

On the Mahler measure of resultants in small dimensions

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Mahler measure of multivariate polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

$$\mathbb{T}^n = S^1 \times \cdots \times S^1$$

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Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \pmod{3} \\ -1 & n \equiv -1 \pmod{3} \\ 0 & n \equiv 0 \pmod{3} \end{cases}$$

Mixed sparse resultant

$$\mathcal{A}_0, \dots, \mathcal{A}_n \subset \mathbb{Z}^n, \mathcal{A}_i := \{\mathbf{a}_{ij}\}_{j=1, \dots, k_i}.$$

$$F_i(t_1, \dots, t_n) := \sum_{j=1}^{k_i} x_{ij} \mathbf{t}^{\mathbf{a}_{ij}} = 0 \quad i = 0, \dots, n \quad (1)$$

$$\mathbf{t}^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \dots t_n^{a_n} \text{ for } \mathbf{a} = (a_1, \dots, a_n).$$

$\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n} \in \mathbb{Z}[x_{ij}]$ irreducible polynomial.

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$\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n} \in \mathbb{Z}[x_{ij}]$ irreducible polynomial.

It vanishes on a specialization of the x_{ij} in K (alg. closed) if the system (1) has a common solution in $(K \setminus \{0\})^n$.

Examples

- ▶ $\mathcal{A}_0 = \{0, \dots, d_0\}$, $\mathcal{A}_1 = \{0, \dots, d_1\} \subset \mathbb{Z}$

$\text{Res}_{\mathcal{A}_0, \mathcal{A}_1}$ = Sylvester resultant of two polynomials of degree d_0 and d_1 .

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- ▶ $\mathcal{A}_0 = \mathcal{A}_1 = \dots = \mathcal{A}_n =$
 $\{(0, \dots, 0), (1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$

$$x_{i0} + x_{i1}t_1 + \dots + x_{in}t_n = 0 \quad i = 0, \dots, n$$

$\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n} = \det(x_{ij})$.

Some previous work

- ▶ Sombra (2004) bounds $h(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})$, $m(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})$ in terms of mixed volumes.

$$H(\text{Res}(f_{(m)}, g_{(n)})) \leq (m+1)^n (n+1)^m$$

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- ▶ D'Andrea and Hare (2004) computed $H(\text{Res}(f, g))$ when $\deg(f) = 2$ and found a tight estimate when $\deg(f) = 3$.

$$H(\text{Res}(f_0 + f_1x + f_2x^2, g_{(n)})) \sim \frac{2.3644}{\sqrt{n\pi}} 1.6180^n - O\left(\frac{1.6180^n}{n\sqrt{n}}\right)$$

$$H(\text{Res}(f_0 + f_1x + f_2x^2 + f_3x^3, g_{(n)})) \sim \frac{8.13488}{n\pi} 1.83928^n - O\left(\frac{1.83928^n}{n^2}\right)$$

$m(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})$ when $\dim(N(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}))$ is low.

Theorem

(Sturmfelds (1994))

$$\dim(N(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})) = k - 2n - 1,$$

where $k = \sum_{i=0}^n k_i$.

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Theorem

$$m(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}) = 0 \iff \dim(N(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})) = 1.$$

$$x_{j1}t_j^{\eta_j} - x_{j2} \quad j = 0, \dots, n$$

$$\dim(N(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})) = 2$$

$$k_0 = 3, k_1 = \dots = k_n = 2.$$

$$\begin{aligned} F_0(t_1, \dots, t_n) &= x_{01} \mathbf{t}^{a_{01}} + x_{02} \mathbf{t}^{a_{02}} + x_{03} \mathbf{t}^{a_{03}} \\ F_1(t_1, \dots, t_n) &= x_{11} t_1^{\eta_1} - x_{12}, \\ &\dots \quad \dots \quad \dots \\ F_n(t_1, \dots, t_n) &= x_{n1} t_n^{\eta_n} - x_{n2}. \end{aligned}$$

Theorem

$$m(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}) = \eta L'(\chi_{-3}, -1),$$

where $\eta := \eta_1 + \eta_2 + \dots + \eta_n$.

$$\begin{aligned}
 F_0 &= x_{01} \mathbf{t}^{a_{01}} + x_{02} \mathbf{t}^{a_{02}} + \dots + x_{0\ell} \mathbf{t}^{a_{0\ell}}, \\
 F_1 &= x_{11} t_1^{\eta_1} - x_{12}, \\
 \dots &\quad \dots \quad \dots \\
 F_n &= x_{n1} t_n^{\eta_n} - x_{n2}.
 \end{aligned}$$

Theorem

$$m(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}) = \eta m(1 + s_1 + s_2 + \dots + s_{\ell-1}).$$

$$\dim(N(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n})) = 3$$

1. $k_0 = 4, k_1 = k_2 = \dots = k_n = 2$. As before,

$$m(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}) = \eta m(1 + s_1 + s_2 + s_3) = \eta \frac{7}{2\pi^2} \zeta(3).$$

2. $k_0 = k_1 = 3, k_2 = k_3 = \dots = k_n = 2$.

$$\begin{aligned}
F_0 &= x_{01} \mathbf{t}^{a_{01}} + x_{02} \mathbf{t}^{a_{02}} + x_{03} \mathbf{t}^{a_{03}}, \\
F_1 &= x_{11} \mathbf{t}^{a_{11}} + x_{12} \mathbf{t}^{a_{12}} + x_{13} \mathbf{t}^{a_{13}}, \\
F_2 &= x_{21} t_2^{\eta_2} - x_{22}, \\
\cdots & \quad \cdots \quad \cdots \\
F_n &= x_{n1} t_n^{\eta_n} - x_{n2}.
\end{aligned}$$

Can be reduced to

Theorem

$$m(\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n}) = \eta m(\text{Res}_{\mathcal{A}'_0, \mathcal{A}'_1})$$

where

$$\mathcal{A}'_0 := \{\alpha_{01}, \alpha_{02}, \alpha_{03}\}, \mathcal{A}'_1 := \{\alpha_{11}, \alpha_{12}, \alpha_{13}\}$$

and $\alpha_{ij} \in \mathbb{Z}$ is the first coordinate of the vector a_{ij} , $i = 0, 1$, $j = 1, 2, 3$.

Theorem

Let $\mathcal{A}'_0 = \mathcal{A}'_1 = \{0, p, q\}$, with $p < q$ and $\gcd(p, q) = 1$.

Then,

$$m(\text{Res}_{\mathcal{A}'_0, \mathcal{A}'_1}) = \frac{2}{\pi^2} (-p\mathcal{L}_3(\varphi^q) - q\mathcal{L}_3(-\varphi^p) + p\mathcal{L}_3(\phi^q) + q\mathcal{L}_3(\phi^p))$$

φ root of $x^q + x^{q-p} - 1 = 0$ s.t. $0 \leq \varphi \leq 1$,

ϕ root of $x^q - x^{q-p} - 1 = 0$ s.t. $1 \leq \phi$.

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$$\mathcal{L}_3(z) = \text{Re} \left(\text{Li}_3(z) - \log |z| \text{Li}_2(z) + \frac{1}{3} \log^2 |z| \text{Li}_1(z) \right).$$

$$\text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k}$$

$$m(\text{Res}_{\{\{0,1,2\},\{0,1,2\}\}}) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\pi^2\zeta(3)}$$

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Use

$$z - \frac{(1-x)^p(1-y)^{q-p}}{(1-xy)^q}$$

An example in dimension 4

$$\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A} := \{(0, 0), (1, 0), (0, 1)\}.$$

(3x3 determinant)

Theorem

$$m(\text{Res}_{\mathcal{A}, \mathcal{A}, \mathcal{A}}) = \frac{9\zeta(3)}{2\pi^2}$$

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Use

$$(x - 1)(y - 1) - (z - 1)(w - 1)$$

Why should we expect such values?

$$\mathcal{X} := \{\text{Res}_{\mathcal{A}_0, \dots, \mathcal{A}_n} = 0\} \subset \mathbb{C}^k.$$

Theorem

The symbol

$$\{x_{01}, \dots, x_{0k_0}, \dots, x_{n1}, \dots, x_{nk_n}\} \in K_k^M(\mathbb{C}(\mathcal{X}))_{\mathbb{Q}}$$

is *trivial*.