An Invariant Property of Mahler Measure

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The definition

For a non-zero rational function $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, we define the (logarithmic) **Mahler measure** of P to be

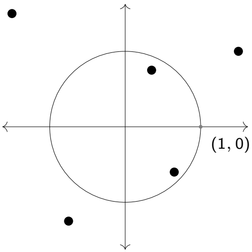
$$\mathfrak{m}(P) := \int_{[0,1]^n} \log \left| P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) \right| \, d\theta_1 \cdots d\theta_n.$$

$\longrightarrow$ Average value of $\log |P|$ over the unit n-torus.

The one-variable case

If $P(x) = A\prod_{j=1}^{d}(x-\alpha_j)$, then Jensen's formula implies

$$\mathfrak{m}(P) = \int_0^1 \log|P(e^{2\pi i\theta})|d\theta = \log|A| + \sum_{\substack{j \\ |\alpha_j|>1}} \log|\alpha_j|.$$



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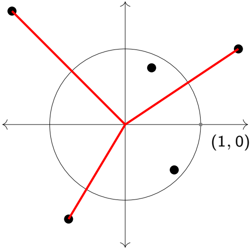


- Thus, if $P(x) \in \mathbb{Z}[x] \implies \mathfrak{m}(P) \geq 0$

Some Properties

- Kronecker's Lemma: $P \in \mathbb{Z}[x]$, $P \neq 0$,

$$\mathfrak{m}(P) = 0 \text{ if and only if } P(x) = x^n \prod_i \Phi_i(x),$$

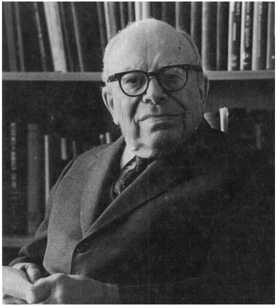
  where $\Phi_i(x)$ are cyclotomic polynomials.
- Lehmer's Question (1933, still open):

  Do we have a constant $\delta > 0$ such that for any $P \in \mathbb{Z}[x]$ with non-zero Mahler measure, we must also have $\mathfrak{m}(P) > \delta$?

$$\mathfrak{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \approx 0.162357612\ldots$$

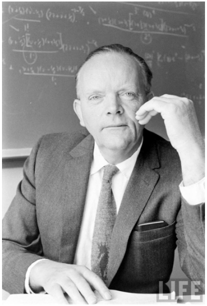
- The Mahler measure of $P(x)$ is related to heights. For an algebraic integer α with logarithmic Weil height $h(\alpha)$,

$$\mathfrak{m}(f_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]h(\alpha).$$



Kurt Mahler

Johan Jensen



Derrick Lehmer

Not just Number theory, it's everywhere!

The Mahler measure makes an appearance in the following areas

- Knot theory
- Hyperbolic Geometry
- Arithmetic Dynamics
- Height functions

More variables, more problems

In general, calculating the Mahler measure of multi-variable polynomials is much more difficult than the univariate case. However, there are more intriguing results concerning such polynomials that suggest that something deeper is in play.

We have the Boyd-Lawton formula for any rational function $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$:

$$\lim_{k_2 \to \infty} \cdots \lim_{k_n \to \infty} \mathfrak{m}(P(x, x^{k_2}, \ldots, x^{k_n})) = \mathfrak{m}(P(x_1, x_2, \ldots, x_n)),$$

where the k_i's vary independently.

Examples

Turns out that for certain polynomials, the Mahler measure is not just any random real number, but in fact a special value of an L-function.
Smyth, 1981:

- $$\mathfrak{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

- $$\mathfrak{m}(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) = -14\zeta'(-2)$$

Examples

Condon, 2004:

- $$\mathfrak{m}(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3) = -\frac{112}{5}\zeta'(-2)$$

Lalín, 2006:

- $$\mathfrak{m}\left(1+x+\left(\frac{1-v}{1+v}\right)\left(\frac{1-w}{1+w}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5) = 124\zeta'(-4)$$

Rogers and Zudilin, 2010:

- $$\mathfrak{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right) = \frac{24}{\pi^2}L(E_{24a3},2) = 4L'(E_{24a3},0)$$

David Boyd

Chris Smyth

Matilde Lalín

Coming up with such Identities

- In general, Mahler measures are arbitrary real values. Only polynomials with a certain structure end up giving interesting values.
- They are commonly associated to evaluating certain *polylogarithms*.
- Oftentimes, such identities are obtained after a numerical experiment on the computer of certain special polynomials.
  For example Boyd looked at polynomials of the type

$$A(x) + B(x)y + C(x)z,$$

where A, B and C are products of cyclotomic polynomials.

Calculations by Brunault and Zudilin

Numerical calculations by Brunault and Zudilin:

$$\left.\begin{array}{l}\mathfrak{m}(x^2+x+1+(x^2-1)(y+z))\\ \mathfrak{m}(x^3-x^2+x-1+(x^3+1)(y+z))\\ \mathfrak{m}(x^4-x^3+x-1+(x^4-x^2+1)(y+z))\\ \mathfrak{m}(x^4-x^3+x-1+(x^4-x^3+x^2-x+1)(y+z))\\ \mathfrak{m}(x^4-x^3+x^2-x+1+(x^4-1)(y+z))\\ \mathfrak{m}(x^4-x^3+x-1+(x^4+1)(y+z))\\ \mathfrak{m}(x^5-x^4+x-1+(x^5+1)(y+z))\end{array}\right\} \stackrel{?}{=} \frac{28}{5\pi^2}\zeta(3).$$

Condon showed

$$\mathfrak{m}(x+1+(x-1)(y+z)) = \frac{28}{5\pi^2}\zeta(3).$$

We will present a change of variables, which when applied to any polynomial, preserves its Mahler measure

Francois Brunault

Wadim Zudilin

Is there some connection?

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X+1)}{X+2}} 2\,\frac{X^2+X+1+(X^2-1)(y+z)}{X+2}$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X^2-X+1)}{-(X^2-X+2)}} 2\,\frac{X^3-X^2+X-1+(X^3+1)(y+z)}{-(X^2-X+2)}$$

$$x+1+(x-1)(y+z) \xrightarrow{x=\frac{X(2X^3-X^2-X+1)}{-(X^3-X^2-X+2)}} 2\,\frac{X^4-X^3+X-1+(X^4-X^2+1)(y+z)}{-(X^3-X^2-X+2)}$$

The transformations

$$x \longrightarrow \frac{X(2X+1)}{X+2}$$

$$x \longrightarrow \frac{X(2X^2-X+1)}{-(X^2-X+2)}$$

$$x \longrightarrow \frac{X(2X^3-X^2-X+1)}{-(X^3-X^2-X+2)}$$

reverse the coefficients of g and multiply by a power of X

$$x = \frac{f(X)}{g(X)}$$

has all roots outside the unit disc

The Main Result

Theorem (Lalín & N., 2022+)

Let $P(x, y_1, \ldots, y_n)$ *be a polynomial over* $\mathbb{C}$ *in the variables* $x, y_1, \ldots, y_n$*. Let* $g(x) \in \mathbb{C}[x]$ *be such that all the roots have absolute value greater than or equal to one, let* k *be an integer such that* $k > \deg(g)$ *and let* $f(x) = \lambda x^k \overline{g}(x^{-1})$*, where* λ *is a complex number with absolute value one. We denote by* $\widetilde{P}$ *the rational function obtained by replacing* x *by* $f(x)/g(x)$ *in* P*. Then*

$$\mathfrak{m}(P) = \mathfrak{m}(\widetilde{P}).$$

- For eg., with $P = x + 1 + (x-1)(y+z)$, we get

$$\mathfrak{m}(f + g + (f-g)(y+z)) = \frac{28}{5\pi^2}\zeta(3) + \mathfrak{m}(g).$$

Matilde Lalín

N.

Some final remarks

- Lalín, 2006:

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5)$$

$$\mathfrak{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_n}{1+x_n}\right)(y+z)\right)$$

  Apply the result to each variable above to get highly non-trivial identities

- Using this theorem, we can obtain the Mahler measure of polynomials with much more complicated geometry

Further questions

- Trying to understand what the $\frac{f}{g}$ transformation means geometrically and how it preserves the L-value.
- Are there any other such transformations that do not change the Mahler measure.
- If the Mahler measure of two polynomials is the same, does that mean they must differ by such a transformation?
- The $\frac{f}{g}$ transformations preserve the unit circle. What if we consider transformations that send the unit circle to a straight line?

THANK YOU!