Mahler Measure and Hyperbolic Volumes

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Mahler Measure

For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is:

$$m(P) = \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$
$$= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Bloch − Wigner Dilogarithm

Dilogarithm:

$$\mathsf{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad z \in \mathbb{C}, \quad |z| < 1$$

$$\mathsf{Li}_2(z) := -\int_0^z \log(1-t)\frac{\mathrm{d}t}{t} \qquad z \in \mathbb{C} \setminus (1,\infty)$$

Bloch − Wigner Dilogarithm:

$$D(z) := \mathrm{Im}(\mathsf{Li}_2(z)) + \arg(1-z)\log|z|$$

real analytic in $\mathbb{C} \setminus \{0,1\}$, continuous in $\mathbb{C}$

Properties:

1. $D(\bar{z}) = -D(z)$ $(\Rightarrow D|_{\mathbb{R}} \equiv 0)$

2. $D(z) = -D\left(\frac{1}{z}\right) = -D(1-z)$

3. $-2\int_0^\theta \log|2\sin t|\mathrm{d}t = D(\mathrm{e}^{2\mathrm{i}\theta})$

Hyperbolic Volumes

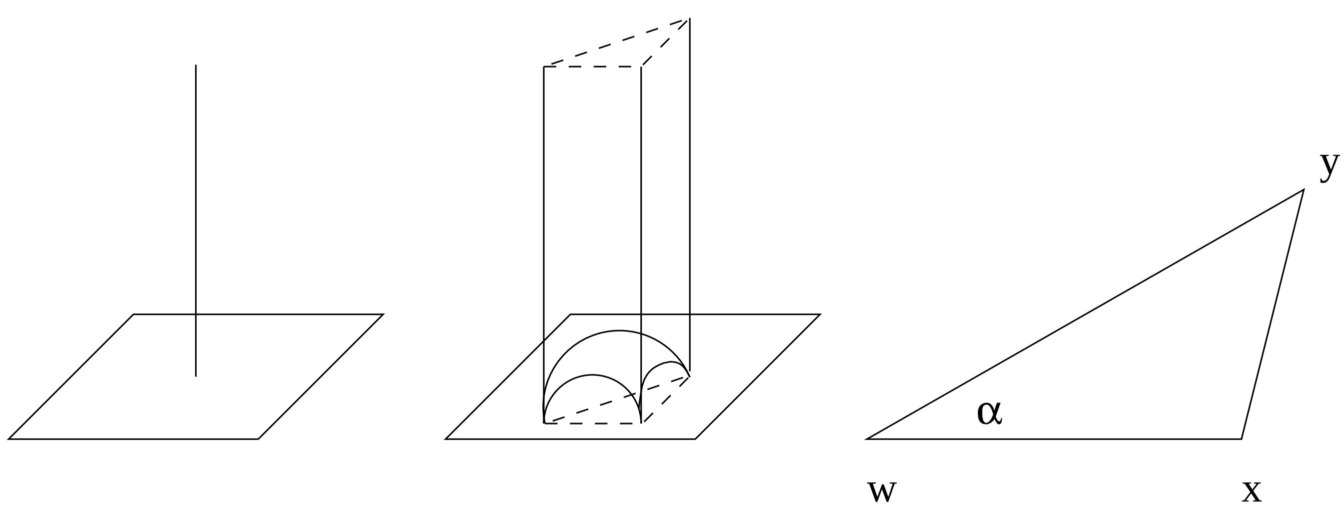
$$\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$$

Volume Element:

$$\frac{dx\,dy\,dz}{z^3}$$

Ideal Tetrahedron:

vertices in $\quad \partial\mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$

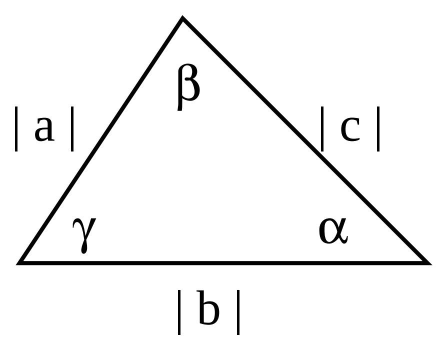


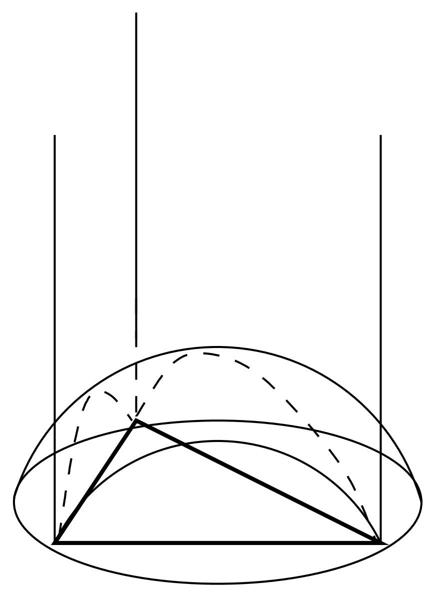
$$\mathrm{Vol}\left(\pi^*\left(\overset{\triangle}{wxy}\right)\right) = D\left(\frac{y-w}{x-w}\right) = D\left(\left|\frac{y-w}{x-w}\right| e^{i\alpha}\right)$$

Cassaigne and Maillot's Example

$$\pi m(a + bx + cy) =$$

$$\begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$





Results

1. Cassaigne and Maillot:

$$y = \frac{ax + b}{c}$$

2. Vandervelde:

$$y = \frac{bx + d}{ax + c}$$

3.

$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \ldots + 1}{t(x^{m-1} + \ldots + 1)}$$

An example

$$y = \frac{x^3 - 1}{t(x^2 - 1)}$$

$$R_t(x, y) = x^2 + x + 1 - t(x + 1)y$$

$m(R_t) - \log |t|$

$$= \begin{cases} \frac{2}{2 \cdot 3 \cdot \pi} \left(\epsilon_1 \mathrm{Vol}(\pi^*(P_1)) + \epsilon_2 \mathrm{Vol}(\pi^*(P_2))\right) & \\ + \frac{\sigma_1 - \sigma_2}{\pi} \log |t| & 0 < t < \frac{3}{2} \\ & \\ \frac{2}{2 \cdot 3 \cdot \pi} \epsilon_1 \mathrm{Vol}(\pi^*(P_1)) + \frac{\sigma_1}{\pi} \log |t| & \frac{3}{2} \le t \end{cases}$$

Computation

$$m(x^2+x+1-t(x+1)y)$$

$$=\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2}\log|x^2+x+1-t(x+1)y|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$=\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1}\log|t(x+1)|\frac{\mathrm{d}x}{x}$$

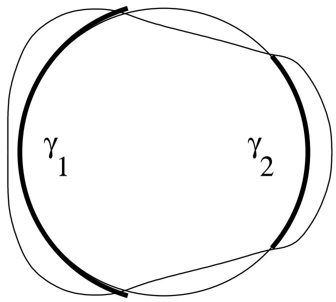
$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2}\log\left|\frac{x^2+x+1}{t(x+1)}-y\right|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$=\log t+\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1}\log^+\left|\frac{x^2+x+1}{t(x+1)}\right|\frac{\mathrm{d}x}{x}$$

Determine for which points we have

$$\left|\frac{x^2+x+1}{t(x+1)}\right|=1$$

$$\frac{x^2+x+1}{t(x+1)}\cdot\frac{x^{-2}+x^{-1}+1}{t(x^{-1}+1)}=1$$



$$x^4 + (2 - t^2)x^3 + (3 - 2t^2)x^2 + (2 - t^2)x + 1 = 0$$

Roots $\alpha_1,\ \alpha_1^{-1},\ \alpha_2,\ \alpha_2^{-1}$

$$\begin{aligned} \operatorname{Re}\alpha_1 &= \frac{t^2 - 2 - t\sqrt{t^2+4}}{4} \quad \text{for} \quad 0 < t \\ \operatorname{Re}\alpha_2 &= \frac{t^2 - 2 + t\sqrt{t^2+4}}{4} \quad \text{for} \quad 0 < t < \frac{3}{2} \end{aligned}$$

Let $\sigma_i = \arg\alpha_i$, $\operatorname{Im}\alpha_i \geq 1$

$$\pi > \sigma_1 > \frac{2\pi}{3}$$

$$\frac{2\pi}{3} > \sigma_2 > 0$$

$$\int_{\alpha}^{\beta} \log|x^n - 1|\frac{\mathrm{d}x}{\mathrm{i}x} = \frac{1}{n}\int_{\alpha^n}^{\beta^n} \log|y-1|\frac{\mathrm{d}y}{\mathrm{i}y}$$

$$= \frac{2}{n}\int_{\frac{\arg \alpha^n}{2}}^{\frac{\arg \beta^n}{2}} \log|2\sin t|\mathrm{d}t = \frac{D(\alpha^n) - D(\beta^n)}{n}$$

For $0 < t < \frac{3}{2}$

$$m(x^2 + x + 1 - t(x+1)y) - \log t$$

$$= \frac{1}{2\pi \mathrm{i}}\int_{\gamma_1 \cup \gamma_2} \log\left|\frac{x^3 - 1}{t(x^2-1)}\right|\frac{\mathrm{d}x}{x}$$

$$= \frac{D(\alpha_1^{-3}) - D(\alpha_1^3) + D(\alpha_2^3) - D(\alpha_2^{-3})}{3(2\pi)}$$

$$- \frac{D(\alpha_1^{-2}) - D(\alpha_1^2) + D(\alpha_2^2) - D(\alpha_2^{-2})}{2(2\pi)}$$

$$- \frac{2(\sigma_1 - \sigma_2)}{2\pi}\log t$$

$$= \frac{3D(\alpha_1^2) - 2D(\alpha_1^3)}{6\pi} - \frac{3D(\alpha_2^2) - 2D(\alpha_2^3)}{6\pi}$$

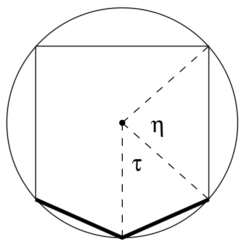
$$-\frac{\sigma_1 - \sigma_2}{\pi} \log t$$

For $\frac{3}{2} \leq t$

$$m(x^2 + x + 1 - t(x+1)y) - \log t$$

$$= \frac{3D(\alpha_1^2) - 2D(\alpha_1^3)}{6\pi} - \frac{\sigma_1}{\pi} \log t$$

For $\alpha_1 \longrightarrow \eta = 2\pi - 2\sigma_1,\ \tau = 3\sigma_1 - 2\pi$



For α_2

$0 < t < \frac{1}{\sqrt{2}}$	$\frac{2\pi}{3} > \sigma_2 > \frac{\pi}{2}$	$\eta = 2\pi - 2\sigma_2$ $\tau = 2\pi - 3\sigma_2$	$3\eta - 2\tau = 2\pi$
$\frac{1}{\sqrt{2}} < t < \frac{2}{\sqrt{3}}$	$\frac{\pi}{2} > \sigma_2 > \frac{\pi}{3}$	$\eta = 2\sigma_2$ $\tau = 2\pi - 3\sigma_2$	$3\eta + 2\tau = 4\pi$
$\frac{2}{\sqrt{3}} < t < \frac{3}{2}$	$\frac{\pi}{3} > \sigma_2 > 0$	$\eta = 2\sigma_2$ $\tau = 3\sigma_2$	$3\eta - 2\tau = 0$