There's something about Mahler measure

Junior Number Theory Seminar – University of Texas at Austin

March 29th, 2005

Matilde N. Lalín

Mahler measure

Definition 1 *For $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic)* Mahler measure *is defined by*

$$m(P) \quad = \quad \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}. \tag{1}$$

This integral is not singular and $m(P)$ always exists.

Because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|, \tag{2}$$

[1]we have a simple expression for the Mahler measure of one-variable polynomials:

$$m(P) = \log |a_d| + \sum_{n=1}^{d} \log^+ |\alpha_n| \quad \text{for} \quad P(x) = a_d \prod_{n=1}^{d} (x - \alpha_n).$$

Examples of Mahler measures in several variables

It is in general very hard to find formulas for Mahler measure of several-variable polynomials. For more than three variables, very little is known.

Theorem 2 *[9] For $n \geq 1$ we have:*

$$\pi^{2n} m \left(1 + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_{2n}}{1 + x_{2n}}\right) z\right)$$
$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} \pi^{2n-2h} (2h)! \frac{2^{2h+1} - 1}{2} \zeta(2h+1) \tag{3}$$

For $n \geq 0$:

$$\pi^{2n+1} m \left(1 + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_{2n+1}}{1 + x_{2n+1}}\right) z\right)$$
$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!} 2^{2h+1} \pi^{2n-2h} (2h+1)! \mathrm{L}(\chi_{-4}, 2h+2) \tag{4}$$

There are analogous (but more complicated) formulas for

$$m \left(1 + x + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_n}{1 + x_n}\right) (1 + y) z\right)$$

$$m \left(1 + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_n}{1 + x_n}\right) x + \left(1 - \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_n}{1 + x_n}\right)\right) y\right)$$

[1] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

Where

$$s_l(a_1,\dots,a_k) = \begin{cases} 1 & \text{if} \quad l=0 \\ \sum_{i_1<\dots<i_l} a_{i_1}\dots a_{i_l} & \text{if} \quad 0<l\le k \\ 0 & \text{if} \quad k<l \end{cases} \tag{5}$$

are the elementary symmetric polynomials, i. e.,

$$\prod_{i=1}^{k}(x+a_i) = \sum_{l=0}^{k} s_l(a_1,\dots,a_k)x^{k-l} \tag{6}$$

For example,

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right) = 24\mathrm{L}(\chi_{-4},4)+\pi^2\mathrm{L}(\chi_{-4},2) \tag{7}$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right) = 62\zeta(5)+\frac{14\pi^2}{3}\zeta(3) \tag{8}$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = 93\zeta(5) \tag{9}$$

$$\tag{10}$$

Polylogarithms

Many examples should be understood in the context of polylogarithms.

Definition 3 *The* kth polylogarithm *is the function defined by the power series*

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty}\frac{x^n}{n^k} \qquad x\in\mathbb{C}, \quad |x|<1. \tag{11}$$

This function can be continued analytically to $\mathbb{C}\setminus[1,\infty)$.

In order to avoid discontinuities, and to extend polylogarithms to the whole complex plane, several modifications have been proposed. Zagier [15] considers the following version:

$$P_k(x) := \mathrm{Re}_k\left(\sum_{j=0}^{k}\frac{2^jB_j}{j!}(\log|x|)^j\mathrm{Li}_{k-j}(x)\right), \tag{12}$$

where B_j is the jth Bernoulli number, $\mathrm{Li}_0(x)\equiv-\frac{1}{2}$ and Re_k denotes Re or Im depending on whether k is odd or even.

This function is one-valued, real analytic in $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ and continuous in $\mathbb{P}^1(\mathbb{C})$. Moreover, P_k satisfy very clean functional equations. The simplest ones are

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1}P_k(x) \qquad P_k(\bar{x}) = (-1)^{k-1}P_k(x).$$

There are also lots of functional equations which depend on the index k. For instance, for $k=2$, we have the Bloch–Wigner dilogarithm,

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

which satisfies the well-known five-term relation

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0. \tag{13}$$

Beilinson's conjectures

One of the main problems in Number Theory is finding rational (or integral) solutions of polynomial equations with rational coefficients (global solutions). In spite of the failure of the local-global principle in general, there are several theorems and conjectures which predict that one may obtain global information from local information and that that relation is made through values of L-functions. These statements include the Dirichlet class number formula, the Birch–Swinnerton-Dyer conjecture, and more generally, Bloch's and Beilinson's conjectures.

Typically, there are four elements involved in this setting: an arithmetic-geometric object X (typically, an algebraic variety), its L-function (which codify local information), a finitely generated abelian group $K \cong H^i_{\mathcal{M}}(X, \mathbb{Q}(j))$, and a regulator map

$$r_{\mathcal{D}} : H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \to H^i_{\mathcal{D}}(X, \mathbb{R}(j))$$

Here $H^i_{\mathcal{D}}(X, \mathbb{R}(i))$ can be thought as a group of differential forms on the variety. Another function $\mathrm{reg} : K \to \mathbb{R}$, is defined and is also called regulator. Basically.

$$\mathrm{reg}(\xi) = \int_{\text{cycle}} r_{\mathcal{D}}(\xi)$$

When K has rank 1, Beilinson's conjectures predict that the $\mathrm{L}'_X(0)$ is, up to a rational number, equal to a value of the regulator reg.

For instance, for a number field F, Dirichlet class number formula states that

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \mathrm{reg}_F}{\omega_F\sqrt{|D_F|}}.$$

Here, $X = \mathcal{O}_F$ (the ring of integers), $\mathrm{L}_X = \zeta_F$, and the group is $\mathcal{O}_F^*$. Hence, when F is a real quadratic field, Dirichlet class number formula may be written as $\zeta'_F(0)$ is equal to, up to a rational number, $\log|\epsilon|$, for some $\epsilon \in \mathcal{O}_F^*$.

An algebraic integration for Mahler measure

The appearance of L-functions in Mahler measures formulas is a common phenomenon. Deninger [5] interpreted the Mahler measure as a Deligne period of a mixed motive. More specifically, in two variables, and under certain conditions, he proved that

$$m(P) = \mathrm{reg}(\xi_i),$$

where reg is the determinant of the regulator matrix, which we are evaluating in some class in an appropriate group in K-theory.

Rodriguez-Villegas [11] has worked out the details for two variables. This was further developed by Boyd and Rodriguez-Villegas [1], [2].

More specifically one has

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y), \tag{14}$$

where

$$\eta(x,y) = \log|x|\mathrm{d}\arg y - \log|y|\mathrm{d}\arg x \tag{15}$$

is a differential form that is "essentially" defined in the curve C determined by the zeros of P. This form is essentially the regulator. It is closed since

$$\mathrm{d}\eta(x,y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}\right).$$

One has a crutial property:

Theorem 4

$$\eta(x, 1-x) = \mathrm{d}D(x). \tag{16}$$

Because of the above property, there is a condition that tells us when $\eta(x,y)$ is exact, namely:

$$x\wedge y = \sum_j r_j \;\; z_j\wedge(1-z_j)$$

in $\bigwedge^2(\mathbb{C}(C)^*)\otimes\mathbb{Q}$, in other words, $\{x,y\}=0$ in $K_2(\mathbb{C}(C))\otimes\mathbb{Q}$.

Under those circunstances,

$$\eta(x,y) = \mathrm{d}\left(\sum_j r_j D(z_j)\right) = \mathrm{d}D\left(\sum_j r_j[z_j]\right).$$

We have $\gamma\subset C$ such that

$$\partial\gamma = \sum_k \epsilon_k[w_k] \quad \epsilon_k = \pm 1$$

where $w_k\in C(\mathbb{C})$, $|x(w_k)| = |y(w_k)| = 1$. Then

$$2\pi m(P) = D(\xi) \quad \text{for } \xi = \sum_k\sum_j r_j[z_j(w_k)].$$

We could summarize the whole picture as follows:

$$\ldots\to(K_3(\bar{\mathbb{Q}})\supset)K_3(\partial\gamma)\to K_2(C,\partial\gamma)\to K_2(C)\to\ldots$$

$$\partial\gamma = C\cap\mathbb{T}^2$$

There are two "nice" situations:

- $\eta(x,y)$ is exact, then $\{x,y\}\in K_3(\partial\gamma)$. In this case we have $\partial\gamma\neq\emptyset$, we use Stokes' Theorem and we finish with an element $K_3(\partial\gamma)\subset K_3(\bar{\mathbb{Q}})$, leading to dilogarithms and zeta functions (of number fields), due to theorems by Borel, Bloch, Suslim and others.

- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(C)$. In this case, we have $\eta(x, y)$ is not exact and we get essentially the L-series of a curve, leading to examples of Beilinson's conjectures.

In general, we may get combinations of both situations.

The three-variable case

We are going to extend this situation to three variables. We will take

$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y|\,\mathrm{d}\log|z| - \mathrm{d}\arg y\,\mathrm{d}\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z|\,\mathrm{d}\log|x| - \mathrm{d}\arg z\,\mathrm{d}\arg x\right)+\log|z|\left(\frac{1}{3}\mathrm{d}\log|x|\,\mathrm{d}\log|y| - \mathrm{d}\arg x\,\mathrm{d}\arg y\right)$$

Then η verifies

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}\wedge\frac{\mathrm{d}z}{z}\right),$$

so it is closed.

We can express the Mahler measure of P

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z).$$

Where

$$\Gamma = \{P(x,y,z)=0\}\cap\{|x|=|y|=1, |z|\geq 1\}.$$

We are integrating on a subset of $S = \{P(x,y,z)=0\}$. The differential form is defined in this surface minus the set of zeros and poles of x, y and z, but that will not interfere our purposes, since we will be dealing with the cases when $\eta(x,y,z)$ is exact and that implies trivial tame symbols thus the element in the cohomology can be extended to S.

As in the two-variable case, we would like to apply Stokes' Theorem.

Let us take a look at Smyth's case, we can express the polynomial as $P(x,y,z) = (1-x)+(1-y)z$. We get:

$$m(P) = m(1-y) + \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathbb{T}^2}\log^+\left|\frac{1-x}{1-y}\right|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = -\frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z).$$

In general, we have

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x,y),$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y + \frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|).$$

Suppose we have

$$x\wedge y\wedge z = \sum r_i\ \ x_i\wedge(1-x_i)\wedge y_i$$

in $\bigwedge^3(\mathbb{C}(S)^*)\otimes\mathbb{Q}$.

Then

$$\int_\Gamma \eta(x,y,z) = \sum r_i\int_\Gamma \eta(x_i, 1-x_i, y_i) = \sum r_i \int_{\partial\Gamma}\omega(x_i,y_i).$$

In Smyth's case, this corresponds to

$$x \wedge y \wedge z = - \ x \wedge (1-x) \wedge y - \ y \wedge (1-y) \wedge x,$$

in other words,

$$\eta(x,y,z) = -\eta(x, 1-x, y) - \eta(y, 1-y, x).$$

Back to the general picture, $\partial\Gamma = \{P(x,y,z) = 0\} \cap \{|x| = |y| = |z| = 1\}$. When $P \in \mathbb{Q}[x,y,z]$, Γ can be thought as

$$\gamma = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}.$$

Note that we are integrating now on a path inside the curve $C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$. The differential form ω is defined in this new curve (this way of thinking the integral over a new curve has been proposed by Maillot). Now it makes sense to try to apply Stokes' Theorem again. We have

$$\omega(x,x) = \mathrm{d}P_3(x).$$

Suppose we have

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.

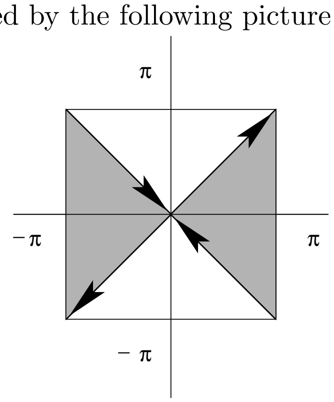
Then, as before:

$$\int_\gamma \omega(x,y) = \sum r_i \ P_3(x_i)|_{\partial\gamma} \,.$$

Back to Smyth's case, in order to compute C we set $\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$ and we get $C = \{x = y\} \cup \{xy = 1\}$ in this example, and

$$-[x]_2 \otimes y - [y]_2 \otimes x = \pm 2[x]_2 \otimes x.$$

We integrate in the set described by the following picture



Then

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y) + \omega(y,x) = \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

and the second condition is

$$[x_i]_2 \otimes y_i = 0 \quad \text{in} \quad H^2(B_{\mathbb{Q}(C)}(3) \otimes \mathbb{Q}) \overset{?}{\cong} K_4^{[1]}(\mathbb{Q}(C))_{\mathbb{Q}}$$

Hence, the conditions can be translated as certain elements in different K-theories must be zero, which is analogous to the two-variable case.

We could summarize this picture as follows. We first integrate in this picture

$$\ldots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \ldots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

As before, we have two situations. All the examples we have talked about fit into the situation when $\eta(x, y, z)$ is exact and $\partial\Gamma \neq \emptyset$. Then we finish with an element in $K_4(\partial\Gamma)$.

Then we go to

$$\ldots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Again we have two possibilities, but in our context, $\omega(x, y)$ is exact and we finish with an element in $K_5(\partial\gamma) \subset K_5(\bar{\mathbb{Q}})$ leading to trilogarithms and zeta functions, due to Zagier's conjecture and Borel's theorem.

Studied examples

- Smyth(1981):
$$\pi^2 m(1 + x + y + z) = \frac{7}{2}\zeta(3)$$
- Smyth(2002):
$$\pi^2 m(1 + x + y^{-1} + (1 + x + y)z) = \frac{14}{3}\zeta(3)$$
- L (2003):
$$\pi^2 m\left(1 + \left(\frac{1 - x_1}{1 + x_1}\right)\left(\frac{1 - x_2}{1 + x_2}\right) z\right) = 7\zeta(3)$$
$$\pi^2 m\left(1 + \left(\frac{1 - x_1}{1 + x_1}\right) x + \left(1 - \left(\frac{1 - x_1}{1 + x_1}\right)\right) y\right)$$
$$= \frac{7}{2}\zeta(3) + \frac{\pi^2 \log 2}{2}$$
- Condon (2003):
$$\pi^2 m\left(z - \left(\frac{1 - x}{1 + x}\right)(1 + y)\right) = \frac{28}{5}\zeta(3)$$
- D'Andrea & L (2003):
$$\pi^2 m\left(z(1 - xy)^{m+n} - (1 - x)^m(1 - y)^n\right)$$
$$= 2n(P_3(\phi_1^m) - P_3(-\phi_2^m)) + 2m(P_3(\phi_2^n) - P_3(-\phi_1^n))$$

$$\pi^2 m((1 - x)(1 - y) - (1 - w)(1 - z)) = \frac{9}{2}\zeta(3)$$

- L (2003):

$$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)x\right.$$
$$\left.+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right)y\right)$$
$$=\frac{21}{4}\zeta(3)+\frac{\pi^2\log 2}{2}$$

References

[1] D. W. Boyd, F. Rodriguez Villegas, Mahler's measure and the dilogarithm (I), *Canad. J. Math.* **54** (2002), 468 - 492.

[2] D. W. Boyd, F. Rodriguez Villegas, with an appendix by N. M. Dunfield, Mahler's measure and the dilogarithm (II), (preprint, July 2003)

[3] J. Condon, Calculation of the Mahler measure of a three variable polynomial, (preprint, October 2003)

[4] C. D'Andrea, M. N. Lalín, On The Mahler measure of resultants in small dimensions. (in preparation).

[5] C. Deninger, Deligne periods of mixed motives, K-theory and the entropy of certain Z^n-actions, *J. Amer. Math. Soc.* **10** (1997), no. 2, 259–281.

[6] A. B. Goncharov, Geometry of Configurations, Polylogarithms, and Motivic Cohomology, *Adv. Math.* **114** (1995), no. 2, 197–318.

[7] A. B. Goncharov, The Classical Polylogarithms, Algebraic K-theory and $\zeta_F(n)$, *The Gelfand Mathematical Seminars 1990-1992, Birkhauser*, Boston (1993), 113 - 135.

[8] A. B. Goncharov, Explicit regulator maps on polylogarithmic motivic complexes, (preprint, March 2000).

[9] M. N. Lalín, Mahler measure of some n-variable polynomial families, (September 2004), to appear in *J. Number Theory*

[10] M. N. Lalín, An algebraic integration for Mahler measure (in preparation).

[11] F. Rodriguez Villegas, Modular Mahler measures I, *Topics in number theory* (University Park, PA 1997), 17-48, Math. Appl., 467, Kluwer Acad. Publ. Dordrecht, 1999.

[12] C. J. Smyth, On measures of polynomials in several variables, *Bull. Austral. Math. Soc. Ser. A* **23** (1981), 49 - 63. Corrigendum (with G. Myerson): *Bull. Austral. Math. Soc.* **26** (1982), 317 - 319.

[13] C. J. Smyth, An explicit formula for the Mahler measure of a family of 3-variable polynomials, *J. Th. Nombres Bordeaux* **14** (2002), 683–700.

[14] D. Zagier, The Bloch-Wigner-Ramakrishnan polylogarithm function. *Math. Ann.* **286** (1990), no. 1-3, 613–624.

[15] D. Zagier, Polylogarithms, Dedekind Zeta functions, and the Algebraic K-theory of Fields, *Arithmetic algebraic geometry* (Texel, 1989), 391–430, Progr. Math., 89, Birkhuser Boston, Boston, MA, 1991.