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ADDENDUM TO: UNIMODULARITY OF ZEROS OF SELF-INVERSIVE POLYNOMIALS

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Abstract. We acknowledge priority of earlier and more general results than Theorem 1 of the paper referred to in the title.

In [4, Theorem 1] we proved the following result.

THEOREM 1. *Let $h(z)$ be a monic complex polynomial of degree n having all its zeros in the closed unit disk $|z| \le 1$. Then for $d > n$ and any λ on the unit circle, the self-inversive polynomial*

$$P^{(\lambda)}(z) = z^{d-n}h(z) + \lambda h^*(z) \tag{1}$$

has all its zeros on the unit circle.

Conversely, given a monic self-inversive polynomial $P(z)$ having all its zeros on the unit circle, there is a polynomial h having all its zeros in $|z| \le 1$ such that P has a representation (1). In particular, we can take $h(z) = \frac{1}{d}P'(z)$.

Here $h^*(z) = z^n\overline{h}(1/z)$. We used this result to prove that a certain family P_k of polynomials have all their zeros on the unit circle. El-Guindy and Raji [2, Theorem 2.2] pointed out that the theorem is also valid for $d = n$ (though we must then allow the possibility that $P^{(\lambda)}(z)$ is identically 0), and that the condition that h be monic is of course unnecessary.

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As we stated in [4], the converse part of the theorem is a well-known result of Cohn. The purpose of this note is to acknowledge that the first part of the theorem was also known earlier. As pointed out by Suzuki in [8], it was stated and proved by Chen [1, Theorem 1] in 1995, essentially in the form above.

In fact, the theorem has been strengthened. For as λ, as in (1), travels anticlockwise around the unit circle from 1 back to 1, the d zeros of $P^{(\lambda)}(z)$, say $z_1, \dots, z_d$, labelled by increasing argument, progress monotonically in argument around the unit circle so that when λ returns to 1 the zero z_1 has progressed to z_2, z_2 has progressed to z_3, and so on, with finally z_d having progressed to z_1. This is proved by a classical winding argument by considering $-z^{d-n}h(z)/h^*(z)$ as z winds around the unit circle. See for instance the proof of [5, Theorem 2.1] for essentially this method of proof, as mentioned by Jankauskas [3, Section 6.1]. In particular, it then follows immediately that for $\lambda_1 \neq \lambda_2$ the zeros of $P^{(\lambda_1)}(z)$ and $P^{(\lambda_2)}(z)$ interlace on the unit circle. Also, Jankauskas unearthed the fact that the stronger version of Theorem 1, incorporating this interlacing property, had already been proved by Schüssler [7] in 1976. See [3, Corollary 17].

Furthermore, Jankauskas [3, Section 6.1] pointed out that these results have an even longer history, albeit in a modified form. Consider a linear fractional transformation that maps the open unit disc $|z| < 1$ to the upper half-plane $\operatorname{Im} z > 0$, and so also maps the unit circle to the real line. Then, on applying this map to z in Theorem 1, with interlacing of the zeros of $P^{(\lambda_1)}(z)$ and $P^{(\lambda_2)}(z)$ for $\lambda_1 \neq \lambda_2$ added, it becomes essentially the classical Hermite-Biehler Theorem:

THEOREM 2 (C. Hermite (1856), M. Biehler (1879); see [6, Theorem 6.3.4]). *Two nonconstant polynomials P and Q with real coefficients, of degrees differing by at most* 1, *have strictly interlacing zeros on the real line if and only if one of $P(z) \pm iQ(z)$ has all its zeros in the upper half-plane* $\operatorname{Im} z > 0$.

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