MAHLER MEASURE OF MULTIVARIABLE POLYNOMIALS

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Abstract. These notes correspond to extended versions of two lectures given by the authors during the BIRS workshop "Women in Numbers 2" at Banff, Canada, 2012

1. Definition of Mahler Measure and Lehmer's question

Looking for large primes, Pierce [Pi17] proposed the following construction. Consider $P \in \mathbb{Z}[x]$ monic, and write

$$P(x) = \prod_i (x - \alpha_i),$$

then look at

$$\Delta_n = \prod_i (\alpha_i^n - 1).$$

The α_i are algebraic integers. By applying Galois theory, it is easy to see that $\Delta_n \in \mathbb{Z}$. Note that if $P = x - 2$, we get the Mersenne sequence $\Delta_n = 2^n - 1$. The idea is to look for primes among the factors of Δ_n. The prime divisors of such integers must satify some congruence conditions that are quite restrictive, hence they are easier to factorize than a randomly given number. Moreover, one can show that $\Delta_m | \Delta_n$ if $m|n$. Then we may look at the numbers

$$\frac{\Delta_p}{\Delta_1} \quad \text{for } p \text{ prime}.$$

Lehmer [Le33] notices that the number of trial divisions would get minimized if the sequence Δ_n grows slowly. Thus, he studied $\frac{|\Delta_{n+1}|}{|\Delta_n|}$, observed that

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1, \\ 1 & \text{if } |\alpha| < 1, \end{cases}$$

and suggested the following definition.

Definition 1. *Given $P \in \mathbb{C}[x]$, such that*

$$P(x) = a \prod_i (x - \alpha_i)$$

define the (Mahler) [1] *measure of P as*

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}.$$

The logarithmic Mahler measure is defined as

$$\mathrm{m}(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|,$$

where $\log^+ |\alpha| = \log \max\{|\alpha|, 1\}$

2010 *Mathematics Subject Classification.* Primary 11R06; Secondary 11R09, 14J27, 14J28.

Key words and phrases. Mahler measure, polynomial, singular $K3$-surfaces, elliptic surfaces.

This work of ML was partially supported by NSERC Discovery Grant 355412-2008 and FQRNT Subvention établissement de nouveaux chercheurs 144987.

[1] The name Mahler came later after the person who successfully extended this definition to the several-variable case.

When does $M(P)=1$ for $P\in\mathbb{Z}[x]$? We have the following result.

Lemma 2. *(Kronecker,* [Kr57]*) Let $P=\prod_i(x-\alpha_i)\in\mathbb{Z}[x]$. If $|\alpha_i|\le 1$, then the α_i are zero or roots of the unity.*

By Kronecker's Lemma, $P\in\mathbb{Z}[x]$, $P\neq 0$, then $M(P)=1$ if and only if P is the product of powers of x and cyclotomic polynomials. This statement characterizes integral polynomials whose Mahler measure is 1.

Lehmer found the example

$$\mathrm{m}(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)=\log(1.176280818\dots)=0.162357612\dots$$

and asked the following (Lehmer's question, 1933):

Is there a constant $C>1$ such that for every polynomial $P\in\mathbb{Z}[x]$ with $M(P)>1$, then $M(P)\ge C$?

Lehmer's question remains open nowadays. His 10-degree polynomial remains the best possible result.

There are several results in the direction of solving Lehmer's question. Some of them consider restricted families of polynomials. The first of such results was found by Breusch [Br51] and (independently) by Smyth [Sm71]. For $P\in\mathbb{Z}[x]$ monic, irreducible, $P\neq\pm P^*$ (nonreciprocal), then

$$M(P)\ge M(x^3-x-1)=\theta=1.324717\dots$$

This result implies in particular that if $P\in\mathbb{Z}[x]$ is monic, irreducible, and of odd degree, then P is nonreciprocal and

$$M(P)\ge\theta.$$

On the other hand, there are results giving lower bounds that depend on the degree. The most fundamental of such results was given by Dobrowolski [Do79]. If $P\in\mathbb{Z}[x]$ is monic, irreducible and noncyclotomic of degree d, then

$$M(P)\ge 1+c\left(\frac{\log\log d}{\log d}\right)^3 \tag{1}$$

where c is an absolute positive constant.

2. Mahler Measure in several variables

Definition 3. *For $P\in\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$, the logarithmic Mahler measure is defined by*

$$\begin{aligned}\mathrm{m}(P) &:= \int_0^1\cdots\int_0^1\log|P(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n})|d\theta_1\dots d\theta_n\\ &= \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\log|P(x_1,\dots,x_n)|\frac{dx_1}{x_1}\dots\frac{dx_n}{x_n},\end{aligned}$$

where $\mathbb{T}^n=\{(x_1,\dots,x_n)\in\mathbb{C}^n\,|\,|x_1|=\dots=|x_n|=1\}$.

It is possible to prove that this integral is not singular and that $\mathrm{m}(P)$ always exists. This definition appeared for the first time in the work of Mahler [Ma62].

Because of Jensen's formula:

$$\int_0^1\log|e^{2\pi i\theta}-\alpha|d\theta=\log^+|\alpha|,$$

we recover the formula for the one-variable case.

Let us mention some elementary properties.

Proposition 4. *For $P,Q\in\mathbb{C}[x_1,\dots,x_n]$*

$$\mathrm{m}(P\cdot Q)=\mathrm{m}(P)+\mathrm{m}(Q).$$

Because of this formula, we can extend the definition of Mahler measure to rational functions.

Proposition 5. *Let $P \in \mathbb{C}[x_1, \ldots, x_n]$ such that $a_{m_1,\ldots,m_n}$ is the coefficient of $x_1^{m_1} \ldots x_n^{m_n}$ and P has degree d_i in x_i. Then*

$$\begin{aligned} |\, a_{m_1,\ldots,m_n}| &\leq \binom{d_1}{m_1} \ldots \binom{d_n}{m_n} M(P), \\ M(P) &\leq L(P) \leq 2^{d_1+\cdots+d_n} M(P), \end{aligned}$$

where $L(P)$ is the length of the polynomial, the sum of the absolute values of the coefficients.

In fact, the reason why Mahler considered this construction is that he was looking for inequalities of the typical polynomial heights (such as $L(P)$ or the maximum absolute value of the coefficients) between the height of a product of polynomials and the heights of the factors. These kinds of inequalities are useful in transcendence theory. The Mahler measure $M(P)$ is multiplicative and comparable to the typical heights, and that makes it possible to deduce such inequalities.

It is also true that $\mathrm{m}(P) \geq 0$ if P has integral coefficients.

Let us mention the following amazing result.

Theorem 6. *(Boyd* [Bo81]*, Lawton* [La83]*) For $P \in \mathbb{C}[x_1, \ldots, x_n]$*

$$\lim_{k_2\to\infty} \ldots \lim_{k_n\to\infty} \mathrm{m}(P(x, x^{k_2}, \ldots, x^{k_n})) = \mathrm{m}(P(x_1, \ldots, x_n)) \tag{2}$$

It should be noted that the limit has to be taken independently for each variable.

Because of the above theorem, Lehmer's question in the several-variable case reduces to the one-variable case. In addition, this theorem shows us that we are working with the "right" generalization of the original definition for one-variable polynomials.

The formula for the one-variable case tells us some information about the nature of the values that Mahler measure can reach. For instance, the Mahler measure of a polynomial in one variable with integer coefficients must be an algebraic number.

It is natural, then, to wonder what happens with the several-variable case. Is there any simple formula, besides the integral from the definition?

3. Examples

We show some examples of formulas for Mahler measures of multivariable polynomials.

- The first results were found by Boyd and Smyth. For example, the following formula in [Sm82]:

$$\mathrm{m}(x + y + 1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1), \tag{3}$$

  where

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \mod 3, \\ -1 & \text{if} \quad n \equiv -1 \mod 3, \\ 0 & \text{if} \quad n \equiv 0 \mod 3, \end{cases}$$

  is a Dirichlet L-function.
- Similar results for three variables were discovered by Smyth as well [Bo81]:

$$\mathrm{m}(x + y + z + 1) = \frac{7}{2\pi^2} \zeta(3). \tag{4}$$

- Rodriguez-Villegas [BDLR-V03] conjectured that the corresponding linear form in four variables satisfies

$$\mathrm{m}(x + y + z + w + 1) \stackrel{?}{=} 6 \left(\frac{\sqrt{-15}}{2\pi i} \right)^5 L(f_{15}, 4),$$

  where[2]

$$f_{15} = \eta(3z)^3 \eta(5z)^3 + \eta(z)^3 \eta(15z)^3$$

  is a CM modular form of weight 3 and level 15.

[2] Throughout this note, the question mark stands for numerical results up to 20 (or more) decimal places.

Shinder and Vlasenko [SV] have recently proved that this Mahler measure is related to a double L-value of modular forms. The definition of a double L-value for modular forms g and h of weight $k+1$ is the following:

$$L(h,g,k,1) = (2\pi)^4 \int_0^\infty g(is) \int_s^\infty \int_{s_1}^\infty \int_{s_2}^\infty \cdots \int_{s_{k-1}}^\infty h(is_k) ds_k \ldots ds_2 ds_1 ds.$$

Shinder and Valsenko proved

$$\mathrm{m}(x+y+z+w+1) = \frac{3\sqrt{5}\Omega_{15}^2}{20\pi} L(g_3,g_1,3,1) - \frac{3\sqrt{5}}{10\pi^3\Omega_{15}^2} L(g_2,g_1,3,1) + \frac{14}{5\pi^2}\zeta(3)$$

for suitable modular forms g_1, g_2, g_3 of weight 4, where Ω_{15} denotes the Chowla–Selberg period for the field $\mathbb{Q}(\sqrt{-15})$.

- There are formulas involving special values of L-functions of elliptic curves. For example, Rogers and Zudilin [RZ] proved the following formula (originally conjectured by Deninger and Boyd).

(5) $$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) = \frac{15}{4\pi^2} L(E_{15}, 2) = L'(E_{15}, 0).$$

  Here E_{15} is an elliptic curve (of conductor 15) that happens to be the algebraic closure of the zero set of the polynomial. We discuss more about these formulas in Section 6.
- Some numerical formulas involving L-functions of elliptic curves evaluated at $s=3$ were found by Boyd (2005):

(6) $$\mathrm{m}(z + (x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15}, -1).$$

  The relationship between E_{15} and this polynomial will be clarified in Section 5.
- There are examples with $K3$-surfaces, mostly due to Bertin [Be07, Be08, Be10, Be12] and more recently, Samart [Sa] and Bertin, Feaver, Fuselier, Lalín and Manes [BFFLM]. See Section 7 for more details.

How could one obtain such formulas? To be concrete, we are going to show the proof of the first example by Smyth (from [Bo81]). By Jensen's formula,

$$\mathrm{m}(x+y+1) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \log|e^{it} + e^{is} + 1| dt ds = \int_{-\pi}^{\pi} \log\max\{|e^{it}+1|, 1\} dt = \frac{1}{2\pi}\int_{-2\pi/3}^{2\pi/3} \log|1+e^{it}| dt.$$

Now we write

$$\log|1+e^{it}| = \mathrm{Re}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} e^{int},$$

and

$$\int_{-2\pi/3}^{2\pi/3} e^{int} dt = \frac{2}{n}\sin\left(\frac{2n\pi}{3}\right) = \frac{\sqrt{3}}{n}\chi_{-3}(n).$$

Thus, we get

$$\mathrm{m}(x+y+1) = \frac{\sqrt{3}}{2\pi}\sum_{n=1}^\infty \frac{(-1)^{n-1}\chi_{-3}(n)}{n^2} = \frac{\sqrt{3}}{2\pi}\left(\sum_{n=1}^\infty \frac{\chi_{-3}(n)}{n^2} - 2\sum_{n=1}^\infty \frac{\chi_{-3}(2n)}{(2n)^2}\right)$$

and use that $\chi_{-3}(2n) = \chi_{-3}(2)\chi_{-3}(n) = -\chi_{-3}(n)$ to obtain the initial formula.

4. Other occurences of Mahler measure

Before going into more detail about these formulas, we will mention the relationship of Mahler measure to low-dimensional topology. A generalization of Smyth's result (3) is due to Cassaigne and Maillot [Ma00]: for $a, b, c \in \mathbb{C}^*$,

$$\pi\mathrm{m}(ax+by+c) = \begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha\log|a| + \beta\log|b| + \gamma\log|c| & \triangle, \\ \pi\log\max\{|a|,|b|,|c|\} & \text{not } \triangle, \end{cases}$$

where $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and in that case, α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively. D is the Bloch - Wigner dilogarithm (see equation (8)). The term with the dilogarithm can be interpreted as the volume of the ideal hyperbolic tetrahedron which has the triangle as basis and the fourth vertex is infinity. Here we use the model of the upper-half space for the hiperbolic space $\mathbb{H}^3$, i.e., $\mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$ with a special metric with constant curvature -1 (the metric is given by $ds^2 = \frac{dx^2+dy^2+dz^2}{z^2}$).

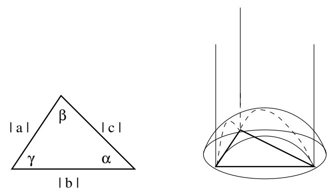

FIGURE 1. Cassaigne and Maillot's formula relate the Mahler measure of $ax + by + c$ with the hyperbolic volume of the ideal tetrahedron that one can construct over the triangle with side lengths $|a|, |b|$, and $|c|$

The relation between Mahler measure and volumes in hyperbolic space does not end in this example. Boyd [Bo02] and later Boyd and Rodriguez-Villegas [BR-V03] explored the relation between Mahler measure of the A-polynomial of a knot and the volume of the knot complement. Let X be a 1-cusped hyperbolic manifold, such as the complement of a hyperbolic knot. The A-polynomial (defined in [CCGLS94]) is an invariant $A(x, y) \in \mathbb{Q}[x, x^{-1}, y, y^{-1}]$ built from the space of representations $\rho : \pi_1(X) \to SL_2(\mathbb{C})$. More precisely, being 1-cusped means that $\pi_1(\partial X) \cong \mathbb{Z}^2$. Let $\lambda, \mu \in \pi_1(\partial X)$ be the longitude and the meridian of the boundary torus. Let L, $\frac{1}{L}$, M, $\frac{1}{M}$ be the eigenvalues of $\rho(\lambda)$ and $\rho(\mu)$. The A-polynomial is such that $A(L, M) = 0$ for all the representations. Typically, Boyd found identities such as

$$\pi \mathrm{m}(A) = \mathrm{Vol}(X).$$

This is true, for instance, for the Fig-8 knot.

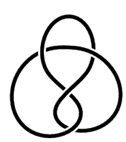

FIGURE 2. The hyperbolic Fig-8 knot

Another topological invariant that appears is the Alexander polynomial. Let K be a knot. Let X be the infinite cyclic cover of the knot complement of K. There is a covering transformation t acting on X. The transformation t acts on the homology and so we can consider $H^1(X)$ as a module over $\mathbb{Z}[t, t^{-1}]$. This is called the Alexander invariant or Alexander module. The module is finitely presentable; a presentation matrix for this module is called the Alexander matrix. If the number of generators, r, is less than or equal to the number of relations, s, then we consider the ideal generated by all $r \times r$ minors of the matrix; this is the Alexander ideal and does not depend on choice of presentation matrix. The Alexander ideal is principal and a generator is the Alexander polynomial of the knot, (unique up to multiplication by $\pm t^n$).

Alexander [Al28] devised a concrete method for constructing this invariant. First, one has to project the knot onto the plane leaving explicit information on when an arc is passing over or under another arc. Such a representation is called a knot diagram (see for example, Figure 2). A knot diagram with n crossings $c_1, \dots, c_n$ will divide the plane into $n + 2$ regions $r_1, \dots, r_{n+2}$. In addition, the knot is given an orientation. An $n \times (n + 2)$ matrix M is created in the following way: each entry $M(u, v)$ takes the values $0, 1, -1, t, -t$ and records the relative configuration of the crossing c_u and the region r_v. The location of the region at the crossing is determined from the perspective of the incoming undercrossing line. The entry is 0 if c_u and r_v are not adjacent. Otherwise the entry is $-t$ (resp. t) if the region is on the left before (resp. after) undercrossing and 1 (resp. -1) if the region is on the right before (resp. after) undercrossing. For example, with Figure 3, we get on the row of the crossing and respective columns i, j, k, l the entries $t, -t, 1, -1$. Then removing two columns of consecutive regions from M gives a square matrix N. The Alexander polynomial $\Delta_K(t)$ is given by $\det N$ eventually divided by $\pm t^n$ to ensure the positivity of the constant term.

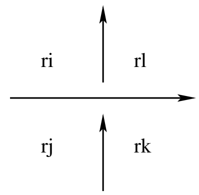

FIGURE 3. Crossing orientation

The Alexander polynomial of the $(-2, 3, 7)$-pretzel knot is Lehmer's polynomial. The Mahler measure of

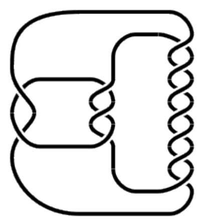

FIGURE 4. The $(-2, 3, 7)$-pretzel knot

Alexander polynomials was further explored by Silver and Williams [SW02, SW04] and by Silver, Stoimenow, and Williams [SSW06].

Finally, we remark that the Mahler measure of other topological polynomial invariants has been studied, such as the Jones polynomial.

Another application is to ergodic theory. Lind, Schmidt and Ward [LSW90] proved that the Mahler measure corresponds to the entropy of certain $\mathbb{Z}^n$-action over the torus $\mathbb{T}^{\mathbb{Z}^n}$.

5. An algebraic integration for Mahler measure: the role of the regulator

The appearance of the L-functions in Mahler measures formulas is a common phenomenon. Deninger [De97] linked this phenomenon to Beilinson's conjectures. Without going into much detail, Beilinson's conjectures are part of a set of statements (some are theorems and other are conjectures) that predict that one may obtain global information from local information and that this relation is made through values of L-functions. These statements include Dirichlet's class number formula, the Birch–Swinnerton-Dyer conjecture, and more generally, Bloch's, Deligne's, and Beilinson's conjectures.

Typically, there are four elements involved in this setting: an arithmetic-geometric object X (typically, an algebraic variety), its L-function (which codifies local information), a finitely generated abelian group K, and a regulator map $K \to \mathbb{R}$. When K has rank 1, Beilinson's conjectures predict that $L'(0)$ is, up to a rational number, equal to a value of the regulator. For instance, for a real quadratic field F, $X = \mathcal{O}_F$ (the ring of integers), $L = \zeta_F$, and $K = \mathcal{O}_F^*$, then Dirichlet's class number formula may be written as $\zeta_F'(0)$ is equal to, up to a rational number, $\log|\varepsilon|$, for some $\varepsilon \in \mathcal{O}_F^*$.

Deninger proved that in favorable cases

$$\mathrm{m}(P) = \mathrm{reg}(\varepsilon),$$

where reg is the determinant of the regulator matrix, which we are evaluating in some class ε in an appropriate group in K-theory.

The relationship with regulators can be exploited to prove identities between Mahler measures of different polynomials such as the following example due to Rodriguez-Villegas [R-V02]

$$\mathrm{m}(y^2+2xy+y-x^3-2x^2-x)=\frac{5}{7}\mathrm{m}(y^2+4xy+y-x^3+x^2).$$

Ideally, the relationship with regulators can be used to connect the Mahler measure with special values of L-functions. Many of these formulas can be proved with elementary methods, and thus, they provide examples of cases of Beilinson's conjectures "in action", which are not easy to find. Other formulas can be only proved using Beilinson's conjectures in the cases that are known, i.e., when the elliptic curve is modular or when it has complex multiplication (such as the results by Mellit, Brunault, and Rodriguez-Villegas described in Section 6). Many other formulas have only been shown numerically.

Rodriguez-Villegas [R-V97] made explicit the relationship between Mahler measure and regulators by computing the regulator for the two-variable case, and using this machinery to explain the formulas for two variables.

For example, let us go back to Smyth's example (3) which we will now write as $P(x,y)=y+x-1$ (this change of coefficients does not affect the Mahler measure). We will reprove it in the context of Beilinson's conjectures. Then its Mahler measure is

$$\mathrm{m}(P)=\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log|y+x-1|\frac{dx}{x}\frac{dy}{y}.$$

By Jensen's equality,

$$\mathrm{m}(P)=\frac{1}{2\pi i}\int_{\mathbb{T}^1}\log^+|1-x|\frac{dx}{x}=\frac{1}{2\pi i}\int_{\gamma}\log|y|\frac{dx}{x}=-\frac{1}{2\pi i}\int_{\gamma}\eta(x,y) \tag{7}$$

where $\gamma=\{P(x,y)=0\}\cap\{|x|=1,|y|\geq 1\}$ and

$$\eta(x,y)=\log|x|di\arg y-\log|y|di\arg x.$$

$\eta(x,y)$ is a closed differential form defined in $X=\{P(x,y)=0\}$ minus the sets of zeros and poles of x and y. It satisfies the following properties:

- $\eta(x,y)=-\eta(y,x)$,
- $\eta(x_1x_2,y)=\eta(x_1,y)+\eta(x_2,y)$.

Without going into technicalities, let us say that this form is the regulator. We would like to apply Stokes Theorem. The question is, when is $\eta(x,y)$ exact? Fortunately, there is

$$\eta(x,1-x)=diD(x),$$

where $D(x)$ denotes the Bloch–Wigner dilogarithm,

$$D(x):=\mathrm{Im}(\mathrm{Li}_2(x))+\arg(1-x)\log|x|. \tag{8}$$

Here

$$\mathrm{Li}_2(x):=\sum_{n=1}^{\infty}\frac{x^n}{n^2}\qquad\text{for } |x|<1,$$

is the classical dilogarithm.

If we use Stokes Theorem, we get

$$\mathrm{m}(P)=-\frac{1}{2\pi}D(\partial\gamma).$$

Now we parametrize

$$\gamma: x=e^{2\pi i\theta}\quad y(\gamma(\theta))=1-e^{2\pi i\theta},\quad \theta\in[1/6\,;5/6]\qquad \partial\gamma=[\bar{\xi_6}]-[\xi_6]$$

where $\xi_6=e^{2\pi i/6}$ is the sixth root of unity (see picture).

Thus we obtain

$$2\pi\mathrm{m}(x+y+1)=D(\xi_6)-D(\bar{\xi_6})=2D(\xi_6)=\frac{3\sqrt{3}}{2}L(\chi_{-3},2).$$

See [La07] for details.

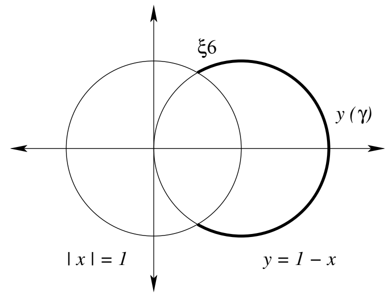

FIGURE 5. Integration path γ for the final integral in equation (7)

The appearance of the dilogarithm may be interpreted as a version of Beilinson's conjectures, or more precisely in this case, Borel's theorem. According to Borel's theorem, the regulator is given by the dilogarithm when we work in K_2 of a number field.

In general, many of the examples can be explained with this setting. For example, many of the zeta values can be seen as special values of polylogarithms, that are defined by

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \qquad \text{for } |x| < 1.$$

When the polynomial has more than two variables the above process is more complicated since there are more steps in the integration. Much of the understanding in this case is due to an idea of Maillot (2003) based on a result of Darboux (1875) in which the Mahler measure can be expressed in terms of an integral over a variety of one dimension less if the initial polynomial P is nonreciprocal. As an example, consider formula (6). In this case, the variety to be considered is

$$\begin{cases} z + (x+1)(y+1) = 0, \\ \frac{1}{z} + \left(\frac{1}{x}+1\right)\left(\frac{1}{y}+1\right) = 0. \end{cases}$$

The intersection of these two equations corresponds to

$$(x+1)^2(y+1)^2 = xy,$$

whose desingularization is an elliptic curve of conductor 15 whose L-function appears in formula (6).

The cases with elliptic curves can be also explained with the aid of Beilinson's conjectures. In this case, the connection is done through the elliptic dilogarithm. Notice that in the case of (5) we can not apply the trick of Maillot since the polynomials are reciprocal.

6. The measures of a family of genus-one curves

An elliptic curve (over $\mathbb{C}$) is roughly speaking a curve (zeros of a two-variable polynomial) that is birationally equivalent to an equation of the form

$$E : Y^2 = X^3 + aX + b.$$

For example, the curve given by the equation

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0,$$

where k is a parameter, corresponds to an elliptic curve. We can see this by applying the change of variables

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}, \tag{9}$$

and we get the equation

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

k	s_k	N
1	1	15
2	1	24
3	2	21
5	6	15
6	1/2	120
7	1/2	231
8	4	24
9	1/2	195
10	-1/8	840

TABLE 1. s_k numerically conjectured values from formula (10). N corresponds to the conductor of the elliptic curve. When $k = 4$ the curve has genus zero.

If the elliptic curve is defined over $\mathbb{Q}$ (i.e., $a, b \in \mathbb{Q}$), one can construct the L-function as follows

$$L(E,s) = \prod_{\text{good}\, p} (1 - a_p p^{-s} + p^{1-2s})^{-1} \prod_{\text{bad}\, p} (1 - a_p p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where for p prime,

$$a_p = 1 + p - \#E(\mathbb{F}_p).$$

The family of two-variable polynomials $P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$ was initially studied by Boyd [Bo81], Deninger [De97], and Rodriguez-Villegas [R-V97] from different points of view. Boyd found many numerical identities of the form

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) \stackrel{?}{=} s_k L'(E_{(k)}, 0) \qquad k \in \mathbb{N} \neq 0, 4 \tag{10}$$

where s_k is a rational number (often integer), and $E_{(k)}$ is the elliptic curve which is the algebraic closure of the zero set of the polynomial (i.e., given by the change of variables (9)). Table 1 shows the first values for s_k conjectured by Boyd. He numerically computed s_k for $k = 1, \dots 40$.

The connection with $L'(E,0)$ was predicted by Deninger using Beilinson's conjectures. However, there are some cases in which this identity can be proved. This happens when Beilinson's conjectures are known, i.e., when the elliptic curve has complex multiplication, or when it is given as a modular curve, and then the Mahler measure may be related to the L-function of a modular form.

In [R-V97], Rodriguez-Villegas expressed this Mahler measure as an Eisenstein–Kronecker series:

$$\begin{aligned} \mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) &= \mathrm{Re}\left(\frac{16\,\mathrm{Im}\,\tau}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m + n4\tau)^2 (m + n4\bar{\tau})}\right) \\ &= \mathrm{Re}\left(-\pi i \tau + 2 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 \frac{q^n}{n}\right), \end{aligned} \tag{11}$$

where the q parameter is coming from

$$q = e^{2\pi i \tau} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)}\right).$$

Rodriguez-Villegas' idea to obtain this formula is as follows. One observes that for $\lambda = -1/k$ such that $|\lambda| < 1/4$, the Mahler measure of this polynomial is given by

$$m(k) = \mathrm{Re}(\tilde{m}(\lambda)),$$

where

$$\begin{aligned}\tilde{m}(\lambda) &= -\log\lambda + \int_{\mathbb{T}^2} \log(1-\lambda(x+x^{-1}+y+y^{-1}))\frac{dx}{x}\frac{dy}{y}\\ &= -\log\lambda - \sum_{n=1}^{\infty}\frac{b_n}{n}\lambda^n,\end{aligned}$$

where b_n is the constant coefficient of the polynomial $(x+x^{-1}+y+y^{-1})^n$. More specifically,

$$b_n = \begin{cases} \binom{n}{n/2}^2 & n \text{ even},\\ 0 & n \text{ odd}.\end{cases}$$

Now consider

$$\begin{aligned}u(\lambda) &= \int_{\mathbb{T}^2} \frac{1}{1-\lambda(x+x^{-1}+y+y^{-1})}\frac{dx}{x}\frac{dy}{y}\\ &= \sum_{n=0}^{\infty} b_n\lambda^n.\end{aligned}$$

Then

$$\tilde{m}(\lambda) = -\log\lambda - \int_0^{\lambda}(u(\delta)-1)d\delta.$$

By construction, $u(\lambda)$ is a period of a holomorphic differential on the curve defined by $1-\lambda(x+x^{-1}+y+y^{-1}) = 0$ (see [Gr69]) hence a solution to a Picard-Fuchs differential equation. Thus, it is not surprising that $\tilde{m}(\lambda)$ has a hypergeometric series form.

Formula (11) may in turn be related to the elliptic dilogarithm (using the techniques of Bloch [Bl00]). Then one has to relate the values of the elliptic dilogarithm to the L-function, which is done through Beilinson's conjectures.

For example, Rodriguez-Villegas [R-V97] proved

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+4\sqrt{2}\right) = L'(E_{32},0). \tag{12}$$

It should be remarked that it suffices that k^2 be an integer for this equation to have an interpretation in terms of Beilinson's conjectures. In this case, the curve has complex multiplication.

Other examples were given by Rogers and Zudilin: in [RZ] they proved

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right) = \frac{15}{4\pi^2}L(E_{15},2) = L'(E_{15},0),$$

and in [RZ12] they proved

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right) = \frac{24}{\pi^2}L(E_{24},2) = 4L'(E_{24},0).$$

The main ideas behind the proof of these results consist on decomposing the cusp form given by the modularity theorem into products of weight one Eisenstein series, and then perform some transformations on the integral that lead to Mahler measure.

It should be noted that there are other families related to elliptic curves that yield similar results that were already numerically studied by Boyd. After Boyd's paper, some identities for these families were proved also using Beilinson's conjectures.

Brunault [Br05, Br06] considered the curve $X_1(11)$ and proved

$$\mathrm{m}\left((1+x)(1+y)(1+x+y)+xy\right) = \frac{77}{4\pi^2}L(E_{11},2) = 7L'(E_{11},0),$$

by giving an explicit version of Beilinson's theorem on modular curves.

Similarly, Mellit [Me] considered the modular curve $X_0(14)$ and proved several identities including, for instance,

$$\mathrm{m}\left(x^3+y^3+1+xy\right)=\frac{7}{\pi^2}L(E_{14},2)=2L'(E_{14},0).$$

What do these polynomials have in common? Boyd noticed that they all satisfy that the faces of their Newton polygon are cyclotomic polynomials (i.e., they have Mahler measure zero). This condition was explained by Rodriguez-Villegas [R-V97] in terms of K-theory. Roughly speaking, this condition guarantees that there is an element $\{x,y\}$ in K_2 of the elliptic curve. The regulator is then evaluated in this element.

7. Mahler measures of families of $K3$-surfaces

In section 6 we exposed results concerning the family of two-variable polynomials defining elliptic curves $x+1/x+y+1/y+k$. It seems natural to try to generalize the results to the family of three-variable polynomials

$$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+k.$$

These polynomials define, after desingularization, $K3$-surfaces. Elliptic curves and $K3$-surfaces belong to the same type of varieties: Calabi-Yau varieties (elliptic curves in dimension $d=1$ and $K3$-surfaces in dimension $d=2$). We recall (see [Yu04]) that a smooth projective variety X of dimension d over $\mathbb{C}$ is called a Calabi-Yau variety if

(1) $H^i(X,\mathcal{O}_X)=0$ for every $0<i<d$ and
(2) the canonical bundle $\mathcal{K}_X$ of X is trivial.

Thus, condition (2) tells us that one of the main properties of a Calabi-Yau variety is to possess a unique, up to scalar, holomorphic d-form which allows the Calabi-Yau variety to have periods satisfying a Picard-Fuchs equation for the variable k. In fact, the main result for our families in dimension 1 and 2 is that the derivative of the logarithmic Mahler measure of the polynomials is a period with respect to the parameter k of the family. Hence, it satisfies the Picard-Fuchs equation of the family. Thus, an expression of the solutions of the Picard-Fuchs differential equation can be integrated to obtain an interesting expression for the Mahler measure.

This point of view has been used by Bertin [Be01] for the family of elliptic curves with 6-torsion and for two families of three-variables polynomials defining $K3$-surfaces [Be08].

Bertin [Be08] obtained an expression for the Mahler measure equivalently either in terms of the function[3]

$$G(q)=\mathrm{Re}\left(-\log(q)+240\sum_{n=1}^{\infty}n^2\log(1-q^n)\right)$$

or in terms of Eisenstein-Kronecker series similar to Rodriguez-Villegas' [R-V97].

By considering the family

$$P_k:=X+\frac{1}{X}+Y+\frac{1}{Y}+Z+\frac{1}{Z}+k$$

and the family

$$Q_k \quad := \quad X+\tfrac{1}{X}+Y+\tfrac{1}{Y}+Z+\tfrac{1}{Z}+XY+\tfrac{1}{XY}+ZY+\tfrac{1}{ZY}+XYZ+\tfrac{1}{XYZ}-k.$$

Bertin obtained [Be08] the following result. Let $k=t+\frac{1}{t}$, let η denote the Dedekind eta function and

$$t=\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^6=q^{1/2}-6q^{3/2}+15q^{5/2}-20q^{7/2}+\dots.$$

Then,

$$\mathrm{m}(P_k)=\mathrm{Re}\left(-\pi i\tau+\sum_{n\geq 1}\sum_{d|n}d^3\left(\frac{4q^n}{n}-\frac{16q^{2n}}{2n}+\frac{36q^{3n}}{3n}-\frac{144q^{6n}}{6n}\right)\right).$$

[3] $G(q)$ was introduced by Rogers in [Ro09].

and

$$\mathrm{m}(P_k) = \frac{\operatorname{Im}\tau}{8\pi^3} \sum_{j\in\{1,2,3,6\}} \sideset{}{'}\sum_{m,\kappa} (-1)^j 4j^2 \left(2\operatorname{Re}\frac{1}{(jm\tau+\kappa)^3(jm\bar{\tau}+\kappa)} + \frac{1}{(jm\tau+\kappa)^2(jm\bar{\tau}+\kappa)^2} \right).$$

Similarly, with $k = -(t+\frac{1}{t}) - 2$ and

$$t = \frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}},$$

on one side,

$$\mathrm{m}(Q_k) = \operatorname{Re}\left(-2\pi i\tau + \sum_{n\geq 1}\sum_{d|n} d^3 \left(\frac{-2q^n}{n} + \frac{32q^{2n}}{2n} + \frac{18q^{3n}}{3n} - \frac{288q^{6n}}{6n} \right) \right)$$

and on the other side

$$\mathrm{m}(Q_k) = \frac{\operatorname{Im}\tau}{8\pi^3} \sum_{j\in\{1,2,3,6\}} \sideset{}{'}\sum_{m,\kappa} a_j \left(2\operatorname{Re}\frac{1}{(jm\tau+\kappa)^3(jm\bar{\tau}+\kappa)} + \frac{1}{(jm\tau+\kappa)^2(jm\bar{\tau}+\kappa)^2} \right),$$

with $a_1 = 2$, $a_2 = -32$, $a_3 = -18$, and $a_4 = 288$.

The first expressions for $\mathrm{m}(P_k)$ and $\mathrm{m}(Q_k)$ were recovered by Rogers [Ro09] in terms of the function $G(q)$.

If X denotes an algebraic $K3$-surface, then $H_2(X,\mathbb{Z})$ is a free group of rank 22. It becomes a lattice $\mathcal{L}$ (known as the $K3$-lattice) with the intersection pairing. In fact we can write

$$H_2(X,\mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

where U_2 is the hyperbolic lattice of rank 2 and E_8 the unimodular lattice of rank 8.

The group $\operatorname{Pic}(X)$ of divisors modulo linear equivalence (if X is $K3$, linear, algebraic, and numerical equivalence are the same) satisfies

$$\operatorname{Pic}(X) \subset H_2(X,\mathbb{Z}) \simeq Hom(H^2(X,\mathbb{Z}),\mathbb{Z}).$$

In fact, it can be viewed also as a sublattice of the $K3$-lattice.

The Picard group is parametrized by the algebraic cycles and we have the following properties

$$\operatorname{Pic}(X) \simeq \mathbb{Z}^{\rho(X)},$$

$$1 \leq \rho(X) \leq 20.$$

The integer $\rho(X)$ is called the Picard number of X. The orthogonal complement of $\operatorname{Pic}(X)$ in the $K3$-lattice is called the transcendental lattice and it has dimension $22-\rho(X)$. If $\{\gamma_1,\cdots,\gamma_{22}\}$ is a $\mathbb{Z}$-basis of $H_2(X,\mathbb{Z})$ and ω the holomorphic 2-form, then

$$\int_{\gamma_i}\omega$$

is called a period of X and

$$\int_{\gamma}\omega = 0 \text{ for } \gamma\in\operatorname{Pic}(X).$$

Let $\{X_z\}$ (with $z\in\mathbb{P}^1$) be a family of $K3$-surfaces with generic Picard number ρ and corresponding holomorphic 2-form ω_z. Then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22-\rho$. For our families $k=3$.

An $\mathcal{M}$-polarized $K3$-surface, with Picard number ≥ 17 has a Shioda-Inose structure. This means there exists an abelian surface $A := E\times E/C_N$ (E an elliptic curve, C_N a group of cyclic isogeny), a Kummer surface $Y = \operatorname{Kum}(A/\pm 1)$, and a canonical involution ι on X with $X/\langle\iota\rangle$ birationally isomorphic to Y.

$$\begin{array}{ccc} X & & A = E\times E/C_N \\ & \searrow \quad\quad \swarrow & \\ & Y = \operatorname{Kum}(A/\pm 1) & \end{array}$$

If the Picard number $\rho = 20$, the $K3$-surface is called singular and the elliptic curve E is CM.

For definitions concerning $K3$-surfaces, see for example [Be08].

We will recall results concerning the modularity of dimension 2 Calabi-Yau varieties, namely $K3$-surfaces, defined over $\mathbb{Q}$.

The Néron-Severi group of X is generated by the algebraic cycles. It is a finitely generated abelian group isomorphic to the Picard group. Moreover, we have the inclusion

$$\mathrm{NS}(X) \subset H^2(X, \mathbb{Z}).$$

Since X is equipped with the perfect pairing induced by the intersection pairing, we can define its orthogonal complement in $H^2(X, \mathbb{Z})$ with respect to this perfect pairing by

$$\mathbb{T}(X) := NS(X)^{\perp}_{H^2(X,\mathbb{Z})}.$$

The lattice $\mathbb{T}(X)$ is called the lattice of transcendental cycles on X.

Consider now a singular (or extremal) $K3$ surface X defined over $\mathbb{Q}$. Its L-series is defined by

$$L(X, s) = (*) \prod_{\text{p good}} P_p(p^{-s})^{-1}.$$

The product runs over all the primes of good reduction, i.e., the primes p for which the reduction modulo p of X still remains a $K3$-surface. The factor $(*)$ corresponds to the bad primes. The polynomial $P_p(T)$ is given by

$$P_p(T) = \det(1 - \mathrm{Frob}_p^* T|_{H^2_{\text{et}}}(\bar{X}, \mathbb{Q}_l)).$$

It is an integral polynomial of degree 22 whose reciprocal roots have absolute value p. The decomposition of the lattice $H^2(X, \mathbb{Z}) = NS(X) \oplus \mathbb{T}(X)$ induces a decomposition of the L-series $L(X, s)$:

$$L(X, s) = L(NS(X) \otimes \mathbb{Q}_l, s) \cdot L(\mathbb{T}(X) \otimes \mathbb{Q}_l, s).$$

Livné [Li95] proved the modularity of singular $K3$-surfaces, that is, if $\rho(X/\mathbb{Q}) = 20$ then

$$L(\mathbb{T}(X) \otimes \mathbb{Q}_l, s) = L(f, s),$$

where f is a modular form of weight $d + 1 = 3$ on $(\Gamma_0(N), \chi)$ for a quadratic character χ or on $\Gamma_1(N)$.

For more details about the modularity of singular $K3$ surfaces we refer to Yui [Yu04].

Schütt [Sc09] determined all CM newforms of weight 3 and rational coefficients. More precisely, he considered the following classifications of singular $K3$-surfaces over $\mathbb{Q}$:

(1) by the discriminant d of the transcendental lattice of the surface up to squares,
(2) by the associated newform up to twisting,
(3) by the level of the associated newform up to squares,
(4) by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.

He proved that all these classifications are equivalent. In particular, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

When the $K3$-surface is singular, by Shioda-Inose and Shioda-Mitani, the corresponding τ defined previously is imaginary quadratic. For example, Boyd computed experimentally for the family P_k

k	0	2	3	6	10	18
τ	$\frac{-3+\sqrt{-3}}{6}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{-3+\sqrt{-15}}{12}$	$\frac{\sqrt{-6}}{6}$	$\frac{\sqrt{-2}}{2}$	$\sqrt{\frac{-5}{6}}$

and for the family Q_k

k	0	12
τ	$\frac{3+\sqrt{-3}}{12}$	$\frac{3+\sqrt{-3}}{6}$.

Bertin [Be08] obtained an expression for the Mahler measure in terms of Hecke L-series for certain Grössencharacter.

We recall that if $K = \mathbb{Q}(\sqrt{d})$ denotes an imaginary quadratic field of discriminant D and ring of integers $\mathcal{O}_K$, a Grössencharacter ϕ of weight $k \geq 2$, and conductor Λ (an ideal of $\mathcal{O}_K$) is defined in the following manner. An homomorphism $\phi : I(\Lambda) \to \mathbb{C}^\times$ satisfying

$$\phi(\alpha\mathcal{O}_K) = \alpha^{k-1} \quad \text{for} \quad \alpha \equiv 1 \mod \Lambda$$

is called a Hecke Grössencharacter of weight k and conductor Λ. The corresponding Hecke L-series is defined by

$$L(\phi, s) := \sum_P \frac{\phi(P)}{N(P)^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $N(P)$ is the norm of the ideal P, the sum being on the prime ideals $P \subset \mathcal{O}_K$ prime to Λ. The ring $\mathcal{O}_K$ may be replaced by an order R of the quadratic field. In all cases it will be specified either $L_{\mathbb{Q}(\sqrt{d})}$ or L_R.

Bertin obtained in [Be08]:

$$\begin{aligned} \mathrm{m}(P_0) &= d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2), \\ \mathrm{m}(P_2) &= \frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3), \\ \mathrm{m}(P_3) &= \frac{15\sqrt{15}}{2\pi^3} L_{\mathbb{Q}(\sqrt{-15})}(\phi, 3), \end{aligned}$$

where $\phi(P) = -\omega$ if $P = (2, \omega)$ and $\omega = \frac{1+\sqrt{-15}}{2}$, ($P$ denotes a representant of the nontrivial ideal class of the number field $\mathbb{Q}(\sqrt{-15})$ of class number 2). Bertin also obtained

$$\mathrm{m}(P_6) = \frac{24\sqrt{6}}{\pi^3} L_{\mathbb{Q}(\sqrt{-6})}(\phi, 3),$$

where $\phi(P) = -2$ si $P = (2, \sqrt{-6})$, (P denotes a representant of the nontrivial ideal class of the number class field $\mathbb{Q}(\sqrt{-6})$ with class number 2).

For the other family, Bertin obtained

$$\mathrm{m}(Q_0) = \frac{12\sqrt{3}}{\pi^3} L_R(\phi, 3)$$

for the order $R = (1, 2\sqrt{-3})$ of class number 1 and

$$\mathrm{m}(Q_{12}) = 4\mathrm{m}(Q_0).$$

All the previous Grössencharacters have weight 3.

In order to apply Livné and Schütt's results, Bertin computed with various methods the determinant of the transcendental lattice of the $K3$-surface Y_k (resp. S_k) defined by the polynomial P_k (resp. Q_k) and obtained

k	2	10
$\det \mathbb{T}(Y_k)$	8	72

k	0	12	-3
$\det \mathbb{T}(S_k)$	12	12	15

The case $\det S_{-3}$ has been computed by Peters, Top and van der Vlugt [PTV92].

Hence Bertin could prove [Be07, Be08, Be10, Be12]

$$\begin{aligned} \mathrm{m}(P_0) &= d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2), \\ \mathrm{m}(P_2) &= 4\frac{|\det \mathbb{T}(Y_2)|^{3/2}}{4\pi^3} L(\mathbb{T}(Y_2), 3) = 4 \cdot \frac{8\sqrt{8}}{4\pi^3} L(g_8, 3), \text{ and} \\ \mathrm{m}(P_{10}) &= \frac{4}{9}\frac{|\det \mathbb{T}(Y_{10})|^{3/2}}{4\pi^3} L(\mathbb{T}(Y_{10}), 3) + 2d_3 = \frac{4}{9} \cdot \frac{72\sqrt{72}}{4\pi^3} L(g_8, 3) + 2d_3, \end{aligned}$$

as well as

$$\begin{aligned}
\mathrm{m}(Q_{-3}) &= \frac{8}{5}d_3,\\
\mathrm{m}(Q_0) &= 2\frac{|\det \mathbb{T}(S_0)\ |^{3/2}}{4\pi^3}L(S_0,3) = 2\cdot\frac{12\sqrt{12}}{4\pi^3}L(g_{12},3), \text{ and}\\
\mathrm{m}(Q_{12}) &= 8\frac{|\det \mathbb{T}(S_{12})\ |^{3/2}}{4\pi^3}L(S_0,3) = 8\cdot\frac{12\sqrt{12}}{4\pi^3}L(g_{12},3),
\end{aligned}$$

where Y_k (resp. S_k) denotes the $K3$-surface associated to the zero set $P_k(x,y,z)=0$ (resp. $Q_k(x,y,z)$), $\mathbb{T}$ denoting its transcendental lattice, and $L(g_N,3)$ the L-series at $s=3$ of a modular form of weight 3 and level N.

In addition, very recent results have been proved by Bertin, Feaver, Fuselier, Lalín, and Manes [BFFLM],

$$\begin{aligned}
\mathrm{m}(P_3) &= 2\frac{|\det \mathbb{T}(Y_3)|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_3),3) = 2\cdot\frac{15\sqrt{15}}{4\pi^3}L(g_{15},3),\\
\mathrm{m}(P_6) &= 2\frac{|\det \mathbb{T}(Y_6)|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_6),3) = 2\cdot\frac{24\sqrt{24}}{4\pi^3}L(g_{24},3), \text{ and}\\
\mathrm{m}(P_{18}) &= \frac{1}{5}\frac{|\det \mathbb{T}(Y_{18})|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_{18}),3) + \frac{14}{5}d_3 = \frac{1}{5}\cdot\frac{120\sqrt{120}}{4\pi^3}L(g_{120},3) + \frac{14}{5}d_3.
\end{aligned}$$

The Mahler measure of polynomials defining singular $K3$-surfaces is also related to values of the hypergeometric functions ${}_5F_4$. Results in that direction are due to Rogers [Ro09] and Samart [Sa]. Rogers' motivation was to generalize Boyd's identities of the following type

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right) = -2\,\mathrm{Re}\left({}_4F_3\left({\textstyle\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};16\right)\right) \stackrel{?}{=} \frac{15}{4\pi^2}L(f,2),$$

where f denotes the modular form

$$f(q) = q\prod_{n=1}^{\infty}(1-q^n)(1-q^{3n})(1-q^{5n})(1-q^{15n}).$$

As noticed by Rogers [Ro09], Kurokawa and Ochiai [KO05] simplified this last conjecture to

$${}_3F_2\left({\textstyle\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop \frac{3}{2},1};\frac{1}{16}\right) \stackrel{?}{=} \frac{15}{\pi^2}L(f,2).$$

In [Ro09], Rogers considered families of logarithmic Mahler measures

$$g_1(u) := \mathrm{m}\left(u+x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}\right)$$

$$g_2(u) := \mathrm{m}\left(-u+4+\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+\left(x+\frac{1}{x}\right)\left(z+\frac{1}{z}\right)+\left(y+\frac{1}{y}\right)\left(z+\frac{1}{z}\right)\right)$$

We remark that $g_1(u)=\mathrm{m}(P_u)$ and after the substitution $\left(xz,\frac{y}{z},\frac{z}{x}\right)\mapsto(x,y,z)$, $g_2(u+4)=m(Q_u)$, where the polynomials P_u and Q_u are those considered by Bertin.

Rogers main results are relations between Mahler measures of three-variable polynomials defining $K3$-surfaces such as, for $|u|$ sufficiently large,

$$g_1(3(u^2+u^{-2})) = \frac{1}{5}\mathrm{m}\left(x^4+y^4+z^4+1+\sqrt{3}\frac{(3+u^4)}{u^3}xyz\right)+\frac{3}{5}\mathrm{m}\left(x^4+y^4+z^4+1+\sqrt{3}\frac{(3+u^{-4})}{u^{-3}}xyz\right)$$

and

$$g_2(u) = -\frac{1}{15}f_3\left(\frac{(16-u)^3}{u^2}\right)+\frac{8}{15}f_3\left(-\frac{(4-u)^3}{u}\right),$$

where

$$f_3(u) := \mathrm{m}(u-(x+x^{-1})^2(y+y^{-1})^2(1+z)^3z^{-2}).$$

Rogers method involves expressing $f_3(s_3(q))$ and $f_4(s_4(q))$ for appropriate modular functions $s_j(q)$ as linear combinations of $G(\pm q^n)$ similar to linear combinations obtained by Bertin [Be08]. Then, Rogers computed

$$G(q) = -5f_4(s_4(q)) - 2f_4(s_4(-q)) + 4f_4(s_4(q^2)).$$

He compared with Bertin's relation and found

$$g_1(t_1(q)) = \frac{1}{20}f_4(s_4(q)) + \frac{3}{20}f_4(s_4(q^2)), \tag{13}$$

with

$$f_4(u) := 4\mathrm{m}(x^4 + y^4 + z^4 + 1 + u^{1/4}xyz),$$

and $t_1(q)$ another modular form.

Then Rogers proved an algebraic relation between the modular functions $s_4(q)$ and $t_1(q)$ satisfied by $t_1(q) = 3(z + z^{-1})$ and $s_4(q) = 9(3 + z^2)^4 z^{-6}$. A similar method was applied to the second relation.

Rogers also expressed the different Mahler measures $f_i(u)$, $2 \le i \le 4$ in terms of values of various hypergeometric functions with rational arguments. Using these expressions and relations like (13), Rogers wrote certain values of hypergeometric functions ${}_5F_4$ as sum of logarithms and L-series of modular forms. For example, he obtained the beautiful results

$${}_5F_4\left({\tfrac{5}{4},\tfrac{3}{2},\tfrac{7}{4},1,1 \atop 2,2,2,2};1\right) = \frac{256}{3}\log(2) - \frac{5120\sqrt{2}}{3\pi^3}L(f,3)$$

for a CM modulo form f of weight 3 and level 8 and

$${}_5F_4\left({\tfrac{4}{3},\tfrac{3}{2},\tfrac{5}{3},1,1 \atop 2,2,2,2};1\right) = 18\log(2) + 27\log(3) - \frac{810\sqrt{3}}{\pi^3}L(g,3)$$

for a CM modulo form g of weight 3 and level 12.

Very recently, Samart [Sa], keeping with the same notations as Rogers', expressed the Mahler measure of certain polynomials defining singular $K3$-surfaces in terms of the L-series of a CM modular form of weight 3 plus a Dirichlet series. Samart gave the following formulas

$$\begin{aligned}
f_2(64) &= \frac{128}{\pi^3}L(h,3),\\
f_2(256) &= \frac{64\sqrt{3}}{\pi^3}L(g_{48},3) + \frac{16}{3\pi}L(\chi_{-4},2),\\
f_3(216) &= \frac{45\sqrt{6}}{\pi^3}L(g_{24}^{(1)},3) + \frac{45\sqrt{3}}{16\pi}L(\chi_{-3},2),\\
f_3(1458) &= \frac{405\sqrt{3}}{4\pi^3}L(g,3) + \frac{15}{2\pi}L(\chi_{-4},2),\\
f_4(648) &= \frac{160}{\pi^3}L(h,3) + \frac{5}{\pi}L(\chi_{-4},2),\\
f_4(2304) &= \frac{80\sqrt{6}}{\pi^3}L(g_{24}^{(2)},3) + \frac{5\sqrt{3}}{\pi}L(\chi_{-3},2),\\
f_4(20736) &= \frac{80\sqrt{10}}{\pi^3}L(g_{40},3) + \frac{32\sqrt{2}}{5\pi}L(\chi_{-8},2),\\
f_4(614656) &= \frac{800\sqrt{2}}{3\pi^3}L(f,3) + \frac{10\sqrt{3}}{\pi}L(\chi_{-3},2),
\end{aligned}$$

where

$$\begin{aligned}
f(\tau) &= \eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2,\\
g(\tau) &= \eta(2\tau)^2\eta(6\tau)^3,\\
h(\tau) &= \eta(4\tau)^6,\\
g_{48}(\tau) &= \frac{\eta(4\tau)^9\eta(12\tau)^9}{\eta(2\tau)^3\eta(6\tau)^3\eta(8\tau)^3\eta(24\tau)^3}.
\end{aligned}$$

In addition, $g_{24}^{(1)}$, $g_{24}^{(2)}$, and g_{40} are twists of CM modular forms of weight 3 of respective levels 24, 24 and 40 given in Schütt's table [Sc09].

Samart's method consists in expressing the different $f_i(s_i(q))$, $2 \le i \le 4$ as Eisenstein-Kronecker series as Bertin did in [Be08] and finding the τ satisfying $s_i(q(\tau)) = k$ for convenient k. As a corollary, Samart extended Rogers results to remarkable new expressions such as

$$ {}_5F_4\left({\frac{5}{4},\frac{3}{2},\frac{7}{4},1,1 \atop 2,2,2,2}; \frac{1}{9}\right) = 768\log(2) + 192\log(3) - 640\left(L'(g_{24}^{(2)},0) + L'(\chi_{-3},-1)\right). $$

Using Rogers' formula,

$$ g_1(10) = \frac{3}{20} f_4(614656) + \frac{1}{20} f_4(256), $$

and the expressions of $f_4(614656)$ and $f_4(256)$ found or easily deduced from Samart's work, one can recover the expression of $g_1(10)$ given by Bertin [Be10].

In both works of Rogers and Samart the connection with the L-series of the geometric object, i.e. the singular $K3$-surface is not done. It would be interesting to compute the determinant of the corresponding transcendental lattice.

Acknowledgement

The authors wish to thank Mathew Rogers for several comments and corrections and the referee for several suggestions that improved the clarity of these notes.

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