

# Regulators and computations of Mahler measures

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## Mahler measure of multivariate polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , the (logarithmic) *Mahler measure* is :

$$m(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

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Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \pmod{3} \\ -1 & n \equiv -1 \pmod{3} \\ 0 & n \equiv 0 \pmod{3} \end{cases}$$

# Polylogarithms

The  $k$ th polylogarithm is

$$\operatorname{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad x \in \mathbb{C}, \quad |x| < 1$$

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Zagier:

$$\widehat{\mathcal{L}}_k(x) := \widehat{\operatorname{Re}}_k \left( \sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \operatorname{Li}_{k-j}(x) \right)$$

$B_j$  is  $j$ th Bernoulli number

$\widehat{\operatorname{Re}}_k = \operatorname{Re}$  or  $i \operatorname{Im}$  if  $k$  is odd or even.

One-valued, real analytic in  $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ , continuous in  $\mathbb{P}^1(\mathbb{C})$ .

$\widehat{\mathcal{L}}_k$  satisfies lots of functional equations

$$\widehat{\mathcal{L}}_k\left(\frac{1}{x}\right) = (-1)^{k-1}\widehat{\mathcal{L}}_k(x) \quad \widehat{\mathcal{L}}_k(\bar{x}) = (-1)^{k-1}\widehat{\mathcal{L}}_k(x)$$

Bloch–Wigner dilogarithm ( $k = 2$ )

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1-x) \log|x|$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

# The relation with regulators

Deninger (1997)

$$m(P) = m(P^*) + \frac{1}{(-2i\pi)^{n-1}} \int_{\Gamma} \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$\eta_n(n)(x_1, \dots, x_n)$  is a  $\mathbb{R}(n-1)$ -valued smooth  $n-1$ -form in  $X(\mathbb{C})$ .

Example:

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$$m(1 + x + y) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |1 + x + y| \frac{dx}{x} \frac{dy}{y}$$

by Jensen's equality:

$$\begin{aligned} &= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ |1 + x| \frac{dx}{x} \\ &= \frac{1}{2\pi i} \int_{\Gamma} \log |y| \frac{dx}{x} = -\frac{1}{2\pi i} \int_{\Gamma} \eta_2(2)(x, y) \end{aligned}$$

where

$$\Gamma = \{1 + x + y = 0\} \cap \{|x| = 1, |y| \geq 1\}$$

## Properties of $\eta_n(n)(x_1, \dots, x_n)$

- ▶ Multiplicative in each variable, anti-symmetric.  
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- ▶  $\eta_n(n)(x, 1-x, x_1, \dots, x_{n-2}) = d\eta_{n-1}(n)(x, x_1, \dots, x_{n-2})$

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$$\eta_3(3)(x, y, z) = \log|x| \left( \frac{1}{3} d \log|y| \wedge d \log|z| + di \arg y \wedge di \arg z \right)$$

$$+ \log|y| \left( \frac{1}{3} d \log|z| \wedge d \log|x| + di \arg z \wedge di \arg x \right)$$

$$+ \log|z| \left( \frac{1}{3} d \log|x| \wedge d \log|y| + di \arg x \wedge di \arg y \right)$$

$$\eta_3(3)(x, 1-x, y) = d\eta_3(2)(x, y)$$

$$\eta_3(2)(x, y)$$

$$= \widehat{D}(x) di \arg y + \frac{1}{3} \log|y| (\log|1-x| d \log|x| - \log|x| d \log|1-x|)$$

First variable in  $\eta_n(n-1)$  behaves like the five-term relation

$$[x] + [y] + [1 - xy] + \left[ \frac{1 - x}{1 - xy} \right] + \left[ \frac{1 - x}{1 - xy} \right]$$

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Now

$$\eta_n(n-1)(x, x, x_1, \dots, x_{n-3}) = d\eta_n(n-2)(x, x_1, \dots, x_{n-3})$$

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$$\eta_n(2)(x, x) = d\eta_n(1)(x)$$

and

$$\eta_n(1)(x) = \widehat{\mathcal{L}}_n(x)$$

## Example in three variables

Smyth (1981)

$$m(1 - x + (1 - y)z) = \frac{7}{2\pi^2}\zeta(3)$$

$$m(P) = m(1 - y) - \frac{1}{(2\pi)^2} \int_{\Gamma} \eta_3(3)(x, y, z).$$

$$x \wedge y \wedge z = -x \wedge (1 - x) \wedge y - y \wedge (1 - y) \wedge x,$$

in other words,

$$\eta_3(3)(x, y, z) = -\eta_3(3)(x, 1 - x, y) - \eta_3(3)(y, 1 - y, x).$$

We have

$$m((1 - x) + (1 - y)z) = \frac{1}{4\pi^2} \int_{\gamma} \eta_3(2)(x, y) + \eta_3(2)(y, x).$$

$$\Gamma = \{P(x, y, z) = 0\} \cap \{|x| = |y| = 1, |z| \geq 1\}$$

$$\partial\Gamma = \{P(x, y, z) = 0\} \cap \{|x| = |y| = |z| = 1\}$$

Maillot:  $P \in \mathbb{R}[x, y, z]$ ,

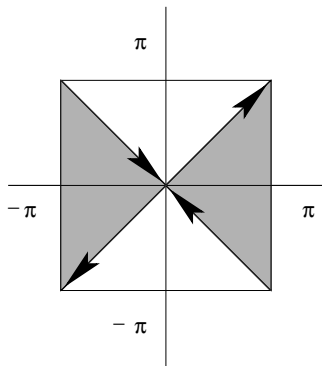
$$\gamma = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}.$$

$$C = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$

$$-\{x\}_2 \otimes y - \{y\}_2 \otimes x = \pm 2\{x\}_2 \otimes x.$$



$$m((1-x) + (1-y)z) = \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2} \zeta(3).$$

## New examples

Boyd & L. (2005)

$$m(x^2 + 1 + (x + 1)y + (x - 1)z) = \frac{L(\chi_{-4}, 2)}{\pi} + \frac{21}{8\pi^2}\zeta(3)$$

$$m(x^2 + x + 1 + (x + 1)y + z) = \frac{\sqrt{3}}{4\pi}L(\chi_{-3}, 2) + \frac{19}{6\pi^2}\zeta(3)$$

## An example in four variables

L.(2003)

$$\pi^3 m \left( 1 + x + \left( \frac{1 - x_1}{1 + x_1} \right) (1 + y)z \right) = 2\pi^2 L(\chi_{-4}, 2) + 8 \sum_{0 \leq j < k} \frac{(-1)^{j+k+1}}{(2j+1)^3 k}$$

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(2005)

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In general, for  $m$  odd,

$$\sum_{0 \leq j < k} \frac{(-1)^{j+k+1}}{(2j+1)^m k}$$
$$= mL(\chi_{-4}, m+1) + \sum_{h=1}^{\frac{m-1}{2}} \frac{(-1)^h \pi^{2h} (2^{2h} - 1)}{(2h)!} B_{2h} L(\chi_{-4}, m - 2h + 1).$$



# Generalized Mahler measure

Gon & Oyanagi (2004)

For  $f_1, \dots, f_r \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ ,

$$m(f_1, \dots, f_r) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \max\{\log |f_1|, \dots, \log |f_r|\} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

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Note

$$m(f_1, f_2) = m(f_1 + zf_2)$$

## Examples

The particular case when  $f_j = P(x_j)$  for some  $P \in \mathbb{C}[x]$ .  
Gon & Oyanagi (2004)

$$m(1 - x_1, \dots, 1 - x_n) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} c_{j,n} \frac{\zeta(2j+1)}{\pi^{2j}}$$

$$m(1 - x_1, 1 - x_2) = \frac{7}{2\pi^2} \zeta(3)$$

$$m(1 - x_1, 1 - x_2, 1 - x_3) = \frac{9}{2\pi^2} \zeta(3)$$

$$m(1 - x_1, 1 - x_2, 1 - x_3, 1 - x_4) = -\frac{93}{2\pi^4} \zeta(5) + \frac{9}{\pi^2} \zeta(3)$$

Can be also computed using regulators.

$|P(x)|$  is monotonous when  $0 \leq \arg x \leq \pi$ .

In this case,  $|P(x)| = 2 \left| \sin \frac{\arg x}{2} \right|$ .

$$m(P(x_1), \dots, P(x_n)) = \frac{n!}{(\pi i)^n} \int_{0 \leq \arg x_n \leq \dots \leq \arg x_1 \leq \pi} \eta(P(x_1), x_1, \dots, x_n)$$

L. (2005)

$$m\left(\frac{1-x_1}{1+x_1}, \dots, \frac{1-x_n}{1+x_n}\right) = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} c'_{j,n} \frac{\zeta(2j+1)}{\pi^{2j}}$$

$$m\left(\frac{1-x_1}{1+x_1}, \frac{1-x_2}{1+x_2}\right) = \frac{7}{\pi^2} \zeta(3)$$

$$m\left(\frac{1-x_1}{1+x_1}, \dots, \frac{1-x_3}{1+x_3}\right) = \frac{21}{2\pi^2} \zeta(3)$$

$$m\left(\frac{1-x_1}{1+x_1}, \dots, \frac{1-x_4}{1+x_4}\right) = -\frac{93}{\pi^4} \zeta(5) + \frac{21}{\pi^2} \zeta(3)$$

$m(1 + x_1 - x_1^{-1}, \dots, 1 + x_n - x_n^{-1}) =$  combination of polylogarithms.

$$m(1 + x_1 - x_1^{-1}) = -\log(\varphi),$$

$$\begin{aligned} & m(1 + x_1 - x_1^{-1}, 1 + x_2 - x_2^{-1}) \\ &= \frac{1}{\pi^2} \operatorname{Re}(\operatorname{Li}_3(\varphi^2) - \operatorname{Li}_3(-\varphi^2) + \operatorname{Li}_3(\varphi^{-2}) - \operatorname{Li}_3(-\varphi^{-2})) \end{aligned}$$

for  $\varphi = \frac{-1+\sqrt{5}}{2}$ .