The Riemann Hypothesis

Matilde N. Lalín

GAME Seminar, Special Series, History of Mathematics
University of Alberta
mlalin@math.ualberta.ca
http://www.math.ualberta.ca/~mlalin

March 5, 2008

Introduction

Fundamental Theorem of Arithmetic $\Rightarrow$ prime numbers bricks of $\mathbb{Z}$

Theorem (Euclid, 300BC)

There are ∞ many primes.

Assume finitely many

$$p_1, \dots, p_n.$$

$$N = p_1 \dots p_n + 1$$

$\exists$ new prime dividing N

Introduction

Fundamental Theorem of Arithmetic $\Rightarrow$ prime numbers bricks of $\mathbb{Z}$

Theorem (Euclid, 300BC)

There are ∞ many primes.

Assume finitely many

$$p_1, \dots, p_n.$$

$$N = p_1 \dots p_n + 1$$

$\exists$ new prime dividing N

Mengoli 1644: (“the Basel problem”)
Find

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Euler 1735:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

infinite product

$$\frac{\sin x}{x} = \prod \left(1 - \frac{x^2}{\pi^2 n^2}\right).$$

Compare coefficients of x^2.

Euler and the zeta function

Mengoli 1644: (“the Basel problem”)
Find

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Euler 1735:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

infinite product

$$\frac{\sin x}{x} = \prod \left(1 - \frac{x^2}{\pi^2 n^2}\right).$$

Compare coefficients of x^2.

Theorem (Euler (1737), Euler's product)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad s \in \mathbb{R}_{>1}$$

(True for $\mathrm{Re}(s) > 1$)

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{1}{1 - p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots,$$

$$\prod_{p \text{ prime} \leq q} \left(1 - \frac{1}{p^s}\right)^{-1} = \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \dots\right) \dots \left(1 + \frac{1}{q^s} + \frac{1}{q^{2s}} + \dots\right)$$

$$= \sum_{n \text{ prime divisors} \leq q} \frac{1}{n^s}.$$

Theorem (Euler (1737), Euler's product)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad s \in \mathbb{R}_{>1}$$

(True for $\mathrm{Re}(s) > 1$)

$$\left(1 - \frac{1}{p^s}\right)^{-1} = \frac{1}{1 - p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots,$$

$$\prod_{p \text{ prime} \leq q} \left(1 - \frac{1}{p^s}\right)^{-1} = \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \dots\right) \dots \left(1 + \frac{1}{q^s} + \frac{1}{q^{2s}} + \dots\right)$$

$$= \sum_{n \text{ prime divisors} \leq q} \frac{1}{n^s}.$$

Another proof for ∞ primes!

$$\ln \sum_{n=1} \frac{1}{n} = \sum_{p \text{ prime}} -\ln(1-p^{-1}) = \sum_{p \text{ prime}} \left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \cdots\right)$$

$$= \left(\sum_{p \text{ prime}} \frac{1}{p}\right) + \sum_{p \text{ prime}} \frac{1}{p^2}\left(\frac{1}{2} + \frac{1}{3p} + \frac{1}{4p^2} + \cdots\right)$$

$$< \left(\sum_{p \text{ prime}} \frac{1}{p}\right) + \sum_{p \text{ prime}} \frac{1}{p^2}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$$

$$= \left(\sum_{p \text{ prime}} \frac{1}{p}\right) + \left(\sum_{p \text{ prime}} \frac{1}{p(p-1)}\right).$$

Euler concluded that $\sum_{p \text{ prime}} \frac{1}{p} = \ln \ln \infty$.

The Prime Number Theorem

$$\sum_{p<x} \frac{1}{p} \sim \ln(\ln x) = \int_1^{\ln x} \frac{\mathrm{d}t}{t} = \int_e^x \frac{\mathrm{d}s}{s \ln s}.$$

p should have density $\frac{1}{\ln s}$

$$\pi(x) = |\{p \leq x \,|\, p \,\text{prime}\}|$$

Gauss 1792:

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\ln t}$$

Conjectured

Theorem (Prime Number Theorem)

$$\pi(x) \sim Li(x)$$

Legendre 1796:

$$\pi(x) \sim \frac{x}{\ln x - 1.08366\ldots}.$$

Another version of Prime Number Theorem

$$\pi(x) \sim \frac{x}{\ln x}.$$

$$\mathrm{Li}(x) = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} + \cdots + \frac{N!x}{(\ln x)^N} + O\left(\frac{x}{(\ln x)^N}\right),$$

x	$\pi(x)$	$\pi(x) - \frac{x}{\ln x}$	$\mathrm{Li}(x) - \pi(x)$	$\frac{x}{\pi(x)}$
10	4	0.3	2.2	2.500
10^2	25	3.3	5.1	4.000
10^3	168	23	10	5.952
10^4	1,229	143	17	8.137
10^5	9,592	906	38	10.425
10^6	78,498	6,116	130	12.740
10^7	664,579	44,158	339	15.047
10^8	5,761,455	332,774	754	17.357
10^9	50,847,534	2,592,592	1,701	19.667
10^{10}	455,052,511	20,758,029	3,104	21.975
10^{15}	29,844,570,422,669	891,604,962,452	1,052,619	33.507
10^{20}	2,220,819,602,560,918,840	49,347,193,044,659,701	222,744,644	45.028

$$\ln 10 = 2.3\ldots$$

Chebyschev 1848, 1850:

- Study $\pi(x)$ by analytic methods, in connection to $\zeta(s)$.
- if $\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln(x)}}$ exists, it is 1.
- $$0.89 < \frac{\pi(x)}{\frac{x}{\ln(x)}} < 1.11$$
- $$\theta(x) = \sum_{p\leq x} \ln p, \qquad \psi(x) = \sum_{p^n \leq x} \ln p.$$

  Prime number theorem is equivalent to

  $$\theta(x) \sim x, \qquad \psi(x) \sim x.$$
- Bertrand's postulate 1845: $\exists$ p prime $n < p < 2n$ for any integer $n \geq 2$.

The work of Riemann

Riemann 1859: "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"

$\zeta(s)$ as a complex function!!!
Gamma function:

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \mathrm{d}x$$

meromorphic, simple poles at $s = -1, -2, \dots,$

$$\Gamma(n+1) = n! \qquad n \in \mathbb{Z}_{>0}$$

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty x^{s-1} e^{-nx} \mathrm{d}x,$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-x}}{1 - e^{-x}} x^{s-1} \mathrm{d}x \quad \mathrm{Re}(s) > 1.$$

extension to $\mathbb{C}$ with a pole at $s = 1$.

Theorem (functional identity)

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}n^s} = \int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}\mathrm{d}x,$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n=1}^\infty e^{-n^2\pi x}\right)\mathrm{d}x$$

$$= \int_1^\infty x^{\frac{s}{2}-1}\omega(x)\mathrm{d}x + \int_1^\infty x^{-\frac{s}{2}-1}\omega\left(\frac{1}{x}\right)\mathrm{d}x$$

Where $\omega(x) = \sum_{n=1}^\infty e^{-n^2\pi x}$ satisfies a functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}-1} + x^{-\frac{s+1}{2}})\omega(x)\mathrm{d}x.$$

Theorem (functional identity)

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{\frac{s}{2}}n^s}=\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}\mathrm{d}x,$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n=1}^\infty e^{-n^2\pi x}\right)\mathrm{d}x$$

$$=\int_1^\infty x^{\frac{s}{2}-1}\omega(x)\mathrm{d}x+\int_1^\infty x^{-\frac{s}{2}-1}\omega\left(\frac{1}{x}\right)\mathrm{d}x$$

Where $\omega(x)=\sum_{n=1}^\infty e^{-n^2\pi x}$ satisfies a functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty(x^{\frac{s}{2}-1}+x^{-\frac{s+1}{2}})\omega(x)\mathrm{d}x.$$

- Euler product no zeros with $\mathrm{Re} > 1$.
- Functional equation no zeros with $\mathrm{Re} < 0$ except trivial zeros: $s = -2, -4, \ldots$.
- All complex zeros are in $0 \leq \mathrm{Re}(s) \leq 1$.

Theorem (conj by Riemann, proved by von Mangold 1905)

Let $N(t) = |\{\sigma + it \,|\, 0 < \sigma < 1, 0 < t < T, \zeta(\sigma + it) = 0\}|$. *Then.*

$$N(T) = \frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi} + O(\ln T).$$

$$\ln \zeta(s) = \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}} = \int_0^\infty x^{-s} \mathrm{d}\Pi(x) = s \int_0^\infty \Pi(x) x^{-s-1} \mathrm{d}x,$$

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi\left(x^{\frac{1}{2}}\right) + \frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) + \cdots$$

Differentiating,

$$\frac{\zeta'(s)}{\zeta(s)} = -\int_0^\infty x^{-s} \mathrm{d}\psi(x) = -s \int_0^\infty \psi(x) x^{-s-1} \mathrm{d}x.$$

After Mellin transform

Theorem (von Mangoldt 1905)

$$\psi(x) = x - \sum_{\rho,\, \zeta(\rho)=0,\, \mathrm{Im}(\rho)\neq 0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln\left(1 - x^{-2}\right)$$

$$\psi(x) = x - \sum_{\rho,\, \zeta(\rho)=0,\, \mathrm{Im}(\rho)\neq 0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \ln\left(1 - x^{-2}\right)$$

- $\frac{\zeta'(0)}{\zeta(0)} = \ln(2\pi)$ corresponds to the pole in $s = 0$
- $-\frac{1}{2} \ln\left(1 - x^{-2}\right) = \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n}$ corresponds to the trivial zeros

The sum can not be absolutely convergent (left side is not continuous).
$\Rightarrow$ Infinitely many ρ.

$$|x^\rho| = x^{\mathrm{Re}\,\rho}$$

$$\mathrm{Re}\,\rho < 1 \Rightarrow \psi \sim x$$

(we had $\mathrm{Re}\,\rho \leq 1$)

Riemann Hypothesis

The nontrivial zeros of $\zeta(s)$ have real part $\frac{1}{2}$

After Riemann

- Hadamard and de la Vallée Poussin in 1896 : Prime Number Theorem

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\ln x}}\right)$$

  for $a > 0$.

- von Koch 1901:
  Riemann hypothesis equivalent to

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\ln x).$$

After Riemann

- Hadamard and de la Vallée Poussin in 1896 : Prime Number Theorem

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\ln x}}\right)$$

  for $a > 0$.

- von Koch 1901:
  Riemann hypothesis equivalent to

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\ln x).$$

- Hilbert 1900: ICM in Paris, includes RH in a list of 23 unsolved problems.
- Early 1900s: Littlewood gets RH as thesis problem(!!!) he can not solve it but he stills does many contributions and gets his PhD.

  (Littlewood is my Greatgreatgreatgreatgrandsupervisor)

- Littlewood 1914: $\pi(n) < \mathrm{Li}(n)$ fails for infinitely many n
- Hardy and Littlewood 1922: RH implies Golbach's weak conjecture

- Hilbert 1900: ICM in Paris, includes RH in a list of 23 unsolved problems.

- Early 1900s: Littlewood gets RH as thesis problem(!!!) he can not solve it but he stills does many contributions and gets his PhD.

  (Littlewood is my Greatgreatgreatgreatgrandsupervisor)

- Littlewood 1914: $\pi(n) < \mathrm{Li}(n)$ fails for infinitely many n
- Hardy and Littlewood 1922: RH implies Golbach's weak conjecture

- Hilbert 1900: ICM in Paris, includes RH in a list of 23 unsolved problems.
- Early 1900s: Littlewood gets RH as thesis problem(!!!) he can not solve it but he stills does many contributions and gets his PhD.

  (Littlewood is my Greatgreatgreatgreatgrandsupervisor)
- Littlewood 1914: $\pi(n) < \mathrm{Li}(n)$ fails for infinitely many n
- Hardy and Littlewood 1922: RH implies Golbach's weak conjecture

- Hilbert 1900: ICM in Paris, includes RH in a list of 23 unsolved problems.
- Early 1900s: Littlewood gets RH as thesis problem(!!!) he can not solve it but he stills does many contributions and gets his PhD.

  (Littlewood is my Greatgreatgreatgreatgrandsupervisor)
- Littlewood 1914: $\pi(n) < \mathrm{Li}(n)$ fails for infinitely many n
- Hardy and Littlewood 1922: RH implies Golbach's weak conjecture

Lindelöf hypothesis

Lindelöf 1890:
For every $\epsilon > 0$,

$$\zeta\left(\frac{1}{2} + it\right) = O\left(t^{\epsilon}\right),$$

as $t \to \infty$.
(consequence of the Riemann Hypothesis).

- Hardy and Littlewood: exponent $\frac{1}{4} + \epsilon$.
- Weyl: exponent $\frac{1}{6} + \epsilon$.

The original question remains unanswered.

Lindelöf hypothesis

Lindelöf 1890:
For every $\epsilon > 0$,

$$\zeta\left(\frac{1}{2}+it\right) = O\left(t^{\epsilon}\right),$$

as $t \to \infty$.
(consequence of the Riemann Hypothesis).

- Hardy and Littlewood: exponent $\frac{1}{4}+\epsilon$.
- Weyl: exponent $\frac{1}{6}+\epsilon$.

The original question remains unanswered.

Mertens' conjecture

$$M(x) = \sum_{n \leq x} \mu(n)$$

Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \dots p_k \\ 0 & \text{if } p^2 | n \end{cases}$$

Mertens 1897:

$$|M(x)| \leq \sqrt{x}$$

Littlewood 1912: implies the Riemann Hypothesis.

- $$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right).$$
- Converges for $\mathrm{Re}(s) > 1$. If the estimate for $M(x)$ holds, then series converges for every s such that $\mathrm{Re}(s) > \frac{1}{2}$.
- Zeros for $\zeta(s)$ in $\mathrm{Re}(s) > \frac{1}{2}$ imply poles for $\frac{1}{\zeta(s)}$.

Odlyzko and te Riele in 1985: Mertens conjecture is FALSE

$$M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$$

is equivalent to RH.

Littlewood 1912: implies the Riemann Hypothesis.

- $$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right).$$
- Converges for $\mathrm{Re}(s) > 1$. If the estimate for $M(x)$ holds, then series converges for every s such that $\mathrm{Re}(s) > \frac{1}{2}$.
- Zeros for $\zeta(s)$ in $\mathrm{Re}(s) > \frac{1}{2}$ imply poles for $\frac{1}{\zeta(s)}$.

Odlyzko and te Riele in 1985: Mertens conjecture is FALSE

$$M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$$

is equivalent to RH.

Littlewood 1912: implies the Riemann Hypothesis.

- $$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right).$$
- Converges for $\mathrm{Re}(s) > 1$. If the estimate for $M(x)$ holds, then series converges for every s such that $\mathrm{Re}(s) > \frac{1}{2}$.
- Zeros for $\zeta(s)$ in $\mathrm{Re}(s) > \frac{1}{2}$ imply poles for $\frac{1}{\zeta(s)}$.

Odlyzko and te Riele in 1985: Mertens conjecture is FALSE

$$M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$$

is equivalent to RH.

"Quantity" of zeros on the critical line

- H. Bohr, Landau 1914: most of the zeros lie in a strip of width ϵ to the right of the critical line $\mathrm{Re}(s) = \frac{1}{2}$.
- Hardy 1914: infinitely many zeros in the critical line.
- Selberg 1942: a positive proportion of zeros in the critical line.
- Conrey 1989: at least 40% of the zeros lie in the critical line.

Computation of zeros

year	number	author
1903	15	Gram
1914	79	Backlund
1925	138	Hutchinson
1935	1,041	Titchmarsh
1953	1,104	Turing
1956	25,000	Lehmer
1958	35,337	Meller
1966	250,000	Lehman
1968	3,500,000	Rosser, Yohe, Schoenfeld
1977	40,000,000	Brent
1979	81,000,001	Brent
1982	200,000,001	Brent, van de Lune, te Riele, Winter
1983	300,000,001	van de Lune, te Riele
1986	1,500,000,001	van de Lune, te Riele, Winter
2001	10,000,000,000	van de Lune (unpublished)
2004	900,000,000,000	Wedeniwski
2004	10,000,000,000,000	Gourdon and Demichel

$\frac{1}{2} + i14.134\ldots, \frac{1}{2} + i21.022\ldots, \frac{1}{2} + i25.011\ldots, \frac{1}{2} + i30.425\ldots$

Random Matrix Theory

- Hilbert and Pólya: search for a Hermitian operator whose eigenvalues were the nontrivial zeros of $\zeta\left(\frac{1}{2}+ti\right)$. (Eigenvalues are real in Hermitian operators).
- Montgomery 1971: distribution of the gaps between zeros of the Riemann zeta function.
  likelihood of a gap of length x is proportional to $1-\frac{\sin^2(\pi x)}{(\pi x)^2}$,
  Dyson: the gaps of eigenvalues of certain random Hermitian matrices (the Gaussian Unitary Ensemble) follow the same path.
  GUE conjecture: all the statistics for the zeros matches the eigenvalues of GUE.
  numerical evidence found by Odlyzko in the 80s

A bigger picture

RH is not an isolated question.
L-functions $\rightarrow$ algebraic-arithmetic objects (variety, number field).
Expected to satisfy:

- Look like Dirichlet series
- Euler product
- Functional equation
- RH
- Zeros and Poles codify information about the algebraic-arithmetic object

$$L(s,\chi_{-3}) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + \dots$$

$$L(s,\chi_{-3}) = \prod_{p\equiv 1 \pmod 3} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p\equiv 2 \pmod 3} \left(1 + \frac{1}{p^s}\right)^{-1},$$

$$\left(\frac{\pi}{3}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi_{-3}) = \left(\frac{\pi}{3}\right)^{-\frac{1-s}{2}} \Gamma\left(\frac{2-s}{2}\right) L(1-s,\chi_{-3}).$$

RH (Generalized Riemann Hypothesis) not proved.

Artin, others: Zeta functions of algebraic varieties over finite fields.
curve C

$$\zeta(s, C) = \sum_{\mathcal{A}>0} \frac{1}{N\mathcal{A}^s},$$

where the $\mathcal{A}$ are the effective divisors that are fixed by the action of the Frobenius automorphism, and the norm is given by $N\mathcal{A} = q^{\deg(\mathcal{A})}$.

Euler product, functional equation, RH (Weil conjectures, by Deligne)

Consequences

- RH: growth rate of the Möbius function (via Mertens function), growth rate of other multiplicative functions like the sum of divisors, statements about Farey sequences, Landau's function (order of elements in symmetric group), Golbach's weak conjecture, etc
- GRH: distribution of prime numbers in arithmetic progressions, existence of small primitive roots modulo p, the Miller-Rabin primality test runs in polynomial time, the Shanks-Tonelli algorithm (for finding roots to quadratic equations in modular arithmetic) runs in polynomial time, etc

In 2000 the Riemann hypothesis was included in the list of the seven Millennium Prize Problems by the Clay Mathematics Institute.

Enrico Bombieri, The Riemann hypothesis. *The millennium prize problems,* 107–124, *Clay Math. Inst.*, Cambridge, MA, 2006.

Borwein, P.; Choi, S.; Rooney, B.; Weirathmueller, A., The Riemann Hypothesis A Resource for the Afficionado and Virtuoso Alike CMS Books in Mathematics 2008, XIV, 538 pp.

J. Brian Conrey, The Riemann hypothesis. *Notices Amer. Math. Soc.* 50 (2003), no. 3, 341–353.

John Derbyshire, Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics. Joseph Henry Press; 448 pages

H. M. Edwards, Riemann's zeta function. Pure and Applied Mathematics, Vol. 58. *Academic Press, New York-London,* 1974. xiii+315 pp.