}{(y_k^2+y_{k-1}^2)}.$$

We have

$$\int_0^\infty \cdots \int_0^\infty \mathfrak{m}(Q_{y_k}) \frac{y_1 \mathrm{d}y_1}{(y_1^2+1)} \cdot \frac{y_2 \mathrm{d}y_2}{(y_2^2+y_1^2)} \cdots \frac{\mathrm{d}y_k}{(y_k^2+y_{k-1}^2)}$$

which can be written as a linear combination of integrals of the form

$$\int_0^\infty \mathfrak{m}(Q_t) \log^j t \frac{\mathrm{d}t}{t^2 \pm 1},$$

and using

$$\int_0^1 \log^k t \, \frac{1}{t-a} \mathrm{d}t = (-1)^{k+1}(k!) \, \mathrm{Li}_{k+1}(1/a),$$

$\rightarrow$ gives zeta values and L-values

Extending these results

Lalín also looked at

$$S_{n,r} = (1+x)z + \left[\left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_n}{1+x_n}\right) \right]^r (1+y).$$

$$\Big\downarrow \text{ compare with}$$

$$Q_\gamma(x,y,z) = (1+x)z + \gamma(1+y)$$

Theorem (Lalín, N., Roy, 2023++)

For $n \geq 1$,

$$\mathfrak{m}(S_{2n,r}) = \sum_{h=1}^{n} \frac{a'_{n,h}}{\pi^{2h}} \,\mathcal{C}_r(h),$$

and for $n \geq 0$,

$$\mathfrak{m}(S_{2n+1,r}) = \sum_{h=0}^{n} \frac{b'_{n,h}}{\pi^{2h+1}} \,\mathcal{D}_r(h)$$

$$
\begin{aligned}
\mathcal{C}_r(h) :=& r(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\\
&+\frac{r^2(2h-1)!}{\pi^2}\times\\
&\left\{\frac{(-1)^{h+1}7B_{2h}\pi^{2h}}{2r^2(2h)!}\zeta(3)\left(2^{2h-1}+(-1)^r2^{2h-1}+(-1)^{r+1}\right)\right.\\
&+(2h+2)(2h+1)\frac{1-2^{-2h-3}}{r^{2h+2}}(1-(-1)^r)\zeta(2h+3)\\
&-\sum_{\ell=0}^{2r-1}(-1)^{\ell}\left[\sum_{t=2}^{2h+2}\left(\frac{(t-1)(t-2)}{2}(-1)^t\left(\mathrm{Li}_t(\xi_{2r}^{\ell})-\mathrm{Li}_t(-\xi_{2r}^{\ell})\right)\right.\right.\\
&\left.\left.\left.-\binom{t-1}{2h-1}(2-2^{1-t})\zeta(t)\right)\times\frac{(2\pi i)^{2h+3-t}}{(2h+3-t)!}B_{2h+3-t}\left(\frac{\ell}{2r}\right)\right]\right\}.
\end{aligned}
$$

Examples

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\right]^2(1+y)z\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right]^2(1+y)z\right)=\frac{96}{\pi^3}L(\chi_{-4},4)-\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left[\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_3}{1+x_3}\right)\right]^2(1+y)z\right)=\frac{31}{2\pi^4}\zeta(5)-\frac{96}{\pi^3}L(\chi_{-4},4)+\frac{21}{2\pi^2}\zeta(3)$$

When $n=1$:

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=\frac{24}{\pi^3}L(\chi_{-4},4)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^2(1+y)z\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^3(1+y)z\right)=-\frac{8}{\pi^3}L(\chi_{-4},4)+\frac{12\sqrt{3}}{\pi^2}L(\chi_{12}(11,\cdot),3)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^4(1+y)z\right)=-\frac{105}{2\pi^2}\zeta(3)+\frac{64\sqrt{2}}{\pi^2}L(\chi_8(5,\cdot),3)$$

Matilde Lalín

Subham Roy

Making some clever transformations!

Why does this work – Möbius transformations?

The transformation

$$\phi(z) = \frac{1-z}{1+z}$$

sends the unit circle to the imaginary axis. For $z = e^{i\theta}$,

$$\frac{1-z}{1+z} = -2i\tan\left(\frac{\theta}{2}\right).$$

Some natural questions:

- Transformations that send unit circle to other lines?
- Those that preserve the unit circle?
  - These are

    $$\phi(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z},$$

    where $a \in \Delta$. Blaschke products?

Other results

Let

$$Q_k(z_1,\dots,z_k,y) = y + \left(\frac{z_1+\alpha}{z_1+1}\right)\cdots\left(\frac{z_k+\alpha}{z_k+1}\right),$$

where $\alpha = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$.

Theorem (N., 2023++)

$$\mathfrak{m}(Q_{2n}) = \sum_{h=1}^{n} \frac{a_{n,h}}{\pi^{2h}}\,\zeta(2h+1) + \sum_{h=0}^{n-1} \frac{b_{n,h}}{\pi^{2h+1}}\, L(\chi_{-3}, 2h+2),$$

and

$$\mathfrak{m}(Q_{2n+1}) = \sum_{h=1}^{n} \frac{c_{n,h}}{\pi^{2h}}\,\zeta(2h+1) + \sum_{h=0}^{n} \frac{d_{n,h}}{\pi^{2h+1}}\, L(\chi_{-3}, 2h+2),$$

where the coefficients $a_{l,k}, b_{l,k}, c_{l,k}, d_{l,k} \in \mathbb{R}$ *are defined recursively.*

Examples

We have the first few examples in this family:

$$\mathfrak{m}(P_1) = \frac{5\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

$$\mathfrak{m}(P_2) = \frac{91}{18\pi^2}\zeta(3) + \frac{5}{4\sqrt{3}\pi} L(\chi_{-3}, 2)$$

$$\mathfrak{m}(P_3) = \frac{91}{36\pi^2}\zeta(3) + \frac{5}{4\sqrt{3}\pi} L(\chi_{-3}, 2) + \frac{153\sqrt{3}}{16\pi^3} L(\chi_{-3}, 4)$$

$$\mathfrak{m}(P_4) = \frac{91}{36\pi^2}\zeta(3) + \frac{3751}{108\pi^4}\zeta(5) + \frac{35}{36\sqrt{3}\pi} L(\chi_{-3}, 2) + \frac{51\sqrt{3}}{8\pi^3} L(\chi_{-3}, 4)$$

Proof

$$\int \cdots \int \mathfrak{m}(P_{y_n}) \frac{y_1 \mathrm{d}y_1}{(y_1^2 + y_1 + 1)} \cdot \frac{y_2 \mathrm{d}y_2}{(y_2^2 + y_2 y_1 + y_1^2)} \cdots \frac{\mathrm{d}y_n}{(y_n^2 + y_n y_{n-1} + y_{n-1}^2)}$$

$$= \sum_{h=1}^{\lfloor n/2 \rfloor} a_h \int_0^1 \log^{2h} t \, \frac{1+t}{1-t^3} \mathrm{d}t \quad + \quad \sum_{h=0}^{\lfloor n/2 \rfloor} b_h \int_0^1 \frac{\log^{2h+1} t}{t^2+t+1} \mathrm{d}t$$

Again, we use

$$\int_0^1 \log^k t \, \frac{1}{t-a} \mathrm{d}t = (-1)^{k+1} (k!) \, \mathrm{Li}_{k+1}(1/a).$$

Relate combinations of polylogs to L-values

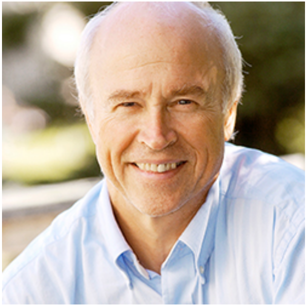
$$\mathrm{Li}_{2n}(\omega) - \mathrm{Li}_{2n}(\omega^2) = \sqrt{3} i \, L(\chi_{-3}, 2n)$$

$$\mathrm{Li}_{2n+1}(\omega) + \mathrm{Li}_{2n+1}(\omega^2) = -\left(1 - \frac{1}{3^{2n}}\right) \zeta(2n+1)$$

Further questions

Regarding the families of polynomials:

- Can we do this for other roots of unity? A general method?
- Do the coefficients have an elegant closed formula?
- Simplifying the polylog expressions



HAPPY BIRTHDAY!