The zeta function strikes back!!!

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Matilde N. Lalín

1. Mahler Measure

Definition 1 *For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic)* Mahler measure *is defined by*

$$m(P) := \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \tag{1}$$

$$= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{2}$$

It is possible to prove that this integral is not singular and that $m(P)$ always exists.

Because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|^1 \tag{3}$$

we have

$$\text{for} \quad P(x) = a \prod_j (x - \alpha_j), \quad \text{then} \quad m(P) = \log |a| + \sum_j \log^+ |\alpha_j| \tag{4}$$

which gives a simple expression for the Mahler measure in the one-variable case, as a function on the roots of the polynomial.

2. Polylogarithms and L-functions

The several-variable case is more complicated. Many examples with explicit formulae have been produced. Before going to these examples, recall the definition of polylogarithm and L-functions.

Definition 2 Multiple polylogarithms *are defined as the power series*

$$\mathrm{Li}_{k_1, \dots, k_m}(x_1, \dots, x_m) := \sum_{0 < n_1 < n_2 < \dots < n_m} \frac{x_1^{n_1} x_2^{n_2} \dots x_m^{n_m}}{n_1^{k_1} n_2^{k_2} \dots n_m^{k_m}} \tag{5}$$

which are convergent for $|x_i| < 1$. The weight of a polylogarithm function is the number $w = k_1 + \dots + k_m$ and its length is the number m.

We see that, for lenght 1, weight $w > 1$ and $x = 1$, we recover the zeta function in w.

[1] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

Definition 3 Hyperlogarithms *are defined as the iterated integrals*

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1:\ldots:a_m:a_{m+1}) :=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{\mathrm{d}t}{t-a_1}\circ\frac{\mathrm{d}t}{t}\circ\ldots\circ\frac{\mathrm{d}t}{t}}_{k_1}\circ\underbrace{\frac{\mathrm{d}t}{t-a_2}\circ\frac{\mathrm{d}t}{t}\circ\ldots\circ\frac{\mathrm{d}t}{t}}_{k_2}\circ\ldots\circ\underbrace{\frac{\mathrm{d}t}{t-a_m}\circ\frac{\mathrm{d}t}{t}\circ\ldots\circ\frac{\mathrm{d}t}{t}}_{k_m}$$

where k_i are integers, a_i are complex numbers, and

$$\int_0^{b_{l+1}} \frac{\mathrm{d}t}{t-b_1}\circ\ldots\circ\frac{\mathrm{d}t}{t-b_l} = \int_{0\le t_1\le\ldots\le t_l\le b_{l+1}} \frac{\mathrm{d}t_1}{t_1-b_1}\cdots\frac{\mathrm{d}t_l}{t_l-b_l}$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C}\setminus\{a_1,\ldots,a_m\}$.

It is easy to see that,

$$\begin{aligned}
\mathrm{I}_{k_1,\ldots,k_m}(a_1:\ldots:a_m:a_{m+1}) &= (-1)^m \mathrm{Li}_{k_1,\ldots,k_m}\left(\frac{a_2}{a_1},\frac{a_3}{a_2},\ldots,\frac{a_m}{a_{m-1}},\frac{a_{m+1}}{a_m}\right)\\
\mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) &= (-1)^m \mathrm{I}_{k_1,\ldots,k_m}\left(\frac{1}{x_1,\ldots,x_m}:\ldots:\frac{1}{x_m}:1\right)
\end{aligned}$$

which gives an analytic continuation to multiple polylogarithms. For instance, with the convention about integrating over a real segment, simple polylogarithms have an analytic continuation to $\mathbb{C}\setminus[1,\infty)$.

There are modified versions of these functions which are analytic in larger sets, like the Bloch-Wigner dilogarithm,

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z|\arg(1-z) \qquad z\in\mathbb{C}\setminus[1,\infty) \tag{6}$$

which can be extended as a real analytic function in $\mathbb{C}\setminus\{0,1\}$ and continuous in $\mathbb{C}$.

Definition 4 *The* L-series *in the character χ is defined to be the function*

$$\mathrm{L}(\chi,s) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$$

We are going to use the real characters

$$\chi_{-f}(n) := \left(\frac{-f}{n}\right)$$

where the symbol in the right is Kronecker's extension to Jacobi's symbol. In particular,

$$\begin{aligned}
\chi_{-3}(n) &= \left(\frac{n}{3}\right)\\
\chi_{-4}(n) &= \begin{cases}\left(\frac{-1}{n}\right) & \text{if } n \text{ odd}\\ 0 & \text{if } n \text{ even}\end{cases}\\
\chi_{-8}(n) &= \begin{cases}\left(\frac{-2}{n}\right) & \text{if } n \text{ odd}\\ 0 & \text{if } n \text{ even}\end{cases}
\end{aligned}$$

3. Examples for two and three variables

- The simplest example with two variables is due to Smyth [13]:

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1) \tag{7}$$

- The above example can be extended to three variables, also due to Smyth:

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) \tag{8}$$

More generally, Smyth [14] computed the Mahler measure of $a+b\frac{1}{x}+cy+(a+bx+cy)z$, for a, b and c any real numbers, from which the above result can be deduced. Another corollary is the formula

$$m\left(1+\frac{1}{x}+y+(1+x+y)z\right) = \frac{14}{3\pi^2}\zeta(3) \tag{9}$$

- The Dilogarithm appears as the Mahler measure of certain polynomials in two variables. Perhaps the simplest example is Maillot's in [10]: for $a,b,c\in\mathbb{C}$,

$$\pi m(a+bx+cy) = \begin{cases} D\left(\left|\frac{a}{b}\right|\mathrm{e}^{\mathrm{i}\gamma}\right)+\alpha\log|a|+\beta\log|b|+\gamma\log|c| & \triangle \\ \\ \pi\log\max\{|a|,|b|,|c|\} & \text{not } \triangle \end{cases} \tag{10}$$

Here $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively.

- Vandervelde [15] studied the example of $axy+bx+cy+d$. He developed a general formula for this case. Some particular cases are:

$$m(1+x+y+\mathrm{i}xy) = \frac{\sqrt{2}}{\pi}\mathrm{L}(\chi_{-8},2) = \frac{1}{4}\mathrm{L}'(\chi_{-8},-1) \tag{11}$$

$$m(1+x+y+\mathrm{e}^{\frac{\pi\mathrm{i}}{3}}xy) = \frac{4\sqrt{2}}{15\pi}\mathrm{L}(\chi_{-8},2) = \frac{1}{15}\mathrm{L}'(\chi_{-8},-1) \tag{12}$$

- Vandervelde also generalized Smyth's example. For $a\in\mathbb{R}_{>0}$,

$$\pi^2 m(1+x+ay+az) = \begin{cases} 2(\mathrm{Li}_3(a)-\mathrm{Li}_3(-a)) & \text{if } a\le 1 \\ \pi^2\log a + 2\left(\mathrm{Li}_3\left(\frac{1}{a}\right)-\mathrm{Li}_3\left(\frac{-1}{a}\right)\right) & \text{if } a\ge 1 \end{cases} \tag{13}$$

This can be also proved by adapting the elementary proof given in Boyd [2]. For $0\le a\le 1$:

$$\pi^2 m(1+x+ay+az) = \pi^2 m(1+ay+x(1+aw)) = \pi^2 m\left(\frac{1+ay}{1+aw}+x\right)$$

$$= \int_0^\pi \int_0^\pi \log^+ \left| \frac{1+a\mathrm{e}^{\mathrm{i}t}}{1+a\mathrm{e}^{\mathrm{i}s}} \right| \mathrm{d}s\,\mathrm{d}t = \int_{0\le t\le s\le \pi} \log|1+a\mathrm{e}^{\mathrm{i}t}| - \log|1+a\mathrm{e}^{\mathrm{i}s}|\,\mathrm{d}s\,\mathrm{d}t$$

$$= \int_0^\pi (\pi - t)\,\log|1+a\mathrm{e}^{\mathrm{i}t}|\mathrm{d}t - \int_0^\pi s\,\log|1+a\mathrm{e}^{\mathrm{i}s}|\mathrm{d}s = -2\int_0^\pi t\,\log|1+a\mathrm{e}^{\mathrm{i}t}|\mathrm{d}t$$

(here we have used that $0 \le a \le 1$, and Jensen's formula).

Now use that

$$\log|1+a\mathrm{e}^{\mathrm{i}t}| = \mathrm{Re}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\, a^n \mathrm{e}^{\mathrm{i}nt} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos(n\,t)}{n} a^n \tag{14}$$

and apply integration by parts,

$$\pi^2 m(1+x+ay+az) = -2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(n\,t)}{n^2}\, a^n t \Big|_0^\pi$$

$$+2\int_0^\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(n\,t)}{n^2}\, a^n \mathrm{d}t = 4\sum_{n=1\,(\mathrm{odd})}^{\infty} \frac{a^n}{n^3} = 2(\mathrm{Li}_3(a) - \mathrm{Li}_3(-a))$$

When $a \ge 1$, use that

$$m(1+x+ay+az) = \log a + m\left(\frac{1}{a} + \frac{x}{a} + y + z\right)$$

4. Examples of higher weight

We have obtained examples of polynomials in several variables whose Mahler measures depend on polylogarithms. The first column of the table shows the polynomials. Here α is a complex number different from zero. The second column indicates the values of the first column for the case $\alpha = 1$.

$\pi m((1+y)+\alpha(1-y)x)$	$2\mathrm{L}(\chi_{-4},2)$
$\pi^2 m((1+w)(1+y)+\alpha(1-w)(1-y)x)$	$7\zeta(3)$
$\pi^3 m((1+v)(1+w)(1+y)+\alpha(1-v)(1-w)(1-y)x)$	$7\pi\zeta(3) + 4\sum_{0\le j<k} \frac{(-1)^j}{(2j+1)^2k^2}$
$\pi^2 m((1+x)+\alpha(y+z))$	$\frac{7}{2}\zeta(3)$
$\pi^3 m((1+w)(1+x)+\alpha(1-w)(y+z))$	$2\pi^2\mathrm{L}(\chi_{-4},2) + 8\sum_{0\le j<k} \frac{(-1)^{j+k+1}}{(2j+1)^3k}$
$\pi^4 m((1+v)(1+w)(1+x)+\alpha(1-v)(1-w)(y+z))$	$93\zeta(5)$
$\pi^2 m((1+w)(1+y)+(1-w)(x-y))$	$\frac{7}{2}\zeta(3) + \frac{\pi^2}{2}\log 2$

Let us observe that all the presented formulae share a common feature. If we assign weight 1 to any Mahler measure and to π, then all the formulae are homogeneous, meaning all the monomials have the same weight, and this weight is equal to the number of variables of the corresponding polynomial.

The idea behind those computations is the following.

1. Let $P_\alpha \in \mathbb{C}[x_1, \dots, x_n]$ whose coefficients depend polynomially on a parameter $\alpha \in \mathbb{C}$. For instance, start with $P_\alpha(x) = 1 + \alpha x$, whose Mahler measure is $\log^+ |\alpha|$.

2. We replace α by $\alpha\frac{1-y}{1+y}$ and obtain a polynomial $\tilde{P}_\alpha \in \mathbb{C}[x_1, \dots, x_n, y]$. In the example, $\tilde{P}_\alpha(x, y) = 1 + y + \alpha(1 - y)x$.

3. The Mahler measure of the second polynomial is a certain integral of the Mahler measure of the first polynomial.
$$m(\tilde{P}_\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} m\left(P_{\alpha\frac{1-y}{1+y}}\right) \frac{\mathrm{d}y}{y}$$

4. If the Mahler measure depends only on the absolute value of $|\alpha|$, we can make a change of variables $u = \left|\alpha\frac{1-y}{1+y}\right|$ (to be precise, first write $y = \mathrm{e}^{\mathrm{i}\theta}$ and then make $u = |\alpha| \tan\left(\frac{\theta}{2}\right)$).We obtain,
$$m(\tilde{P}_\alpha) = \frac{2}{\pi} \int_0^\infty m\left(P_u\right) \frac{|\alpha|\,\mathrm{d}u}{u^2 + |\alpha|^2} = \frac{\mathrm{i}}{\pi} \int_0^\infty m\left(P_u\right) \left(\frac{1}{u + \mathrm{i}|\alpha|} - \frac{1}{u - \mathrm{i}|\alpha|}\right) \mathrm{d}u$$
In the example,
$$\begin{aligned} m(1 + y + \alpha(1 - y)x) &= \frac{\mathrm{i}}{\pi} \int_0^\infty \log^+ u \left(\frac{1}{u + \mathrm{i}|\alpha|} - \frac{1}{u - \mathrm{i}|\alpha|}\right) \mathrm{d}u \\ &= \frac{\mathrm{i}}{\pi} \int_0^1 \int_s^1 \frac{\mathrm{d}t}{t} \left(\frac{1}{s + \frac{\mathrm{i}}{|\alpha|}} - \frac{1}{s - \frac{\mathrm{i}}{|\alpha|}}\right) \mathrm{d}s \\ = \frac{\mathrm{i}}{\pi}\left(\mathrm{I}_2\left(-\frac{\mathrm{i}}{|\alpha|} : 1\right) - \mathrm{I}_2\left(\frac{\mathrm{i}}{|\alpha|} : 1\right)\right) &= -\frac{\mathrm{i}}{\pi}(\mathrm{Li}_2(\mathrm{i}|\alpha|) - \mathrm{Li}_2(-\mathrm{i}|\alpha|)) \end{aligned}$$

5. The example with 5 variables

By applying the method described above, we can prove that
$$\begin{aligned} \pi^4 m((1 + v)(1 + w)(1 + x) + \alpha(1 - v)(1 - w)(y + z)) &= 7\pi^2\zeta(3) + 8(\mathrm{Li}_{3,2}(1, 1) - \mathrm{Li}_{3,2}(-1, 1)) \\ &\quad + 8(\mathrm{Li}_{3,2}(1, -1) - \mathrm{Li}_{3,2}(-1, -1)) \quad (15) \end{aligned}$$
Now we use formula (75) of [1], which in this particular case, states that
$$\mathrm{Li}_{3,2}(x, y) = -\frac{1}{2}\mathrm{Li}_5(x\,y) + \mathrm{Li}_3(x)\,\mathrm{Li}_2(y) + 3\mathrm{Li}_5(x) + 2\mathrm{Li}_5(y) - \mathrm{Li}_2(x\,y)(\mathrm{Li}_3(x) + 2\mathrm{Li}_3(y))$$
for $x, y = \pm 1$.

Taking into account that
$$\mathrm{Li}_k(1) = \zeta(k) \quad \text{and} \quad \mathrm{Li}_k(-1) = \left(\frac{1}{2^{k-1}} - 1\right)\zeta(k) \tag{16}$$
we get
$$\mathrm{Li}_{3,2}(1, 1) - \mathrm{Li}_{3,2}(-1, 1) + \mathrm{Li}_{3,2}(1, -1) - \mathrm{Li}_{3,2}(-1, -1) = -\frac{21}{4}\zeta(2)\zeta(3) + \frac{93}{8}\zeta(5)$$
We obtain the result by using that $\zeta(2) = \frac{\pi^2}{6}$

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