Functional equations for Mahler measures of genus-one curves

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Mahler measure of polynomials

Definition 1 *For* $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, *the (logarithmic)* Mahler measure *is defined by*

$$m(P) \quad = \quad \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{1}$$

This integral is not singular and $m(P)$ always exists.
Because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \, \mathrm{d}\theta = \log^+ |\alpha| \tag{2}$$

[2]we have a simple expression for the Mahler measure of one-variable polynomials:

$$m(P) = \log |a_d| + \sum_{n=1}^{d} \log^+ |\alpha_n| \quad \text{for} \quad P(x) = a_d \prod_{n=1}^{d} (x - \alpha_n)$$

Properties

Here are some general properties (see [6])

- For $P, Q \in \mathbb{C}[x_1, \dots, x_n]$, we have $m(P \cdot Q) = m(P) + m(Q)$. Because of that, it makes sense to talk about the Mahler measure of rational functions.
- $m(P) \geq 0$ if P has integral coefficients.
- Mahler measure is related to heights. Indeed, if α is an algebraic number, and P_α is its minimal polynomial over $\mathbb{Q}$, then

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}] \, h(\alpha),$$

where h is the logarithmic Weil height.

- By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $m(P) = 0$ if and only if P is the product of powers of x and cyclotomic polynomials.
- For $P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots x^{k_n})) = m(P(x_1, \dots x_n)) \tag{3}$$

(this result is due to Boyd and Lawton see [2], [9]).

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[2] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

- Lehmer [10] studied this example in 1933:

  $$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1) = \log(1.176280818\ldots) = 0.162357612\ldots$$

  This example is special because its Mahler measure is very small, but still greater than zero.

  The following questions are still open: Is there a lower bound for positive Mahler measure of polynomials in one variable with integral coefficients? Does this degree 10 polynomial reach the lowest bound?

  Boyd–Lawton result implies that Lehmer's problem in several variables reduces to the one variable case.

Examples

For one-variable polynomials, the Mahler measure has to do with the roots of the polynomial. However, it is very hard to compute explicit formulas for examples in several variables. The first and simplest ones were computed by Smyth:

- Smyth [13]

  $$m(x+y+1) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1), \tag{4}$$

  where

  $$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{if} \quad n \equiv 0 \bmod 3 \end{cases}$$

- Smyth [2]

  $$m(x+y+z+1) = \frac{7}{2\pi^2}\zeta(3). \tag{5}$$

- D'Andrea & L. (2003):

  $$m\left(z(1-xy)^2-(1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)\pi^2}$$

- Boyd & L. (2005):

  $$m(x^2+1+(x+1)y+(x-1)z) = \frac{\mathrm{L}(\chi_{-4},2)}{\pi} + \frac{21}{8\pi^2}\zeta(3)$$

  L.(2003):

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  $$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right) = \frac{24}{\pi^3}\mathrm{L}(\chi_{-4},4) + \frac{\mathrm{L}(\chi_{-4},2)}{\pi}$$

-
  $$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\ldots\left(\frac{1-x_4}{1+x_4}\right)z\right) = \frac{62}{\pi^4}\zeta(5) + \frac{14}{3\pi^2}\zeta(3)$$

- $$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=\frac{24}{\pi^3}\mathrm{L}(\chi_{-4},4)$$

- $$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=\frac{93}{\pi^4}\zeta(5)$$

The measures of a family of genus-one curves

The family of two-variable polynomials $P_k(x,y)=x+\frac{1}{x}+y+\frac{1}{y}+k$ was studied by Boyd [2], Deninger [5], and Rodriguez-Villegas [11] from different points of view. They found

$$\begin{aligned} m\left(x+\frac{1}{x}+y+\frac{1}{y}+k\right) &\stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{s_k} \qquad k\in\mathbb{N} \\ m\left(x+\frac{1}{x}+y+\frac{1}{y}+4\right) &= 2\mathrm{L}'(\chi_{-4},-1) \end{aligned} \tag{6}$$

Where s_k is a rational number (often integer), and E_k is the elliptic curve with corresponds to the zero set of the polynomial. When $k=4$ the curve has genus zero. The conection with $\mathrm{L}'(E,0)$ can be predicted by Beilinson's conjectures. The question mark stands for numerical results. However, there are some cases in which this identity can be proved (when Beilinson's conjectures are known). For instance, consider the case of $k=4\sqrt{2}$. In this case the curve has complex multiplication and

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+4\sqrt{2}\right)=\mathrm{L}'(E_{4\sqrt{2}},0)$$

If $k=3\sqrt{2}$ we get the modular curve $X_0(24)$ and again,

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+3\sqrt{2}\right)=\frac{5}{2}\mathrm{L}'(E_{3\sqrt{2}},0)$$

Let $m(k):=m\left(x+\frac{1}{x}+y+\frac{1}{y}+k\right)$.

Theorem 2 *(Rodriguez-Villegas [11])*

$$m(k)=\mathrm{Re}\left(\frac{16y_\mu}{\pi^2}\sum_{m,n}{}'\frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})}\right)=\mathrm{Re}\left(-\pi\mathrm{i}\mu+2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2\frac{q^n}{n}\right)$$

where $j(E_k)=j\left(-\frac{1}{4\mu}\right)$,

$$q=\mathrm{e}^{2\pi\mathrm{i}\mu}=q\left(\frac{16}{k^2}\right)=\exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

and y_μ *is the imaginary part of* μ

This result is achieved by computing the Mahler measure as a function on the parameter $\lambda = \frac{1}{k}$. The result is a hypergeometric series in λ which satisfies a Picard-Fuchs differential equation.

Here is our main result.

Theorem 3 *For $h \in \mathbb{R}^*$,*

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right). \tag{7}$$

For $h \in \mathbb{R}^*$, $|h| < 1$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(\mathrm{i}h + \frac{1}{\mathrm{i}h}\right)\right) = m\left(\frac{4}{h^2}\right). \tag{8}$$

These identitites can provide information that allows us to compute more Mahler measures. For instance,

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

These results were discovered by Rogers. We will interpret them directly from regulators.

The elliptic regulator

Let F be a field. By Matsumoto's Theorem, $K_2(F)$ is generated by the symbols $\{a, b\}$ for $a, b \in F$ with the bilinearity relations $\{a_1a_2, b\} = \{a_1, b\}\{a_2, b\}$ and $\{a, b_1b_2\} = \{a, b_1\}\{a, b_2\}$, and the Steinberg relation $\{a, 1 - a\} = 1$ for all $a \neq 0$.

Recall that for a field F with discrete valuation v and maximal ideal $\mathcal{M}$ the tame symbol is

$$(x, y)_v \equiv (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \bmod \mathcal{M}$$

(see [11]). In the case when $F = \mathbb{Q}(E)$ (from now on E denotes an elliptic curve), a valuation is determined by de order of the rational functions in each point $S \in E(\bar{\mathbb{Q}})$. We will denote the valuation determined by a point $S \in E(\bar{\mathbb{Q}})$ by v_S.

The tame symbol is then a map $K_2(\mathbb{Q}(E)) \to \mathbb{Q}(S)^*$.

We have

$$0 \to K_2(E) \otimes \mathbb{Q} \to K_2(\mathbb{Q}(E)) \otimes \mathbb{Q} \to \coprod_{S \in E(\bar{\mathbb{Q}})} \mathbb{Q}(S)^* \otimes \mathbb{Q},$$

where the last arrow corresponds to the coproduct of tame symbols.

Hence an element $\{x, y\} \in K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$ can be seen as an element in $K_2(E) \otimes \mathbb{Q}$ when $(x, y)_{v_S} = 1$ for all $S \in E(\bar{\mathbb{Q}})$.

(Beilinson, Bloch) regulator map can be defined by

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R})$$

$$\{x, y\} \to \left\{\gamma \to \int_\gamma \eta(x, y)\right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$ and

$$\eta(x, y) := \log|x| \,\mathrm{d} \arg y - \log|y| \,\mathrm{d} \arg x$$

Here we think of $H^1(E,\mathbb{R})$ as the dual of $H_1(E,\mathbb{Z})$. The function is well defined because $\eta(x,1-x) = \mathrm{d}D(x)$ where

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$$

is the Bloch-Wigner dilogarithm.

Assume that E is defined over $\mathbb{R}$. Because of the way that complex conjugation acts on η the regulator map is trivial for the classes in $H_1(E,\mathbb{Z})^+$, the cycles that remain invariant by complex conjugation. Therefore it suffices to consider the regulator as a funcion on $H_1(E,\mathbb{Z})^-$.

Bloch defines the regulator function by a Kronecker-Eisenstein series

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}\alpha}\right) = \frac{y_\tau^2}{\pi}\sideset{}{'}\sum_{m,n\in\mathbb{Z}}\frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)} \tag{9}$$

if $\alpha = a+b\tau$ and y_τ is the imaginary part of τ.

Let $J(z) = \log|z|\log|1-z|$.

We write $E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ we have $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$ where $z \bmod \Lambda = \mathbb{Z}+\tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$.

Definition 4 *We consider the following function on $E(\mathbb{C}) \cong \mathbb{C}^*/q^{\mathbb{Z}}$:*

$$J_\tau(z) = \sum_{n=0}^{\infty} J(zq^n) - \sum_{n=1}^{\infty} J(z^{-1}q^n) + \frac{1}{3}\log^2|q| B_3\left(\frac{\log|z|}{\log|q|}\right) \tag{10}$$

where $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ is the third Bernoulli polynomial.

On the other hand, the elliptic dilogarithm is defined by

$$D_\tau(z) := \sum_{n\in\mathbb{Z}} D(zq^n) \tag{11}$$

Then the regulator function (see [1]) is given by

$$R_\tau = D_\tau - \mathrm{i}J_\tau \tag{12}$$

By linearity R_τ can be extended to the divisors with support in $E(\mathbb{C})$. Let $\mathbb{Z}[E(\mathbb{C})]^-$ mean that $[-P] \sim -[P]$. Because R_τ is an odd function, we obtain a map

$$\mathbb{Z}[E(\mathbb{C})]^- \to \mathbb{C}.$$

Let x, y be non-constant functions on E with divisors

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

Following [1] and the notation in [11], we recall the diamond operation $\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$

$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$

Theorem 5 *x, y are non-constant functions in $\mathbb{Q}(E)$ with $\{x,y\} \in K_2(E)$, then*

$$-\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x)\diamond(y)\right)\right) \tag{13}$$

where Ω_0 is the real period and $\Omega = \int_\gamma \omega$.

PROOF. (Inspired in Deninger's proof) $\eta(x,y)$ is an element of the one dimensional vector space $H^1(E/\mathbb{R},\mathbb{R})$ and so is $\mathrm{i}([\omega]-[\bar{\omega}])$. Then we may write

$$\eta(x,y) = \alpha \mathrm{i}([\omega]-[\bar{\omega}]),$$

from where

$$\int_\gamma \eta(x,y) = 2\alpha \mathrm{i}\,\mathrm{Im}(\Omega).$$

On the other hand, we have

$$\int_{E(\mathbb{C})} \eta(x,y)\wedge\bar{\omega} = \alpha \mathrm{i}\int_{E(\mathbb{C})} \omega\wedge\bar{\omega} = -\alpha 2\Omega_0^2 y_\tau$$

Beilinson proves

$$\int_{E(\mathbb{C})} \eta(x,y)\wedge\bar{\omega} = \Omega_0 R_\tau((x)\diamond(y)).$$

The proof is completed by observing that $R_\tau((x)\diamond(y))$ is real in this case since $\eta(x,y)$ changes its sign under conjugation. □

The relation with Mahler measures

Let us recall the relation of Mahler measure and the regulator (proved by Deninger [5], studied by Rodriguez-Villegas [11]):

$$m(k) \sim_{\mathbb{Z}} \frac{1}{2\pi} r(\{x,y\})(\gamma) \tag{14}$$

For our particular family we write

$$yP_k(x,y) = (y - y_{(1)}(x))(y-y_{(2)}(x)),$$

we have

$$m(k) = m(yP_k(x,y)) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1} (\log^+|y_{(1)}(x)| + \log^+|y_{(2)}(x)|)\frac{\mathrm{d}x}{x}.$$

The last equality is the result of applying Jensen's formula respect to the variable y. When the polynomial does not intersect the torus (when $|k|>4$ or $k \notin \mathbb{R}$), we may forget the "+" in the log as each $y_{(i)}(x)$ is always inside or outside the unit circle. Indeed there is always a branch inside the unit circle and a branch outside. Then we may write

$$m(k) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1}\log|y|\frac{\mathrm{d}x}{x} = -\frac{1}{2\pi}\int_{\mathbb{T}^1}\eta(x,y), \tag{15}$$

then $\mathbb{T}^1$ is interpreted as a cycle in the homology of the elliptic curve defined by $P_k(x,y)=0$, namely, $H_1(E,\mathbb{Z})$.

If $|k|\le 4$ and k real, we can still write the above equation, but it is more subtle.

Functional identities for the regulator

We recall a result by Bloch [1] which studies the modularity of R_τ:

Proposition 6 *Let* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ *and let* $\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$*, such that*

$$\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}$$

Then:

$$R_{\tau'}\left(\mathrm{e}^{2\pi \mathrm{i}(a'+b'\tau')}\right) = \frac{1}{\gamma\bar{\tau}+\delta} R_\tau\left(\mathrm{e}^{2\pi \mathrm{i}(a+b\tau)}\right).$$

We will need to use functional equations for J_τ. First let us recall the following trivial property.

$$J(z) = p \sum_{x^p=z} J(x) \tag{16}$$

Proposition 7 *Let* p *prime, and let* $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$*, and* $q_j = \mathrm{e}^{\frac{2\pi \mathrm{i}(\tau+j)}{p}}$ *for* $j = 0, \ldots, p-1$*.*

$$(1 + \chi_{-4}(p)p^2)J_{4\tau}(q) = \sum_{j=0}^{p-1} pJ_{\frac{4(\tau+j)}{p}}(q_j) + \chi_{-4}(p)J_{4p\tau}(q^p) \tag{17}$$

In addition,

$$J_{\frac{2\tau+1}{2}}\left(\mathrm{e}^{\pi \mathrm{i}\tau}\right) = J_{2\tau}\left(\mathrm{e}^{\pi \mathrm{i}\tau}\right) - J_{2\tau}\left(-\mathrm{e}^{\pi \mathrm{i}\tau}\right) \tag{18}$$

Proof of the result

First we will write the equation

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

in Weierstrass form. We consider the rational transformation

$$X = -\frac{1}{xy} \qquad Y = \frac{(y-x)\left(1+\frac{1}{xy}\right)}{2xy}$$

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX+2Y}{2X(X-1)},$$

which leads to

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

Let us note that there is a torsion point of order 4 in $\mathbb{Q}(k)$, namely $P = \left(1, \frac{k}{2}\right)$. Note that $2P = (0,0)$ and $3P = \left(1, -\frac{k}{2}\right)$.

Now

$$(X) = 2(2P) - 2O,$$

and

$$(x) = (P) - (2P) - (3P) + O,$$

$$(y) = -(P) - (2P) + (3P) + O.$$

We compute

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$

Note that P corresponds to $\frac{3w_1}{4}$ for $k \in \mathbb{R}$. On the other hand, τ is purely imaginary for $k \in \mathbb{R}, |k| > 4$, and with real part $\frac{1}{2}$ for $k \in \mathbb{R}$ and $|k| < 4$.

The next step is to understand the cycle $|x| = 1$ as an element of $H_1(E, \mathbb{Z})$ in order to compute the value of Ω. It runs out that $\Omega = \tau\Omega_0$ for $k \in \mathbb{R}$.

Then we get

$$m(k) = \frac{4}{\pi} \operatorname{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right).$$

for $k \in \mathbb{R}$.

For the case of k real, take $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$. Then

$$R_\tau(-\mathrm{i}) = R_\tau\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{4}}\right) = \bar{\tau} R_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{4\tau}}\right)$$

then,

$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{4\tau}}\right)$$

If we let $\mu = -\frac{1}{4\tau}$, then

$$m(k) = -\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right) = \operatorname{Im}\left(\frac{1}{\pi y_\mu} R_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right)\right)$$

Thus we have recovered the result for $k \in \mathbb{R}$. For $k \in \mathbb{C}$ we obtain the result by using continuity and holomorphicity.

We will deduce our equations directly from the identity

$$m(k) = -\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right)$$

In the p-identity for J set $p = 2$, we obtain,

$$J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right) = 2J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) + 2J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}(\mu+1)}{2}}\right)$$

which translates into

$$\frac{1}{y_{4\mu}} J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right) = \frac{1}{y_{2\mu}} J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) + \frac{1}{y_{2\mu}} J_{2\mu}\left(-\mathrm{e}^{\pi\mathrm{i}\mu}\right)$$

setting $\tau = -\frac{1}{2\mu}$, and assuming that $|h| < 1$ so that μ is purely imaginary,

$$D_{\frac{\tau}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) + \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}(\mu+1)}{2}}\right)$$

This is actually the content of the first identity.

For the second, we use that

$$J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) = J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) - J_{2\mu}\left(-\mathrm{e}^{\pi\mathrm{i}\mu}\right)$$

Set $\tau = -\frac{1}{2\mu}$ and use $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

$$D_{\frac{\tau-1}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) - \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)}\left(-\mathrm{e}^{\frac{2\pi\mathrm{i}(\mu+1)}{2}}\right)$$

Putting things together,

$$2D_\tau(-\mathrm{i}) = D_{\frac{\tau}{2}}(-\mathrm{i}) + D_{\frac{\tau-1}{2}}(-\mathrm{i})$$

which proves the other equality.

It turns out that

$$m(k) = \mathrm{Re}\left(-\pi\mathrm{i}\mu - \pi\mathrm{i}\int_{\mathrm{i}\infty}^{\mu}(e(z)-1)\,\mathrm{d}z\right)$$

where

$$e(\mu) = 1 - 4\sum_{n=1}^{\infty}\sum_{d\ n}\chi_{-4}(d)d^2q^n$$

is an Eisenstein series. Hence the equations can be also deduced from identities of Hecke operators.

It remains to express the identities in terms of the original parameter k. For this it is necessary to use Rodriguez-Villegas expression. More precisely,

$$q = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

Then the second degree modular equation implies for h real, $|h| < 1$,

$$q^2\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(h^4\right).$$

The substitution $h \to \mathrm{i}h$ allows us to deduce

$$-q\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right).$$

Then the equation with J becomes

$$m\left(q\left(\left(\frac{2h}{1+h^2}\right)^2\right)\right) + m\left(q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right)\right) = m\left(q\left(h^4\right)\right).$$

Finally,

$$m\left(2\left(h+\frac{1}{h}\right)\right) + m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right) = m\left(\frac{4}{h^2}\right).$$

The identity with $h = \frac{1}{\sqrt{2}}$

If we set $h = \frac{1}{\sqrt{2}}$ in the first identity, we obtain

$$m(8) = m\left(3\sqrt{2}\right) + m\left(\mathrm{i}\sqrt{2}\right)$$

We prove that

$$3m\left(3\sqrt{2}\right) = 5m\left(\mathrm{i}\sqrt{2}\right)$$

Consider the function $f = \frac{\sqrt{2}Y - X}{2} \in \mathbb{R}(E_{3\sqrt{2}})$ and $1 - f$. Their divisors are

$$\left(\frac{\sqrt{2}Y - X}{2}\right) = (2P) + 2(P+Q) - 3O,$$

$$\left(1 - \frac{\sqrt{2}Y - X}{2}\right) = (P) + (Q) + (3P+Q) - 3O,$$

Where $Q = \left(-\frac{1}{h^2}, 0\right)$ is an element of order 2. The diamond operation yields

$$(f) \diamond (1-f) = 6(P) - 10(P+Q).$$

But $(f) \diamond (1-f)$ is trivial in K-theory, then

$$6(P) \sim 10(P+Q).$$

It turns out that $E_{3\sqrt{2}}$ and $E_{\mathrm{i}\sqrt{2}}$ are isomorphic,

$$\phi : E_{3\sqrt{2}} \to E_{\mathrm{i}\sqrt{2}} \qquad (X, Y) \to (-X, \mathrm{i}Y)$$

$$r_{\mathrm{i}\sqrt{2}}(\{x, y\}) = r_{3\sqrt{2}}(\{x \circ \phi, y \circ \phi\})$$

But

$$(x \circ \phi) \diamond (y \circ \phi) = 8(P+Q)$$

This implies

$$6r_{3\sqrt{2}}(\{x, y\}) = 10r_{\mathrm{i}\sqrt{2}}(\{x, y\})$$

and

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}).$$

Consequently,

$$m(8) = \frac{8}{5}m(3\sqrt{2})$$

$$m(2) = \frac{2}{5}m(3\sqrt{2})$$

Other families

Other identities discovered by Rogers that we can also interpret in terms of regulators are:

- For the Hesse family $h(a^3) = m\left(x^3 + y^3 + 1 - \frac{3xy}{a}\right)$ (studied by Rodriguez-Villegas)

$$h(u^3) = \sum_{j=0}^{2} h\left(1 - \left(\frac{1-\xi_3^j u}{1+2\xi_3^j u}\right)^3\right) \qquad |u| \text{ small}$$

- More complicated equations for examples studied by Stienstra:

$$m\left((x+1)(y+1)(x+y) - \frac{xy}{t}\right)$$

and Zagier and Stienstra:

$$m\left((x+y+1)(x+1)(y+1) - \frac{xy}{t}\right)$$

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