Mahler Measure and Hyperbolic Volumes

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1. Introduction to Mahler measure

Definition 1 *For $P \in \mathbb{C}[x_1, \dots, x_n]$, the (logarithmic)* Mahler measure *is defined by*

$$m(P) := \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{1}$$

In particular, given a polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \dots + a_0 = a_d \prod_{n=1}^d (x - \alpha_n)$ with complex coefficients, it is true that

$$m(P) := \log |a_d| + \sum_{n=1}^d \log^+ |\alpha_n| \tag{2}$$

because of Jensen's equality $\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha| = \log \max\{1, |\alpha|\}$

It is possible to prove that this integral is not singular and that $m(P)$ always exists.

For the several-variable case, it seems that there is no simpler general formula than the integral defining the measure. However, many examples have been found relating the Mahler measure of polynomials in two variables to special values of L-functions in quadratic characters, L-functions on elliptic curves and dilogarithms.

2. Dilogarithm and volumes of ideal tetrahedra

Definition 2 *The* Dilogarithm *is the function defined by the power series*

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad z \in \mathbb{C}, \quad |z| < 1 \tag{3}$$

It has an analytic continuation to $\mathbb{C} \setminus (1, \infty)$ given by

$$\mathrm{Li}_2(z) := -\int_0^z \log(1-t) \frac{\mathrm{d}t}{t}$$

The dilogarithm jumps by $2\pi \mathrm{i} \log |z|$ as z crosses the cut by $(1, \infty)$. Hence, it is natural to consider the following function:

Definition 3 *The* Bloch – Wigner Dilogarithm *is defined by*

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z) \log |z| \tag{4}$$

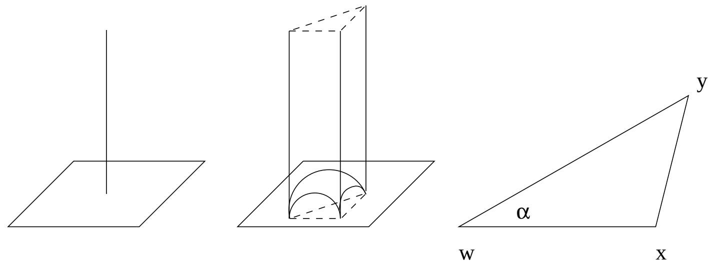

Figure 1: The volume of the hyperbolic tetrahedron over $\overset{\triangle}{wxy}$ is $D\left(\frac{y-w}{x-w}\right) = D\left(\left|\frac{y-w}{x-w}\right| \mathrm{e}^{\mathrm{i}\alpha}\right)$.

This function has several properties. It is real analytic in $\mathbb{C} \setminus \{0,1\}$ and continuous in $\mathbb{C}$. It also satisfies

$$D(\bar{z}) = -D(z) \quad (\Rightarrow D|_{\mathbb{R}} \equiv 0) \tag{5}$$

$$D(z) = -D\left(\frac{1}{z}\right) = -D(1-z) \tag{6}$$

$$-2\int_0^\theta \log|2\sin t|\mathrm{d}t = D(\mathrm{e}^{2\mathrm{i}\theta}) = \sum_{n=1}^\infty \frac{\sin(2n\theta)}{n^2} \tag{7}$$

among other important relations.

Now consider the space $\mathbb{H}^3$ which can be represented as $\mathbb{C}\times\mathbb{R}_{\geq 0}\cup\{\infty\}$. In this space the geodesics are either vertical lines or semicircles in vertical planes with endpoints in $\mathbb{C}\times\{0\}$. An *ideal tetrahedron* is a tetrahedron whose vertices are all in $\mathbb{C}\times\{0\}\cup\{\infty\} = \mathbb{P}^1(\mathbb{C})$. Such a tetrahedron is always equivalent to a tetrahedron Δ with vertices $0, 1, \infty, z$ and $\operatorname{Im} z > 0$ and has a hyperbolic volume equal to

$$\operatorname{Vol}(\Delta) = D(z) \tag{8}$$

(The volume element in this space is $\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{z^3}$).

The invariance of the formula by the action of $PSL_2(\mathbb{C})$ is in agreement with the fact that this is the group of isometries (preserving orientation) of $\mathbb{H}^3$.

These ideal tetrahedra are important because any completed oriented 3-manifold with finite volume can be decomposed in ideal tetrahedra (which may be degenerated).

3. Mahler measure and hyperbolic volumes

One of the simplest examples of Mahler measure in two variables is Cassaigne and Maillot's formula:

Theorem 4

$$\pi m(ax+by+c) = \begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{i\gamma}\right) + \alpha\log|a| + \beta\log|b| + \gamma\log|c| & \triangle \\ \pi \log\max\{|a|,|b|,|c|\} & \text{not } \triangle \end{cases} \tag{9}$$

Here $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively.

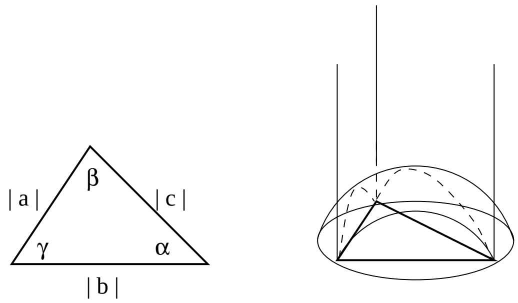

Figure 2: The main term in Cassaigne – Maillot formula is the volume of the ideal hyperbolic tetrahedron over the triangle.

In this formula, the polynomial equation can be written as

$$y = \frac{ax+b}{c}$$

and the dilogarithm term is the volume of the ideal tetrahedron that can be built over the triangle of sides $|a|$, $|b|$ and $|c|$. See figure 2.

Another example was due to Vandervelde. He studied the polynomials whose equation can be expressed as

$$y = \frac{bx+d}{ax+c}$$

When $a, b, c, d \in \mathbb{R}^*$, the Mahler measure of this polynomial is the sum of some logarithms and two dilogarithm terms, which can be interpreted as the volume of the ideal polyhedra built over a cyclic quadrilateral of sides $|a|$, $|b|$, $|c|$ and $|d|$.

We have studied the case of

$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \cdots + 1}{t(x^{m-1} + \cdots + 1)}$$

and obtained a similar result.

To be concrete, we are going to describe a particular example with $n = 3$, and $m = 2$.

$$y = \frac{x^3 - 1}{t(x^2 - 1)}, \qquad R_t(x, y) = x^2 + x + 1 - t(x+1)y \tag{10}$$

in full detail.

The result is

$$m(R_t) - \log|t| = \begin{cases} \frac{2}{2\cdot 3\cdot \pi}(\epsilon_1 \mathrm{Vol}(\pi^*(P_1)) + \epsilon_2 \mathrm{Vol}(\pi^*(P_2))) + \frac{\sigma_1 - \sigma_2}{\pi} \log|t| & 0 < t < \frac{3}{2} \\ \frac{2}{2\cdot 3\cdot \pi} \epsilon_1 \mathrm{Vol}(\pi^*(P_1)) + \frac{\sigma_1}{\pi} \log|t| & \frac{3}{2} \leq t \end{cases}$$

where $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$.

We may suppose that $t > 0$ because the Mahler measure only depends on $|t|$ in this case.

Let us compute the Mahler measure:

$$m(x^2 + x + 1 - t(x+1)y) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log|x^2 + x + 1 - t(x+1)y| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

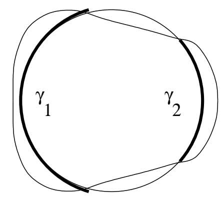

Figure 3: We integrate over the arcs γ_i where $|y| \geq 1$. The extremes of these arcs occur in points where y crosses the unit circle.

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log|t(x+1)| \frac{\mathrm{d}x}{x} + \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log\left| \frac{x^2+x+1}{t(x+1)} - y \right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$
$$= \log t + \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ \left| \frac{x^2+x+1}{t(x+1)} \right| \frac{\mathrm{d}x}{x}$$

by Jensen's formula. We need to compute the integral on the right. We will integrate in the set where $\left|\frac{x^2+x+1}{t(x+1)}\right| \geq 1$. So we need to determine for which points we have

$$\left| \frac{x^2+x+1}{t(x+1)} \right| = 1, \qquad \frac{x^2+x+1}{t(x+1)} \cdot \frac{x^{-2}+x^{-1}+1}{t(x^{-1}+1)} = 1$$

since $|x| = 1$ (recall that we have restricted to $x \in \mathbb{T}^1$).

In other words, we need to solve

$$Q(x) = x^4 + (2-t^2)x^3 + (3-2t^2)x^2 + (2-t^2)x + 1 = 0$$

In fact, it is easy to see that the roots of Q in the unit circle are at most four and they are of the form α_1, α_1^{-1}, α_2, α_2^{-1}, such that

$$\operatorname{Re}\alpha_1 = \frac{t^2-2-t\sqrt{t^2+4}}{4} \quad \text{for} \quad 0 < t \tag{11}$$
$$\operatorname{Re}\alpha_2 = \frac{t^2-2+t\sqrt{t^2+4}}{4} \quad \text{for} \quad 0 < t < \frac{3}{2} \tag{12}$$

Let $\sigma_i = \arg \alpha_i$, say that $\operatorname{Im}\alpha_i \geq 0$, we have

$$\pi > \sigma_1 > \frac{2\pi}{3} \tag{13}$$
$$\frac{2\pi}{3} > \sigma_2 > 0 \tag{14}$$

Back to the Mahler measure calculation, first note that

$$\int_\alpha^\beta \log|x^n - 1| \frac{\mathrm{d}x}{\mathrm{i}x} = \frac{1}{n} \int_{\alpha^n}^{\beta^n} \log|y-1| \frac{\mathrm{d}y}{\mathrm{i}y} = \frac{2}{n} \int_{\frac{\arg\alpha^n}{2}}^{\frac{\arg\beta^n}{2}} \log|2\sin t| \mathrm{d}t = \frac{D(\alpha^n) - D(\beta^n)}{n}$$

So, for $0 < t < \frac{3}{2}$

$$m(x^2+x+1-t(x+1)y) - \log t = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1 \cup \gamma_2} \log\left| \frac{x^3-1}{t(x^2-1)} \right| \frac{\mathrm{d}x}{x}$$

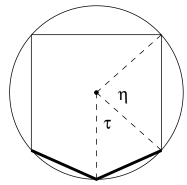

Figure 4: The case of α_1 corresponds to the ordinary convex polygon. Note that α_1 exists for any $t > 0$ and the same is true for the polygon. In every picture, the bold segments correspond to sides of length t, (opposite to angles τ) and the others are sides of length 1 (opposite to angles η).

where γ_1 is the arc from α_1 to α_1^{-1} and γ_2 is the arc from α_2^{-1} to α_2 (see picture 3)

$$= \frac{D(\alpha_1^{-3}) - D(\alpha_1^3) + D(\alpha_2^3) - D(\alpha_2^{-3})}{3(2\pi)} - \frac{D(\alpha_1^{-2}) - D(\alpha_1^2) + D(\alpha_2^2) - D(\alpha_2^{-2})}{2(2\pi)} - \frac{2(\sigma_1 - \sigma_2)}{2\pi} \log t$$

$$= \frac{3D(\alpha_1^2) - 2D(\alpha_1^3)}{6\pi} - \frac{3D(\alpha_2^2) - 2D(\alpha_2^3)}{6\pi} - \frac{\sigma_1 - \sigma_2}{\pi} \log t$$

For $\frac{3}{2} \leq t$, the integral is over the arc γ_1 alone, so

$$m(x^2 + x + 1 - t(x+1)y) - \log t = \frac{3D(\alpha_1^2) - 2D(\alpha_1^3)}{6\pi} - \frac{\sigma_1}{\pi} \log t$$

Our claim is that each of these terms with dilogarithms correspond (up to a sign) to the volume of a hyperbolic orthoscheme built over a cyclic polygon. Each polygon has 3 sides of length 1 and 2 sides of length t. The sides of length 1 are opposite to a central angle η and the sides of length t are opposite to a central angle τ. Also $0 < \eta, \tau \leq \pi$.

For the α_1 term, take $\eta = 2\pi - 2\sigma_1$ and $\tau = 3\sigma_1 - 2\pi$. Then $3\eta + 2\tau = 2\pi$. This corresponds to the convex pentagon which is inscribed in a circle (see figure 4).

The case of α_2 splits into three subcases according to the values of t, as shown in the following table.

$0 < t < \frac{1}{\sqrt{2}}$	$\frac{2\pi}{3} > \sigma_2 > \frac{\pi}{2}$	$\eta = 2\pi - 2\sigma_2$ $\tau = 2\pi - 3\sigma_2$	$3\eta - 2\tau = 2\pi$
$\frac{1}{\sqrt{2}} < t < \frac{2}{\sqrt{3}}$	$\frac{\pi}{2} > \sigma_2 > \frac{\pi}{3}$	$\eta = 2\sigma_2$ $\tau = 2\pi - 3\sigma_2$	$3\eta + 2\tau = 4\pi$
$\frac{2}{\sqrt{3}} < t < \frac{3}{2}$	$\frac{\pi}{3} > \sigma_2 > 0$	$\eta = 2\sigma_2$ $\tau = 3\sigma_2$	$3\eta - 2\tau = 0$

Figure 5 illustrates the polygons corresponding to each of these subcases.

We would like to point out that in every case, the polygon exists if and only if the corresponding term shows up in the formula.

The cases of $t = \frac{1}{\sqrt{2}}$ and $t = \frac{2}{\sqrt{3}}$ are limit cases and we get the transition figures of picture 6. Figure 6.d is the intermediate figure between 5.a and 5.b and the same is true for 6.e, which is between 5.b and 5.c.

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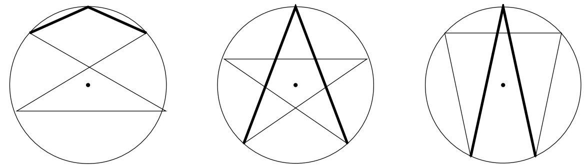

Figure 5: Case α_2: a) $0 < t < \frac{1}{\sqrt{2}}$ b) $\frac{1}{\sqrt{2}} < t < \frac{2}{\sqrt{3}}$ c) $\frac{2}{\sqrt{3}} < t < \frac{3}{2}$

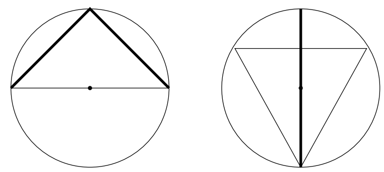

Figure 6: d) $t = \frac{1}{\sqrt{2}}$ e) $t = \frac{2}{\sqrt{3}}$

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