Higher Mahler measures

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Zetas and Limit Laws in Okinawa 2008

Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log \left| P\left(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}\right)\right| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi\mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

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By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$
- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

Higher Mahler measure

The k-higher Mahler measure of P is defined by

$$m_k(P) = \int_0^1 \dots \int_0^1 \log^k \left| P\left(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}\right)\right| d\theta_1 \cdots d\theta_n.$$

$$k = 1 : \qquad m_1(P) = m(P),$$

and

$$m_0(P) = 1.$$

The simplest example

$$m_2(1-x) = \frac{\zeta(2)}{2} = \frac{\pi^2}{12}.$$

$$m_3(1-x) = -\frac{3\zeta(3)}{2}.$$

$$m_4(1-x) = \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}.$$

$$m_5(1-x) = -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}.$$

$$m_6(1-x) = \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6.$$

$$Z(s,P)=\int_0^1\dots\int_0^1\left|P\left(\mathrm{e}^{2\pi\mathrm{i}\theta_1},\dots,\mathrm{e}^{2\pi\mathrm{i}\theta_n}\right)\right|^s \mathrm{d}\theta_1\dots\mathrm{d}\theta_n.$$

$$Z(s,P)=\sum_{k=0}^{\infty}\frac{m_k(P)s^k}{k!}.$$

Akatsuka (2007): $Z(s, x-c)$

An example

Theorem

$$Z(s, x-1) = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k(1-2^{1-k})\zeta(k)}{k} s^k\right)$$

around $s = 0$.

$$\begin{aligned} Z(s,x-1) &= \int_0^1 \left|1-e^{2\pi i\theta}\right|^s d\theta = \int_0^1 (2\sin\pi\theta)^s d\theta \\ &= 2^{s+1}\int_0^{1/2} (\sin\pi\theta)^s d\theta. \end{aligned}$$

$t = \sin^2\pi\theta$:

$$= \frac{2^s}{\pi}\int_0^1 t^{\frac{s-1}{2}}(1-t)^{-1/2}dt.$$

$$\begin{aligned} &= \frac{2^s}{\pi} B\left(\frac{s+1}{2},\frac{1}{2}\right) \\ &= \frac{2^s}{\pi}\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} = \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2} \end{aligned}$$

Weierstrass product:

$$\Gamma(s+1)^{-1} = \mathrm{e}^{\gamma s} \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right) \mathrm{e}^{-\frac{s}{n}}$$

yields

$$\begin{aligned} Z(s,x-1) &= \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2} = \prod_{n=1}^{\infty} \frac{\left(1+\frac{s}{2n}\right)^2}{1+\frac{s}{n}} \\ &= \exp\left(\sum_{n=1}^{\infty}\left(2\log\left(1+\frac{s}{2n}\right) - \log\left(1+\frac{s}{n}\right)\right)\right) \\ &= \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=1}^{\infty}\left(2\left(\frac{1}{2n}\right)^k - \frac{1}{n^k}\right)s^k\right) \\ &= \exp\left(\sum_{k=2}^{\infty}\frac{(-1)^k(1-2^{1-k})\zeta(k)}{k}s^k\right). \end{aligned}$$

$$Z(s, x-1) = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k(1-2^{1-k})\zeta(k)}{k} s^k\right)$$

$$\begin{aligned}
m_2(x-1) &= \frac{\zeta(2)}{2} = \frac{\pi^2}{12}. \\
m_3(x-1) &= -\frac{3\zeta(3)}{2}. \\
m_4(x-1) &= \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}. \\
m_5(x-1) &= -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}.
\end{aligned}$$

...

An example in two variables

Theorem

$$m_2(1+x+y) = \frac{5\pi^2}{54} = \frac{5}{9}\zeta(2)$$

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2)$$

Theorem

$$m_2(1+x+y(1-x)) = \frac{4\mathrm{i}}{\pi}(\mathrm{Li}_{2,1}(-\mathrm{i},-\mathrm{i}) - \mathrm{Li}_{2,1}(\mathrm{i},\mathrm{i}))$$
$$+\frac{6\mathrm{i}}{\pi}(\mathrm{Li}_{2,1}(\mathrm{i},-\mathrm{i}) - \mathrm{Li}_{2,1}(-\mathrm{i},\mathrm{i}))$$
$$+\frac{\mathrm{i}}{\pi}(\mathrm{Li}_{2,1}(1,-\mathrm{i}) - \mathrm{Li}_{2,1}(1,\mathrm{i})) - \frac{7\zeta(2)}{16} + \frac{\log 2}{\pi}\mathrm{L}(\chi_{-4},2)$$

Smyth (1981)

$$m(1-x+y(1+x)) = \frac{2}{\pi}\mathrm{L}(\chi_{-4},2)$$

$$\mathrm{Li}_{2,1}(x,y) = \sum_{0<m<n} \frac{x^m y^n}{m^2 n}$$

Multiple Mahler measure

Let $P_1, \dots, P_k \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$\begin{aligned} m(P_1, \dots, P_k) &= \int_0^1 \cdots \int_0^1 \log \left| P_1 \left(e^{2\pi i \theta_1}, \cdots, e^{2\pi i \theta_n} \right) \right| \\ &\quad \cdots \log \left| P_k \left(e^{2\pi i \theta_1}, \cdots, e^{2\pi i \theta_n} \right) \right| d\theta_1 \cdots d\theta_n \end{aligned}$$

$$m(P_1) \cdots m(P_k) = m(P_1, \dots, P_k)$$

when the variables of P_j's are algebraically independent.

Higher Mahler measure for several linear polynomials

Theorem

For $0 \leq \alpha \leq 1$

$$m(1-x, 1-e^{2\pi i\alpha}x) = \frac{\pi^2}{2}\left(\alpha^2 - \alpha + \frac{1}{6}\right).$$

Examples

$$\begin{aligned} m(1-x, 1+x) &= -\frac{\pi^2}{24}, \\ m(1-x, 1 \pm ix) &= -\frac{\pi^2}{96}, \\ m(1-x, 1-e^{2\pi i\alpha}x) &= 0 \Leftrightarrow \alpha = \frac{3 \pm \sqrt{3}}{6}. \end{aligned}$$

Jensen's formula for multiple Mahler measure

$$m(1-\alpha x) = \begin{cases} 0 & \text{if } |\alpha| \leq 1, \\ \log|\alpha| & \text{if } |\alpha| \geq 1. \end{cases}$$

$$m(1-\alpha x, 1-\beta x) = \begin{cases} \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\alpha\bar{\beta}\right) & \text{if } |\alpha|, |\beta| \leq 1, \\ \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\frac{\alpha\beta}{|\alpha|^2}\right) & \text{if } |\alpha| \geq 1, |\beta| \leq 1, \\ \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\frac{\alpha\bar{\beta}}{|\alpha\beta|^2}\right) + \log|\alpha|\log|\beta| & \text{if } |\alpha|, |\beta| \geq 1. \end{cases}$$

Application to Jensen's formula

$$\begin{aligned}
m_2(x+y+1) &= \frac{1}{(2\pi \mathrm{i})^2}\int_{|y|=1}\int_{|x|=1}\log^2|x+y+1|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\\
&= \frac{1}{2\pi \mathrm{i}}\int_{|x|=1,|x+1|\le 1}\frac{1}{2}\mathrm{Li}_2\left(|1+x|^2\right)\frac{\mathrm{d}x}{x}\\
&+ \frac{1}{2\pi \mathrm{i}}\int_{|x|=1,|x+1|\ge 1}\left(\frac{1}{2}\mathrm{Li}_2\left(\frac{1}{|1+x|^2}\right)+\log^2|1+x|\right)\frac{\mathrm{d}x}{x}\\
&= \frac{1}{2\pi}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\mathrm{Li}_2\left(4\cos^2\left(\frac{\theta}{2}\right)\right)\mathrm{d}\theta+\frac{\pi^2}{9}\\
&= \frac{\sqrt{3}}{2\pi}\int_0^1\sum_{n=1}^{\infty}\frac{1}{n^2}\binom{2n}{n}\frac{s^n(1-s)^n}{1-s(1-s)}\mathrm{d}s+\frac{\pi^2}{9}.
\end{aligned}$$

Lemma

For $|t| \le \frac{1}{4}$*, we have*

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{t^k}{k^2} = 2\mathrm{Li}_2\left(\frac{1-\sqrt{1-4t}}{2}\right) - \left(\log\left(\frac{1+\sqrt{1-4t}}{2}\right)\right)^2.$$

Now, if we set $t = s(1-s)$,

$$\begin{aligned}
&= \frac{\sqrt{3}}{2\pi}\int_0^1 (2\mathrm{Li}_2(s) - \log^2(1-s))\frac{\mathrm{d}s}{1-s(1-s)} + \frac{\pi^2}{9} \\
&= -\frac{\sqrt{3}}{\pi}\int_{0\le s_1\le s_2\le s\le 1} \frac{\mathrm{d}s_1}{s_1-1}\frac{\mathrm{d}s_2}{s_2}\frac{\mathrm{d}s}{1-s+s^2} \\
&- \frac{\sqrt{3}}{\pi}\int_{0\le s_1\le s_2\le s\le 1} \frac{\mathrm{d}s_1}{s_1-1}\frac{\mathrm{d}s_2}{s_2-1}\frac{\mathrm{d}s}{1-s+s^2} + \frac{\pi^2}{9}.
\end{aligned}$$

But

$$\frac{1}{1-s+s^2} = \frac{1}{\sqrt{3}\mathrm{i}}\left(\frac{1}{s-\omega} - \frac{1}{s-\bar{\omega}}\right), \qquad \omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{6}}$$

$$
\begin{aligned}
&= \frac{\mathrm{i}}{\pi}\int_{0\le s_1\le s_2\le s\le 1}\frac{\mathrm{d}s_1}{s_1-1}\frac{\mathrm{d}s_2}{s_2}\left(\frac{1}{s-\omega}-\frac{1}{s-\bar{\omega}}\right)\mathrm{d}s\\
&+ \frac{\mathrm{i}}{\pi}\int_{0\le s_1\le s_2\le s\le 1}\frac{\mathrm{d}s_1}{s_1-1}\frac{\mathrm{d}s_2}{s_2-1}\left(\frac{1}{s-\omega}-\frac{1}{s-\bar{\omega}}\right)\mathrm{d}s+\frac{\pi^2}{9}\\
&= \frac{\mathrm{i}}{\pi}(\mathrm{Li}_{2,1}(\omega,\bar{\omega})-\mathrm{Li}_{2,1}(\bar{\omega},\omega)-\mathrm{Li}_{1,1,1}(1,\omega,\bar{\omega})+\mathrm{Li}_{1,1,1}(1,\bar{\omega},\omega))+\frac{\pi^2}{9}.\\
&= \frac{7\pi^2}{162}-\frac{5\pi^2}{81}+\frac{\pi^2}{9}=\frac{5\pi^2}{54}.
\end{aligned}
$$

$$Z(s_1,\dots,s_k;P_1,\dots,P_k) = \int_0^1 \cdots \int_0^1 \left|P_1\left(\mathrm{e}^{2\pi\mathrm{i}\theta_1},\cdots,\mathrm{e}^{2\pi\mathrm{i}\theta_n}\right)\right|^{s_1} \cdots \left|P_k\left(\mathrm{e}^{2\pi\mathrm{i}\theta_1},\cdots,\mathrm{e}^{2\pi\mathrm{i}\theta_n}\right)\right|^{s_k} \mathrm{d}\theta_1\cdots\mathrm{d}\theta_n$$

The Taylor coefficients yield multiple higher Mahler measure.

Theorem

- $$Z(s,t;x-1,x+1) = \frac{\Gamma(s+1)\Gamma(t+1)}{\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{t}{2}+1\right)\Gamma\left(\frac{s+t}{2}+1\right)}$$
$$= \exp\left(\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\zeta(k)\left((1-2^{-k})(s^k+t^k)-2^{-k}(s+t)^k\right)\right)$$

- $$m(\underbrace{x-1,\dots,x-1}_{k},\underbrace{x+1,\dots,x+1}_{l})$$

belongs to $\mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\zeta(7),\dots]$ *for integers* $k,l\geq 0$.

$$m(x-1,x+1) = -\frac{\zeta(2)}{4} = -\frac{\pi^2}{24},$$

$$m(x-1,x-1,x+1) = 2\frac{\zeta(3)}{8} = \frac{\zeta(3)}{4},$$

$$m(x-1,x+1,x+1) = 2\frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}.$$

$$m_3(x-1) = -\frac{3\zeta(3)}{2}.$$

Properties of zeta Mahler measures

- $\lambda > 0$,

$$Z(s, \lambda P) = \lambda^s Z(s, P)$$

- $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad P = P^*, \quad |\lambda| \leq 1/||P||_\infty$,

$$\begin{aligned} Z(s, 1 + \lambda P) &= \sum_{k=0}^{\infty} \binom{s}{k} Z(k, P) \lambda^k, \\ m(1 + \lambda P) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} Z(k, P) \lambda^k. \end{aligned}$$

More generally,

$$m_j(1 + \lambda P) = j! \sum_{0 < k_1 < \cdots < k_j} \frac{(-1)^{k_j - j}}{k_1 \dots k_j} Z(k_j, P) \lambda^{k_j}.$$

Let $c \geq 2$.

$$Z(s, x+y+c) = c^s \sum_{j=0}^{\infty} \binom{s/2}{j}^2 \frac{1}{c^{2j}} \binom{2j}{j}.$$
$$= c^s {}_3F_2\left(\begin{matrix} -\frac{s}{2}, -\frac{s}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \,\middle|\, \frac{4}{c^2} \right),$$

where the generalized hypergeometric series ${}_3F_2$ is defined by

$${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} \,\middle|\, z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j (a_3)_j}{(b_1)_j (b_2)_j j!} z^j,$$

with the Pochhammer symbol defined by $(a)_j = a(a+1)\cdots(a+j-1)$.

$$
\begin{aligned}
Z(s, x+y+c) &= Z\left(\frac{s}{2}, (x+y+c)\left(x^{-1}+y^{-1}+c\right)\right) \\
&= c^s Z\left(\frac{s}{2}, \left(1+\frac{1}{c}(x+y)\right)\left(1+\frac{1}{c}\left(x^{-1}+y^{-1}\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{c^s}{(2\pi \mathrm{i})^2}\int_{|y|=1}\int_{|x|=1}\left(1+\frac{x+y}{c}\right)^{s/2}\left(1+\frac{x^{-1}+y^{-1}}{c}\right)^{s/2}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} \\
&= c^s\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{s/2}{j}\binom{s/2}{k}\frac{1}{(2\pi \mathrm{i})^2}\int_{|y|=1}\int_{|x|=1}\left(\frac{x+y}{c}\right)^j\left(\frac{x^{-1}+y^{-1}}{c}\right)^k \\
&= c^s\sum_{j=0}^{\infty}\binom{s/2}{j}^2\frac{1}{c^{2j}}\binom{2j}{j}.
\end{aligned}
$$

In particular, we obtain the special values

- $$m_2(x+y+2) = \frac{\zeta(2)}{2},$$

- $$m_3(x+y+2) = \frac{9}{2}(\log 2)\zeta(2) - \frac{15}{4}\zeta(3).$$

Proof uses

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{t^k}{k^2} = 2\mathrm{Li}_2\left(\frac{1-\sqrt{1-4t}}{2}\right) - \log^2\left(\frac{1+\sqrt{1-4t}}{2}\right).$$

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A family related with Dyson integrals

Consider the following family of polynomials

$$\begin{aligned} P_N(x_1,\ldots,x_N) &= \prod_{1\le h\ne j\le N}\left(1-\frac{x_h}{x_j}\right) \\ &= \prod_{h<j}\left(2-\frac{x_h}{x_j}-\frac{x_j}{x_h}\right) \\ &= 2^{N(N-1)}\prod_{h<j}\sin^2\pi(\theta_h-\theta_j), \qquad (x_h=\mathrm{e}^{2\pi\mathrm{i}\theta_h}). \end{aligned}$$

Then we have the following result

$$\begin{aligned} Z(k,P_N) &= \int_0^1\cdots\int_0^1 P_N(\mathrm{e}^{2\pi\mathrm{i}\theta_1},\ldots,\mathrm{e}^{2\pi\mathrm{i}\theta_N})^k\mathrm{d}\theta_1\cdots\mathrm{d}\theta_N \\ &= \frac{(Nk)!}{(k!)^N} \end{aligned}$$

due to Dyson.

$$Z(s, 1+\lambda P_N) \quad = \quad {}_N F_{N-1}\left(\begin{array}{c} -s, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N} \\ 1, \dots, 1 \end{array} \middle| \frac{\lambda}{N^N}. \right)$$

$$\begin{aligned} m(1+\lambda P_N) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{(Nk)!}{(k!)^N} \lambda^k, \\ m_2(1+\lambda P_N) &= \sum_{k=1}^{\infty} \frac{(-1)^{k}}{k} \left(1 + \dots + \frac{1}{k-1}\right) \frac{(Nk)!}{(k!)^N} \lambda^k. \end{aligned}$$

In particular, for $N = 2$,

$$\begin{aligned} m(1+\lambda P_2) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \binom{2k}{k} \lambda^k, \\ m_2(1+\lambda P_2) &= \sum_{k=2}^{\infty} \frac{(-1)^{k}}{k} \left(1 + \dots + \frac{1}{k-1}\right) \binom{2k}{k} \lambda^k. \end{aligned}$$

Further question

Why do we get such numbers?

Is there an explanation in terms of regulators?

Mahler measures under variations of the base group

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By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

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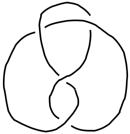
- Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \frac{\mathrm{Vol(Fig8)}}{2\pi}$$



- Boyd (1998)

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 1\right) \stackrel{?}{=} \mathrm{L}'(E_1, 0)$$

E_1 elliptic curve, projective closure of $x + \frac{1}{x} + y + \frac{1}{y} - 1 = 0$.
(50 decimal places)

Also studied by Deninger, Rodriguez-Villegas

Examples in several variables

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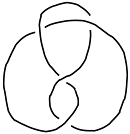

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Rodriguez-Villegas (1997)

$$P_\lambda(x,y) = 1 - \lambda P(x,y) \qquad P(x,y) = x + \frac{1}{x} + y + \frac{1}{y}$$

Reciprocal

$$m(P,\lambda) := m(P_\lambda)$$

$$m(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log |1 - \lambda P(x,y)| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}.$$

Note

$$|\lambda P(x,y)| < 1, \qquad \lambda \quad \text{small}, \quad x, y \in \mathbb{T}^2$$

$$\tilde{m}(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log(1 - \lambda P(x,y)) \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} P(x,y)^n \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = -\sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n}$$

$$a_n := [P(x,y)^n]_0$$

Note

$$|\lambda P(x,y)| < 1, \qquad \lambda \quad \text{small}, \quad x, y \in \mathbb{T}^2$$

$$\tilde{m}(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log(1-\lambda P(x,y)) \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} P(x,y)^n \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = -\sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n}$$

$$a_n := [P(x,y)^n]_0$$

Let

$$u(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \frac{1}{1-\lambda P(x,y)} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = \sum_{n=0}^{\infty} a_n \lambda^n$$

$$\frac{\mathrm{d}\tilde{m}(P,\lambda)}{\mathrm{d}\lambda} = -\frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \frac{P(x,y)}{1-\lambda P(x,y)} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = \sum_{n=0}^{\infty} a_{n+1} \lambda^n$$

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In the case $P = x + \frac{1}{x} + y + \frac{1}{y}$,

$$a_n = \begin{cases} \binom{2m}{m}^2 & n = 2m \\ 0 & \textit{otherwise} \end{cases}$$

Definition

Γ *finitely generated group with generators* $x_1, \dots, x_l$

$$Q = Q(x_1, \dots, x_l) = \sum_{g \in \Gamma} c_g g \in \mathbb{C}\Gamma,$$

$$Q^* = \sum_{g \in \Gamma} \overline{c_g} g^{-1} \in \mathbb{C}\Gamma \text{ reciprocal.}$$

$P = P(x_1, \dots, x_l) \in \mathbb{C}\Gamma$, $P = P^*$, $|\lambda|^{-1} >$ *length of* P,

$$m_\Gamma(P, \lambda) = -\sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n},$$

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We also write

$$u_\Gamma(P,\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$$

for the generating function of the a_n.

$Q(x_1,\dots,x_l) \in \mathbb{C}\Gamma$

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for λ real and positive and $1/\lambda$ larger than the length of QQ^*.

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Volume of hyperbolic knots

K knot: smooth embedding $S^1 \subset S^3$.

$$\Gamma = \pi_1(S^3 \setminus K) = \langle x_1, \ldots, x_g \mid r_1, \ldots, r_{g-1} \rangle$$

Derivation: mapping $\mathbb{C}\Gamma \to \mathbb{C}\Gamma$ (any group)

- $D(u + v) = Du + Dv$.
- $D(u \cdot v) = D(u)\epsilon(v) + uD(v)$

$$\epsilon : \mathbb{C}\Gamma \to \mathbb{C} \qquad \sum_g c_g g \to \sum_g c_g.$$

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Fox (1953) $\{x_1, \dots\}$ generators, there is $\frac{\partial}{\partial x_i}$ such that

$$\frac{\partial x_j}{\partial x_i} = \delta_{i,j}.$$

Back to knots,
Let

$$F = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{g-1}}{\partial x_1} & \cdots & \frac{\partial r_{g-1}}{\partial x_g} \end{pmatrix} \in M^{(g-1)\times g}(\mathbb{C}\Gamma)$$

Fox matrix.

Delete a column $F \rightsquigarrow A \in M^{(g-1)\times(g-1)}(\mathbb{C}\Gamma)$.

Theorem (Lück, 2002)

Suppose K is a hyperbolic knot. Then, for λ sufficiently small

$$\frac{1}{3\pi}\mathrm{Vol}(S^3 \setminus K) = -(g-1)\ln\lambda - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_{\mathbb{C}\Gamma}\left((1-\lambda AA^*)^n\right).$$

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Γ of order m

$$\alpha : \Gamma \to \mathbb{C} \qquad \alpha(g) = \overline{\alpha(g^{-1})} \quad \forall g \in \Gamma$$

Weighted Cayley graph:

- Vertices $g_1, \ldots, g_m$.
- (directed) Edge between g_i and g_j has weight $\alpha(g_i^{-1}g_j)$.

Weighted adjacency matrix

$$A(\Gamma, \alpha) = \{\alpha(g_i^{-1}g_j)\}_{i,j}$$

The Mahler measure over finite groups

$$P = \sum_i (\delta_i S_i + \overline{\delta_i} S_i^{-1}) + \sum_j \eta_j T_j \in \mathbb{C}\Gamma$$

$\delta_i \in \mathbb{C}$, $\eta_j \in \mathbb{R}$, and $S_i, T_j \in \Gamma$,

$$a_n = \frac{\mathrm{tr}(A^n)}{|\Gamma|}$$

Theorem

For Γ finite

$$m_\Gamma(P, \lambda) = \frac{1}{|\Gamma|} \log \det(I - \lambda A),$$

A is the adjacency matrix of the Cayley graph (with weights) and $\frac{1}{\lambda} > \rho(A)$.

Analytic continuation for $m_\Gamma(P, \lambda)$ to $\mathbb{C} \setminus \mathrm{Spec}(A)$.

Spectrum of a Cayley Graph

Let $\chi_1, \dots, \chi_h$ be the irreducible characters of Γ of degrees $n_1, \dots, n_h$.

Theorem (Babai, 1979)

The spectrum of $A(\Gamma, \alpha)$ can be arranged as

$$\mathcal{S} = \{\sigma_{i,j} : i = 1, \dots, h; j = 1, \dots, n_i\}.$$

such that $\sigma_{i,j}$ has multiplicity n_i and

$$\sigma_{i,1}^t + \cdots + \sigma_{i,n_i}^t = \sum_{g_1,\dots,g_t \in \Gamma} \left(\prod_{s=1}^{t} \alpha(g_s) \right) \chi_i \left(\prod_{s=1}^{t} g_s \right).$$

Finite Abelian Groups

$$\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$$

Corollary

$$m_\Gamma(P,\lambda) = \frac{1}{|\Gamma|}\log\left(\prod_{j_1,\dots,j_l}\left(1-\lambda P(\xi_{m_1}^{j_1},\dots,\xi_{m_l}^{j_l})\right)\right)$$

where ξ_k is a primitive root of unity.

Theorem

For small λ*,*

$$\lim_{m_1,\dots,m_l\to\infty} m_{\mathbb{Z}/m_1\mathbb{Z}\times\cdots\times\mathbb{Z}/m_l\mathbb{Z}}(P,\lambda) = m_{\mathbb{Z}^l}(P,\lambda).$$

Where the limit is with $m_1,\dots,m_l$ going to infinity independently.

Dihedral groups

$$\Gamma = D_m = \langle \rho, \sigma \mid \rho^m, \sigma^2, \sigma\rho\sigma\rho \rangle.$$

Theorem

Let $P \in \mathbb{C}[D_m]$ be reciprocal. Then

$$[P^n]_0 = \frac{1}{2m} \sum_{j=1}^{m} \left(P^n\left(\xi_m^j, 1\right) + P^n\left(\xi_m^j, -1\right) \right),$$

where P^n is expressed as a sum of monomials ρ^k, $\sigma\rho^k$ before being evaluated.

For $\Gamma = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y \,|\, x^m, y^2, [x,y] \rangle$,

$$[P^n]_0 = \frac{1}{2m} \sum_{j=1}^{m} \left(P\left(\xi_m^j, 1\right)^n + P\left(\xi_m^j, -1\right)^n \right).$$

Compare D_m and $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $x = \rho$ and $y = \sigma$ in D_m.

Theorem

Let

$$P = \sum_{k=0}^{m-1} \alpha_k x^k + \sum_{k=0}^{m-1} \beta_k y x^k$$

with real coefficients and reciprocal in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (therefore it is also reciprocal in D_m). Then

$$m_{\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P, \lambda) = m_{D_m}(P, \lambda).$$

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$$m_{\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P, \lambda) = m_{D_m}(P, \lambda).$$

Corollary

Let $P \in \mathbb{R}[\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ *be reciprocal. Then*

$$m_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P, \lambda) = m_{D_\infty}(P, \lambda),$$

where $D_\infty = \langle \rho, \sigma \mid \sigma^2, \sigma\rho\sigma\rho \rangle$.

Γ_m are quotients of Γ:

Theorem

Let $P \in \Gamma$ reciprocal.

- *For $\Gamma = D_\infty$, $\Gamma_m = D_m$,*

$$\lim_{m\to\infty} m_{D_m}(P,\lambda) = m_{D_\infty}(P,\lambda).$$

- *For $\Gamma = PSL_2(\mathbb{Z}) = \langle x, y \,|\, x^2, y^3\rangle$, $\Gamma_m = \langle x, y \,|\, x^2, y^3, (xy)^m\rangle$,*

$$\lim_{m\to\infty} m_{\Gamma_m}(P,\lambda) = m_{PSL_2(\mathbb{Z})}(P,\lambda).$$

- *For $\Gamma = \mathbb{Z} * \mathbb{Z} = \langle x, y\rangle$, $\Gamma_m = \langle x, y \,|\, [x,y]^m\rangle$,*

$$\lim_{m\to\infty} m_{\Gamma_m}(P,\lambda) = m_{\mathbb{Z}*\mathbb{Z}}(P,\lambda).$$

For $P_{1,l} = x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$,

$$u_{\mathbb{F}_l}(P_{1,l}, \lambda) = g_{2l}(\lambda).$$

where

$$g_d(\lambda) = \frac{2(d-1)}{d - 2 + d\sqrt{1 - 4(d-1)\lambda^2}}.$$

is the generating function of the circuits of a d-regular tree (Bartholdi, 1999).

For $P_{2,l} = (1 + x_1 + \cdots + x_{l-1})\left(1 + x_1^{-1} + \cdots + x_{l-1}^{-1}\right)$,

$$u_{\mathbb{F}_{l-1}}(P_{2,l}, \lambda) = g_l(\lambda).$$

In particular,

$$m_{\mathbb{F}_l}(P_{1,l}, \lambda) = m_{\mathbb{F}_{2l-1}}(P_{2,2l}, \lambda).$$

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Abelian case.

For $P_{1,l} = x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$,

$$\left[P_{1,l}^n\right]_0 = \sum_{a_1+\cdots+a_l=n} \frac{(2n)!}{(a_1!)^2 \ldots (a_l!)^2},$$

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$$\left[P_{1,l}^{2n}\right]_0 = \binom{2n}{n} \left[P_{2,l}^n\right]_0$$

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$$\left[P_{1,l}^{2n}\right]_0 = \binom{2n}{n} \left[P_{2,l}^n\right]_0$$

Now $P = x + x^{-1} + y + y^{-1}$.

$$u_{\mathbb{Z}\times\mathbb{Z}}(P,\lambda) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \lambda^{2n} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 16\lambda^2\right)$$

$$u_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P,\lambda) = \sum_{n=0}^{\infty} \binom{4n}{2n} \lambda^{2n}$$

$$u_{\mathbb{Z}*\mathbb{Z}}(P,\lambda) = \frac{3}{1 + 2\sqrt{1 - 12\lambda^2}}$$

Recurrence relations $x + x^{-1} + y + y^{-1}$

Coefficients satisfy recurrence relations

$$\mathbb{Z} \times \mathbb{Z} : \quad n^2 a_{2n} - 4(2n-1)^2 a_{2n-2} = 0$$

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} : \quad n(2n-1)a_{2n} - 2(4n-1)(4n-3)a_{2n-2} = 0$$

$$\mathbb{Z} * \mathbb{Z} : \quad na_{2n} - 2(14n-9)a_{2n-2} + 96(2n-3)a_{2n-4} = 0$$

- $\mathbb{Z}^l$

  Rodriguez - Villegas: $u(\lambda)$ periods of a differential in the curve defined by $1 = \lambda P(x, y)$. By Griffiths (1969)

  $$A_k(\lambda)u^{(k)} + A_{k-1}(\lambda)u^{(k-1)} + \cdots + A_0(\lambda)u = 0,$$

  Picard-Fuchs differential equation (A_j polynomials).
  $\Rightarrow$ Recurrence of the coefficients.
  Wilf and Zeilberger: a_n multisums, generating series is hypergeometric.

- This recurrence result extends to the case of Γ finitely generated abelian group.

- Finite groups :

$$a_n = \frac{\mathrm{tr}(A^n)}{|\Gamma|}$$

minimal polynomial of A.

- $\mathbb{F}_l$

By Haiman (1993): $u(\lambda)$ is algebraic.

Algebraic functions in non-commuting variables.

$P = x + x^{-1} + y + y^{-1}$

$$\Gamma = \langle x, y \,|\, x^2 y = y x^2, y^2 x = x y^2 \rangle$$

Domb (1960)

$$a_{2n} = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k}$$

Same as ordinary Mahler measure for

$$1 - \lambda \left(x + x^{-1} + z\left(y + y^{-1}\right)\right)\left(x + x^{-1} + z^{-1}\left(y + y^{-1}\right)\right)$$

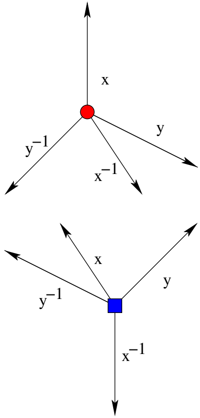
$$n^3 a_{2n} - 2(2n-1)(5n^2-5n+2)a_{2n-2} + 6(n-1)^3 a_{2n-4} = 0$$

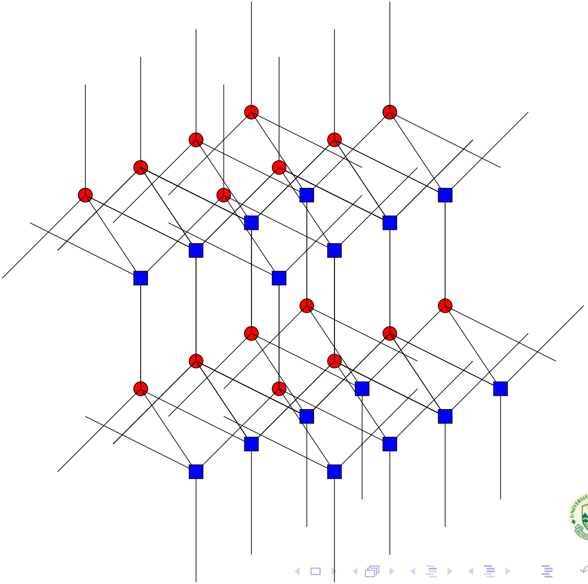
Rogers (2007)

$$1-\lambda\left(4+\left(x+x^{-1}\right)\left(y+y^{-1}\right)+\left(y+y^{-1}\right)\left(z+z^{-1}\right)+\left(z+z^{-1}\right)\left(x+x^{-1}\right)\right)$$

$$_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};1,1;-\frac{108\lambda}{(1-16\lambda)^3}\right) = (1-16\lambda)\sum_{n=0}^{\infty} a_{2n}\lambda^n$$

The diamond lattice



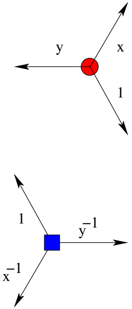


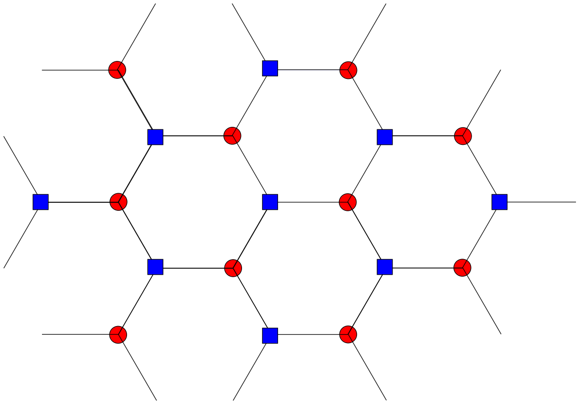
$$Q = (1 + x + y)\left(1 + x^{-1} + y^{-1}\right)$$

$$[Q^n]_0 = a_n$$

$$n^2 a_n - (10n^2 - 10n + 3)a_{n-1} + 9(n-1)^2 a_{n-2} = 0,$$

Honeycomb lattice $(1+x+y)(1+x^{-1}+y^{-1})$





$$P = x + x^{-1} + y + y^{-1} + xy^{-1} + x^{-1}y$$

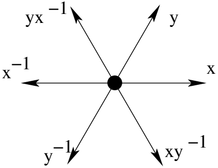
$$[P^n]_0 = b_n$$

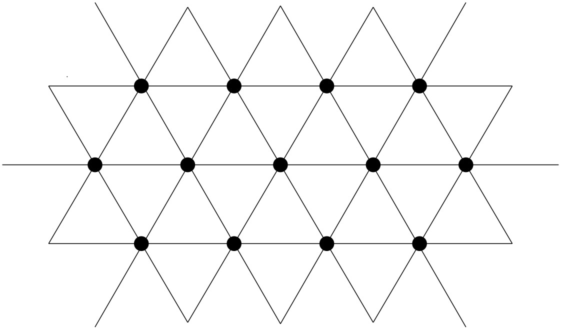
$$n^2 b_n - n(n-1)b_{n-1} - 24(n-1)^2 b_{n-2} - 36(n-2)(n-1)b_{n-3} = 0.$$

$$Q = 3 + P$$

$$b_n = \sum_{j=0}^{n} \binom{n}{j} (-3)^{n-j} a_j$$

Triangular lattice $x + x^{-1} + y + y^{-1} + xy^{-1} + x^{-1}y$





$m\left(P, \frac{1}{l^1(P)}\right)$ related to $h(G)$

where G is the Cayley graph and h is the tree entropy

$$h(G) := \log \deg_G(o) - \sum_{n=1}^{\infty} \frac{p_n(o, G)}{n},$$

- o fixed vertex
- $p_n(o, G)$ is the probability that a simple random walk started at o on G is again at o after n steps.

Lyons (2005)

G_n are finite graphs that tend to a fixed transitive infinite graph G, then

$$h(G) = \lim_{n\to\infty} \frac{\log \tau(G_n)}{|V(G_n)|},$$

where $\tau(G)$ is the complexity, i.e., the number of spanning trees.

Compare to

Conjecture ((Volume Conjecture) Kashaev, H. Murakami, J. Murakami (1997))

Let K be a hyperbolic knot, and $J_n(K,q)$ its normalized colored Jones polynomial. Then

$$\frac{1}{2\pi}\mathrm{Vol}(S^3 \setminus K) = \lim_{n\to\infty} \frac{\log \left| J_n\left(K, \mathrm{e}^{\frac{2\pi i}{n}}\right)\right|}{n}$$