Math 103. Section 205. Midterm 1a

February 8st, 2007

NAME:

STUDENT NUMBER:

1. This exam consists of 5 pages, including this one.
2. Ensure that **your *full* name and student number** appear on this page.
3. Read the questions carefully before starting to work.
4. Give complete arguments and explanations for all your calculations in questions 3, 4, 5, and 6.
5. Continue on the back of the page if you run out of space.
6. Attempt to answer all questions for partial credit.
7. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
8. Five minutes before the end of the test period you will be given a verbal notice. After that time, you must remain seated until all test papers have been collected.
9. When the test period is over, you will be instructed to put away writing implements. Put away all pens and pencils at this point.
10. Please remain seated and pass your test paper down the row to the nearest indicated aisle.

Question:	1	2	3	4	5	6	Total
Points:	16	15	10	12	22	25	100
Score:							

Summation Formulae

$$\sum_{k=1}^{N} C = NC \qquad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2 \qquad \sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r}$$

1. Multiple choice (only one option is correct in each case)

   (a) (5 points) If $\int_1^2 f(x)dx = 7$ and $\int_1^3 f(x)dx = 4$, what is $\int_2^3 f(x)dx =$?

   - ☐ $\int_2^3 f(x)dx = 3$
   - ☐ $\int_2^3 f(x)dx = 11$
   - ☐ $\int_2^3 f(x)dx = -3$
   - ☐ $\int_2^3 f(x)dx = \frac{11}{3}$

   Solution:

   $$\int_2^3 f(x)dx = -3$$

   (b) (5 points) If $\int_2^5 g(x)dx = 8$ and $\int_5^8 g(x)dx = 7$, what is the average value of g in $2 \leq x \leq 8$?

   - ☐ $\overline{g} = \frac{1}{3}$
   - ☐ $\overline{g} = \frac{15}{8}$
   - ☐ $\overline{g} = 15$
   - ☐ $\overline{g} = \frac{5}{2}$

   Solution:

   $$\overline{g} = \frac{5}{2}$$

   (c) (6 points) What is $\lim_{N\to\infty} \sum_{k=1}^{N} \left(3 + \sqrt{\frac{5k}{N}}\right) \frac{5}{N}$?

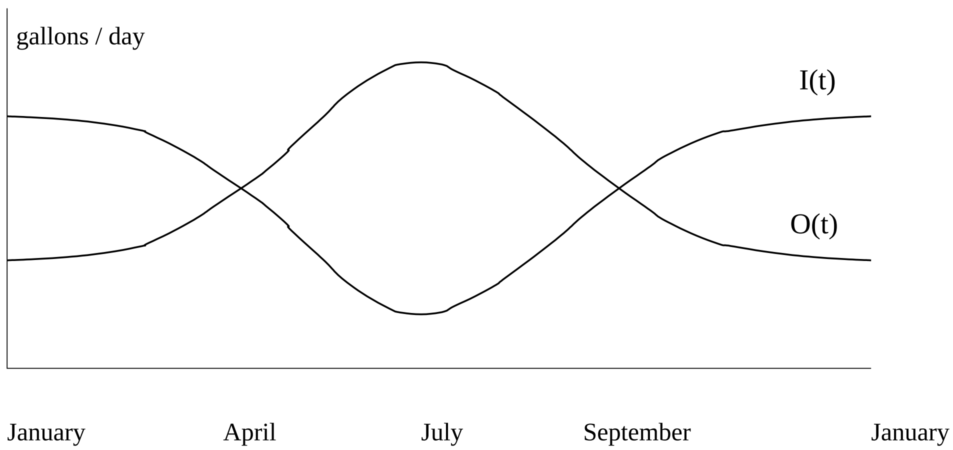
   - ☐ $\int_0^5 \left(3 + \sqrt{x}\right) dx$

- ☐ $\displaystyle\int_3^8 \sqrt{x}\,dx$
- ☐ 3
- ☐ $\displaystyle\int_0^1 \left(3+\sqrt{5x}\right)dx$

Solution:

$$\lim_{N\to\infty}\sum_{k=1}^{N}\left(3+\sqrt{\frac{5k}{N}}\right)\frac{5}{N}=\int_0^5 \left(3+\sqrt{x}\right)dx$$

2. The rate at which water flows in and out of Capilano Reservoir is described by two functions. $I(t)$ is the rate at which the water flows in to the reservoir (in gallons per day) and $O(t)$ is the rate at which water flows out (in gallons per day). Both functions follow a cyclic pattern that repeats every year. See sketch.



Answer True or False

(a) (5 points) The quantity of water is smallest in September.

Solution: TRUE

(b) (5 points) The maximum for the rate at which water flows out of the reservoir is in July.

Solution: TRUE

(c) (5 points) The quantity of water is greatest in January.

Solution: FALSE

3. (10 points) Calculate the following sum

$$\sum_{k=3}^{10} \frac{3}{2^k} =$$

Solution:

$$\sum_{k=3}^{10} \frac{3}{2^k} = 3\sum_{k=3}^{10} \frac{1}{2^k} = 3\left(\sum_{k=0}^{10} \frac{1}{2^k} - \sum_{k=0}^{2} \frac{1}{2^k}\right)$$

$$3\left(\left(\frac{1-\frac{1}{2^{11}}}{1-\frac{1}{2}}\right)-\left(\frac{1-\frac{1}{2^3}}{1-\frac{1}{2}}\right)\right)=3\left(2-\frac{1}{2^{10}}-2+\frac{1}{4}\right)=\frac{3}{4}-\frac{3}{2^{10}}=\frac{3\cdot 255}{1024}=\frac{765}{1024}$$

4. (12 points) Compute the following integral

$$\int_{-2}^{1}|x+1|dx =$$

Solution:

Since $x+1>0$ for $x>-1$ and $x+1<0$ for $x<-1$, we have

$$\int_{-2}^{1}|x+1|dx=\int_{-2}^{-1}-(x+1)dx+\int_{-1}^{1}(x+1)dx$$

$$=\left(-\frac{x^2}{2}-x\right)\Bigg|_{-2}^{-1}+\left(\frac{x^2}{2}+x\right)\Bigg|_{-1}^{1}=-\frac{1}{2}+1-\left(-\frac{4}{2}+2\right)+\frac{1}{2}+1-\left(\frac{1}{2}-1\right)=\frac{5}{2}$$

5. The speed of a mail delivery truck moving north-south on Granville St. is given by $v(t) = t(t-3)$ km /hour. Positive speed means that the truck is moving north. Negative speed means that the truck is moving south. Assume that the truck leaves the post-office at time $t = 0$.

(a) (7 points) Determine the distance between the truck and the post-office after 3 hours. Is the truck north or south of the post-office?

Solution:
The position is integral of the velocity, therefore

$$\int_0^3 t(t-3)dt = \left(\frac{t^3}{3} - 3\frac{t^2}{2}\right)\bigg|_0^3 = -\frac{9}{2}$$

Since the sign is negative, the truck is 4.5 km (or $\frac{9}{2}$ km) to the south.

(b) (7 points) Determine the distance that the truck travels between $t = 3$ and $t = 6$.

Solution:

$$\int_3^6 t(t-3)dt = \left(\frac{t^3}{3} - 3\frac{t^2}{2}\right)\bigg|_3^6 = \left(\frac{6^3}{3} - 3\frac{36}{2}\right) - \left(-\frac{9}{2}\right) = 18 + \frac{9}{2} = \frac{45}{2}$$

The truck covers 22.5 km (or $\frac{45}{2}$ km) between $t = 3$ and $t = 6$.

(c) (8 points) If the truck consumes 1 litre of gas every 10 km, calculate the amount of gas consumed in the first 6 hours. How far is the truck from the post-office at $t = 6$? Is it south or north?

Solution:
Total distance is $\frac{9}{2} + \frac{45}{2} = 27$ km. It consumes 2.7 litres of gas. The truck is $\frac{45}{2} - \frac{9}{2} = 18$ km away, to the north.

6. (25 points) Find the area of the region enclosed by the graphs of $y = f(x) = \frac{4}{x}$ and $y = g(x) = 5 - x$.

Solution:

We solve $5 - x = \frac{4}{x}$. Multiplying by x, we get, $5x - x^2 = 4$.

Observe that $x^2 - 5x + 4 = (x-1)(x-4)$. Therefore the intersections are in $x = 1, 4$. Now note that $f(2) = 2$, while $g(2) = 3$. Then we need to integrate $g - f$.

We write

$$\int_1^4 \left(5 - x - \frac{4}{x}\right) dx = 5x - \frac{x^2}{2} - 4\ln|x|\Big|_1^4 = \left(20 - \frac{16}{2} - 4\ln 4\right) - \left(5 - \frac{1}{2} - 4\ln 1\right)$$

$$= \frac{15}{2} - 4\ln 4 = \frac{15}{2} - 8\ln 2$$