Some aspects of Mahler Measure

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Mahler measure and Lehmer's question

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

ω

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma:

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$$= \log(1.176280818\ldots) = 0.162357612\ldots$$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

$$\text{there exists} \quad C > 0, \quad \text{for all} \quad P(x) \in \mathbb{Z}[x]$$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the polynomial above the best possible?

Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

Properties

- $m(P \cdot Q) = m(P) + m(Q)$

- $m(P) \geq 0$ if P has integral coefficients.

- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,
$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$
where h is the logarithmic Weil height.

- Boyd & Lawton : $P \in \mathbb{C}[x_1, \dots, x_n]$
$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n}))$$
$= m(P(x_1, \dots, x_n))$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

$$
\begin{aligned}
m(1+x+y) &= \frac{3\sqrt{3}}{4\pi}\mathsf{L}(\chi_{-3},2) = \mathsf{L}'(\chi_{-3},-1) \\
m(1+x+y+z) &= \frac{7}{2\pi^2}\zeta(3)
\end{aligned}
$$

$$
\mathsf{L}(\chi_{-3},s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}
$$

∞

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
$$

Boyd & Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathsf{L}'(E_k,0)}{B_k} \qquad k \in \mathbb{N}$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathsf{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathsf{L}'(A,0)$$

$$A : y^2 = x^3 - 44x + 112$$

More examples in several variables

L (2003)

$$\pi^n m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)z\right)$$

$=$ combination of $\zeta(\text{odd})$ / $\mathsf{L}(\chi_{-4},\text{even})$

$$\pi^n m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

$=$ combination of $\zeta(\text{odd})$ / $\mathsf{L}(\chi_{-4},\text{even})$, polylogarithms

$$\pi^n m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)\right)y\right)$$

$=$ combination of $\zeta(\text{odd})$

Examples

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$= 24\mathsf{L}(\chi_{-4},4)+\pi^2\mathsf{L}(\chi_{-4},2)$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)=62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F number field)

- L-function ($L_X = \zeta_F$)

- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)

- Regulator map

$$r : K \to \text{smooth differential forms}$$

($r = \log|\cdot|$)

Conjecture: special value of $\mathsf{L}_X \sim_{\mathbb{Q}^*} \int_\gamma r(\xi)$

(E.g. Dirichlet class number formula, F real quadratic,
$\zeta_F'(0) \sim_{\mathbb{Q}^*} \log|\epsilon| \qquad \epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) :

$$P(x,y) = y + x - 1 \qquad C = \{P(x,y) = 0\}$$

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log |y + x - 1| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

by Jensen's equality:

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{\mathrm{d}x}{x}$$

$$= \frac{1}{2\pi \mathsf{i}} \int_\gamma \log|y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_\gamma \eta(x, y)$$

where

$$\gamma = C \cap \{|x| = 1, |y| \geq 1\}$$

and

$$\eta(x, y) = \log|x| \mathrm{d} \arg y - \log|y| \mathrm{d} \arg x$$

- $\eta(x,y) = -\eta(y,x)$

- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$

Theorem 1

$$\eta(x, 1-x) = \mathrm{d}D(x)$$

Bloch–Wigner dilogarithm:

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

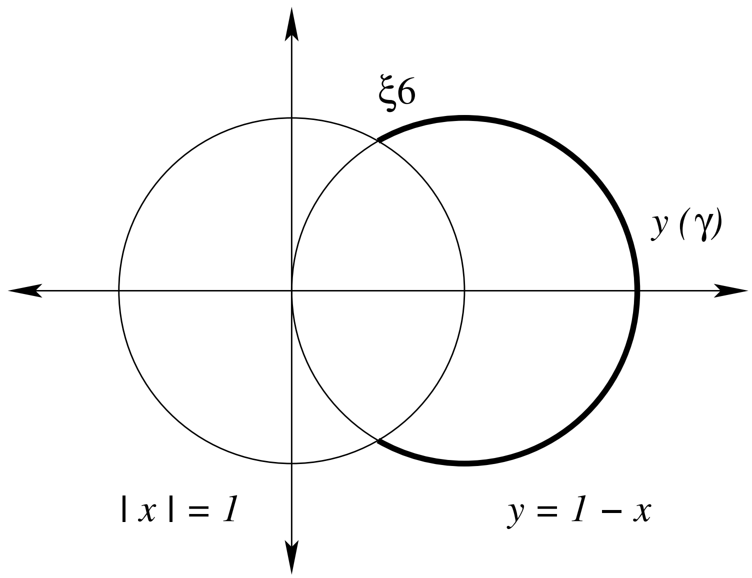
$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad |x| < 1$$

Use Stokes Theorem:

$$m(P) = -\frac{1}{2\pi} D(\partial\gamma)$$

$$x = \mathsf{e}^{2\pi \mathsf{i}\theta} \qquad y(\gamma(\theta)) = 1 - \mathsf{e}^{2\pi \mathsf{i}\theta}, \quad \theta \in [1/6\,;5/6]$$

$$\partial\gamma = [\bar{\xi_6}] - [\xi_6]$$



$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi_6}) = 2D(\xi_6) = \frac{3\sqrt{3}}{2}\mathsf{L}(\chi_{-3}, 2)$$

In general, $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

Need $\{x,y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$.

$$x \wedge y = \sum_j r_j \;\; z_j \wedge (1 - z_j)$$

in $\bigwedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$, then

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}$$

Big picture

$$\ldots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C, \partial\gamma) \to K_2(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x, y)$ is exact, then $\{x, y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

  $\rightsquigarrow$ dilogarithms, zeta function

- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(C)$. We have $\eta(x, y)$ is not exact.

  $\rightsquigarrow$ L-series of a curve

We may get combinations of both situations.

The three-variable case

$$P(x,y,z) = (1-x) + (1-y)z \qquad S = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3}\int_{\mathbb{T}^3} \log\left|z - \frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log^+\left|\frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = -\frac{1}{(2\pi)^2}\int_{\Gamma} \log|z|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \eta(x,y,z)$$

$$\Gamma = S \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$\eta(x,y,z) = \mathsf{log}\,|x|\left(\frac{1}{3}\mathsf{d}\,\mathsf{log}\,|y|\,\mathsf{d}\,\mathsf{log}\,|z| - \mathsf{d}\,\mathsf{arg}\,y\,\mathsf{d}\,\mathsf{arg}\,z\right)$$

$$+\,\mathsf{log}\,|y|\left(\frac{1}{3}\mathsf{d}\,\mathsf{log}\,|z|\,\mathsf{d}\,\mathsf{log}\,|x| - \mathsf{d}\,\mathsf{arg}\,z\,\mathsf{d}\,\mathsf{arg}\,x\right)$$

$$+\,\mathsf{log}\,|z|\left(\frac{1}{3}\mathsf{d}\,\mathsf{log}\,|x|\,\mathsf{d}\,\mathsf{log}\,|y| - \mathsf{d}\,\mathsf{arg}\,x\,\mathsf{d}\,\mathsf{arg}\,y\right)$$

Theorem 2

$$\eta(x, 1-x, y) = \mathsf{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x)\mathsf{d}\,\mathsf{arg}\,y$$

$$+\frac{1}{3}\,\mathsf{log}\,|y|(\mathsf{log}\,|1-x|\mathsf{d}\,\mathsf{log}\,|x| - \mathsf{log}\,|x|\mathsf{d}\,\mathsf{log}\,|1-x|)$$

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

Maillot: if $P \in \mathbb{Q}[x,y,z]$,

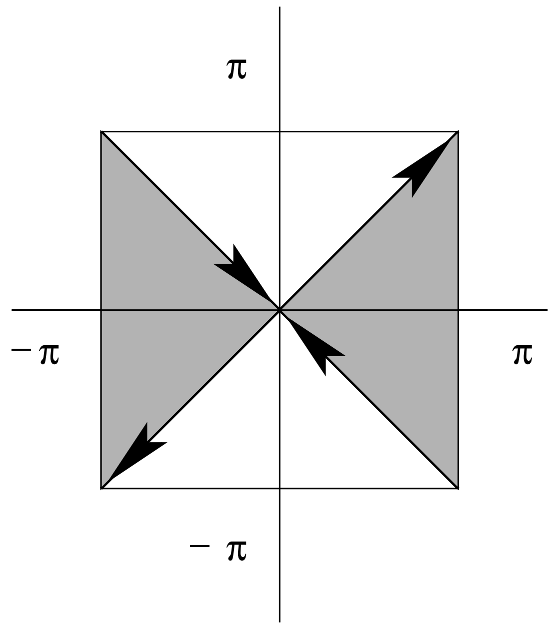
$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$

ω defined in

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1 \qquad C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

Theorem 3

$$\omega(x,x) = \mathrm{d}P_3(x)$$

$$= \frac{1}{4\pi^2}8(P_3(1)-P_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_\Gamma \eta(x,y,z)$$

Need $\{x,y,z\} = 0$ in $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$.

$$x \wedge y \wedge z = \sum r_i \; x_i \wedge (1 - x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$, then

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_\Gamma \eta(x_i, 1 - x_i, y_i) = \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Let

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] = 0$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.

F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\{[0],[\infty],R_2(x,y)\}$$

Need

$$[x]_2 \otimes y = \sum r_i[x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.

Goncharov: zero element in $K_4^{[1]}(\mathbb{Q}(C))$.

Then

$$\int_\gamma \omega(x,y) = \sum r_i \, P_3(x_i)|_{\partial\gamma}$$

Big picture II

$$\ldots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \ldots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\ldots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.

Examples from the world of resultants

D'Andrea & L (2003).

- $m(\mathrm{Res}_{\{0,m,n\}}) = m(\mathrm{Res}_t(x + yt^m + t^n, z + wt^m + t^n))$

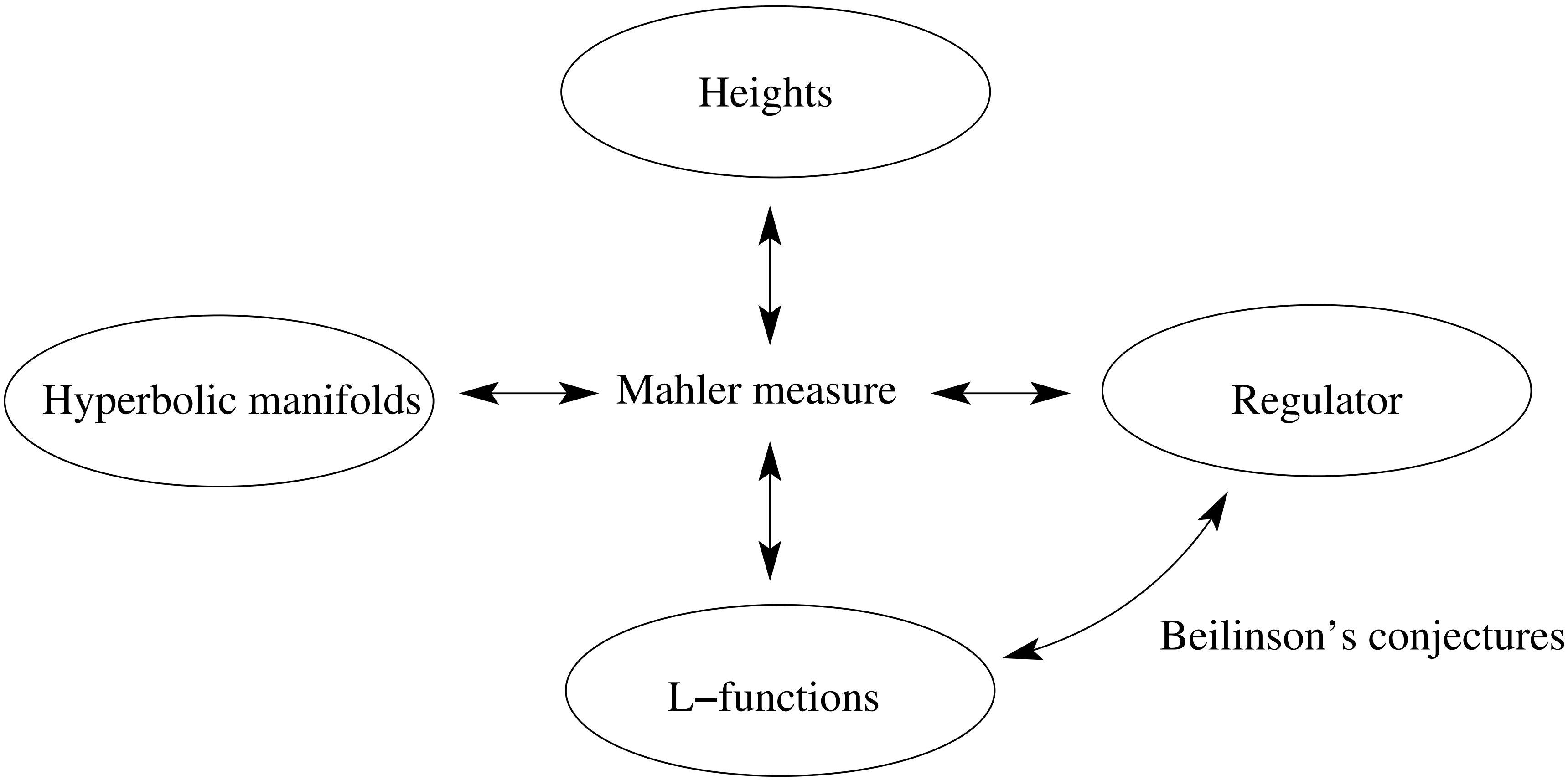
$$= \frac{2}{\pi^2}(-mP_3(\varphi^n) - nP_3(-\varphi^m) + mP_3(\phi^n) + nP_3(\phi^m))$$

$$0 \leq \varphi \leq 1 \quad \text{root of} \quad x^n + x^{n-m} - 1 = 0$$

$$1 \leq \phi \quad \text{root of} \quad x^n - x^{n-m} - 1 = 0$$

- $m(\mathrm{Res}_{\{(0,0),(1,0),(0,1)\}}) = m\left(\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix}\right)$

$$= m((1-x)(1-y) - (1-z)(1-w)) = \frac{9\zeta(3)}{2\pi^2}$$


Heights
Hyperbolic manifolds
Mahler measure
Regulator
L−functions
Beilinson’s conjectures