HIGHER MAHLER MEASURE OF AN n-VARIABLE FAMILY

MATILDE N. LALÍN AND JEAN-SÉBASTIEN LECHASSEUR

ABSTRACT. We prove formulas for the k-higher Mahler measure of a family of rational functions with an arbitrary number of variables. Our formulas reveal relations with multiple polylogarithms evaluated at certain roots of unity.

1. INTRODUCTION

For k a positive integer, the k-higher Mahler measure of a non-zero, n-variable, rational function $P(x_1, \dots, x_n) \in \mathbb{C}(x_1, \dots, x_n)$ is given by

$$\mathrm{m}_k\left(P(x_1, \dots, x_n)\right) = \int_0^1 \dots \int_0^1 \log^k \left|P\left(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}\right)\right| d\theta_1 \dots d\theta_n.$$

We observe that the case $k = 1$ recovers the formula for the "classical" Mahler measure. This function, originally defined as a height on polynomials, has attracted considerable interest in the last decades due to its connection to special values of the Riemann zeta function, and of L-functions associated to objects of arithmetic significance such as elliptic curves as well as special values of polylogarithms and other special functions. Part of such phenomena has been explained in terms of Beilinson's conjectures via relationships with regulators by Deninger [Den97] (see also the crutial articles by Boyd [Boy97] and Rodriguez-Villegas [R-V97]).

Higher (and multiple) Mahler measures were originally defined in [KLO08] and subsequently studied by several authors [Sas10, BS11, LS11, BBSW12, BS12, Sas12, Bis14, BM14]. A related object, the Zeta Mahler measure, was first studied by Akatsuka in [Aka09]. As remarked by Deninger, higher Mahler measures are expected to yield different regulators than the ones that appear in the case of the usual Mahler measure and they may reveal a more complicated structure at the level of the periods (see [Lal10] for more details).

In order to continue this program of understanding periods that arise from higher Mahler measure, an essential component is to generate examples

2010 *Mathematics Subject Classification.* Primary 11R06; Secondary 11M06, 11R09.
Key words and phrases. Mahler measure, Higher Mahler measure, special values of $\zeta(s)$ and Dirichlet L-functions, polylogarithms.

of formulas of higher Mahler measure involving special functions that can be easily expressed as periods, such as polylogarithms. In the present work we consider the family of rational functions in $\mathbb{C}(x_1, \ldots, x_m, z)$ given by

$$R_m(x_1, \ldots, x_m, z) := z + \left(\frac{1 - x_1}{1 + x_1}\right) \cdots \left(\frac{1 - x_m}{1 + x_m}\right).$$

Let $\zeta(s)$ be the Riemann zeta function, and let $L(\chi_{-4}, s)$ be the Dirichlet L-function in the character of conductor 4, defined, for $\mathrm{Re}(s) > 1$, as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

$$L(\chi_{-4}, s) := \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^s}, \quad \chi_{-4}(n) = \begin{cases} \left(\frac{-1}{n}\right) & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases}$$

We also consider the functions

$$\mathcal{L}_{n_1,\ldots,n_m}(w_1, \ldots, w_m) = \sum_{(r_1,\ldots,r_m)\in\{0,1\}^m} (-1)^{r_m} \mathrm{Li}_{n_1,\ldots,n_m}((-1)^{r_1} w_1, \ldots, (-1)^{r_m} w_m), \tag{1.1}$$

given by combinations of multiple polylogarithms (of length m), defined for n_i positive integers by

$$\mathrm{Li}_{n_1,\ldots,n_m}(w_1, \ldots, w_m) = \sum_{0<j_1<\cdots<j_m} \frac{w_1^{j_1} \cdots w_m^{j_m}}{j_1^{n_1} \cdots j_m^{n_m}}.$$

The series above is absolutely convergent for $|w_i| \leq 1$ and $n_m > 1$.

Finally, for $a_1, \ldots a_m \in \mathbb{C}$, consider

$$s_\ell(a_1, \ldots, a_m) = \begin{cases} 1 & \text{if } \ell = 0, \\ \sum_{i_1<\cdots<i_\ell} a_{i_1} \cdots a_{i_\ell} & \text{if } 0 < \ell \leq m, \\ 0 & \text{if } m < \ell. \end{cases}$$

In the present article, we prove the following result.

Theorem 1.1. *We have, for $n \geq 1$, the following formula*

$$\mathrm{m}_k\left(R_{2n}\right) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2, 4^2, \ldots, (2n-2)^2)}{(2n-1)!} \left(\frac{2}{\pi}\right)^{2h} \mathcal{A}_k(h),$$

where

$$\begin{aligned}\mathcal{A}_k(h) :=&(2h+k-1)!\left(1-\frac{1}{2^{2h+k}}\right)\zeta(2h+k)\\ &+(-1)^k k! \sum_{\frac{k}{2}-1\leq n\leq k-2}\frac{1}{2^{k-n-1}}\\ &\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}}(2h-1)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h}(1,\dots,1)\\ &+\sum_{j=1}^{k-2}\frac{(-1)^{k+j}k!}{j!}\sum_{\frac{k-j}{2}-1\leq n\leq k-j-2}\frac{1}{2^{k-n-j}}\\ &\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}}(2h+j-1)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+j}(1,\dots,1).\end{aligned}$$

We have, for $n\geq 0$, the following formula

$$\mathrm{m}_k\left(R_{2n+1}\right)=\sum_{h=0}^{n}\frac{s_{n-h}(1^2,3^2,\dots,(2n-1)^2)}{(2n)!}\left(\frac{2}{\pi}\right)^{2h+1}\mathcal{B}_k(h)$$

where

$$\begin{aligned}\mathcal{B}_k(h) :=&(2h+k)!L(\chi_{-4},2h+k+1)\\ &+(-1)^{k+1} k! \sum_{\frac{k}{2}-1\leq n\leq k-2}\frac{1}{2^{k-n-1}}\\ &\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}}i(2h)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+1}(1,\dots,1,i,i)\\ &+\sum_{j=1}^{k-2}\frac{(-1)^{k+j+1}k!}{j!}\sum_{\frac{k-j}{2}-1\leq n\leq k-j-2}\frac{1}{2^{k-n-j}}\\ &\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}}i(2h+j)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+j+1}(1,\dots,1,i,i).\end{aligned}$$

For the sake of clarity, we record here the case of $k=2$.

$$\mathcal{A}_2(h) :=(2h+1)!\left(1-\frac{1}{2^{2h+2}}\right)\zeta(2h+2)+(2h-1)!\mathcal{L}_{2,2h}(1,1)$$
$$\mathcal{B}_2(h) :=(2h+2)!L(\chi_{-4},2h+3)-i(2h)!\mathcal{L}_{2,2h+1}(i,i)$$

The case of $k = 1$ clearly yields formulas that only depend on $\zeta(s)$, $L(\chi_{-4}, s)$ and powers of π. This is equivalent to saying that all the terms can be expressed in terms of polylogarithms of length one. There is another case in which we can prove a similar formula.

Corollary 1.2. *The previous result includes the following particular case*

$$\mathrm{m}_2(R_2) = -\frac{31\pi^2}{360}+\frac{28}{\pi^2}\log 2\zeta(3)+\frac{32}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{4}{3\pi^2}\log^2 2(\log^2 2-\pi^2), \tag{1.2}$$

where all the terms are product of polylogarithms of length one.

Our method of proof for Theorem 1.1 relies in the ideas of [Lal06a] combined with key properties of the Zeta Mahler function constructed by Akatsuka [Aka09].

The same method yields a formula for a simpler polynomial. Let

$$Q_m(x_1, \dots, x_m) := \left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_m-1}{x_m+1}\right).$$

We can express the higher Mahler measure of this family in terms of rational combinations of powers of π. More precisely, we obtain the following result.

Proposition 1.3. *We have, for $n \geq 1$, the following formula*

$$\begin{aligned}&\mathrm{m}_{2k}(Q_{2n})\\ &= \pi^{2k}\sum_{h=1}^{n}\frac{s_{n-h}(2^2, 4^2, \dots, (2n-2)^2)}{(2n-1)!}\frac{(-1)^{k+h+1}}{2(k+h)}2^{2h}\left(2^{2k+2h}-1\right)B_{2(k+h)}.\end{aligned}$$

We have, for $n \geq 0$, the following formula

$$\mathrm{m}_{2k}(Q_{2n+1}) = \left(\frac{\pi}{2}\right)^{2k}\sum_{h=0}^{n}\frac{s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)}{(2n)!}(-1)^{k+h}E_{2(k+h)}.$$

In addition, for $k \geq 0$ and $m \geq 1$,

$$\mathrm{m}_{2k+1}(Q_m) = 0.$$

In the above expressions, B_m and E_m are the Bernoulli and Euler numbers respectively. Formulas for $\mathrm{m}_k(Q_1)$ where recovered in [BBSW12].

It is not necessary to use the method of [Lal06a] to find formulas for $\mathrm{m}_k(Q_m)$. By using simple properties for the higher Mahler measure of a product of polynomials with different variables, we recover alternative expressions for the same formulas. By comparing with the results of Proposition 1.3, we obtain the following identities between Bernoulli and Euler numbers which generalize some results from the Appendix in [Lal06a].

Corollary 1.4.

$$\sum_{h=1}^{n} \frac{s_{n-h}(2^2, 4^2, \dots, (2n-2)^2)}{(2n-1)!} \frac{(-1)^{h+1}}{2(k+h)} 2^{2k+2h} \left(2^{2k+2h} - 1\right) B_{2(k+h)}$$
$$= \sum_{j_1+\cdots+j_{2n}=k} \binom{2k}{2j_1, \cdots, 2j_{2n}} E_{2j_1} \cdots E_{2j_{2n}},$$

and

$$\sum_{h=0}^{n} \frac{s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)}{(2n)!} (-1)^h E_{2(k+h)}$$
$$= \sum_{j_1+\cdots+j_{2n+1}=k} \binom{2k}{2j_1, \cdots, 2j_{2n+1}} E_{2j_1} \cdots E_{2j_{2n+1}}.$$

The present article is organized as follows. In Section 2 we recall previous results on the classical Mahler measure of R_n, as well as similar results for other rational functions that were obtained in [Lal06a]. We outline the general method of proof in Section 3. In Section 4 we discuss some properties of the Zeta Mahler measure. Sections 5 and 6 treat certain technical simplifications of the involved integrals. We discuss properties of polylogatithms in Section 7, and we present the final details of the proofs in Section 8. Some technical results were already part of [Lal06a] but we include them in this article for the sake of completeness.

2. Description of similar results for Mahler measure

In this section we present the previous results that were obtained with this method for the classical Mahler measure.

Theorem 2.1. *(*[Lal06a]*) We have the following identities. For $n \geq 1$,*

$$\mathrm{m}\left(R_{2n}\right) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2, 4^2, \dots, (2n-2)^2)}{(2n-1)!} \left(\frac{2}{\pi}\right)^{2h} \mathcal{A}_1(h),$$

where

$$\mathcal{A}_1(h) := (2h)! \left(1 - \frac{1}{2^{2h+1}}\right) \zeta(2h+1).$$

For $n \geq 0$,

$$\mathrm{m}\left(R_{2n+1}\right) = \sum_{h=0}^{n} \frac{s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)}{(2n)!} \left(\frac{2}{\pi}\right)^{2h+1} \mathcal{B}_1(h),$$

where

$$\mathcal{B}_1(h) := (2h+1)! L(\chi_{-4}, 2h+2).$$

In addition, similar results were proved in [Lal06a] for other rational functions by the same method. Let

$$\begin{aligned} &S_m(x_1,\dots,x_m,x,y,z)\\ &:=(1+x)z+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)(1+y),\\ &T_m(x_1,\dots,x_m,x,y)\\ &:=1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)\right)y. \end{aligned}$$

Theorem 2.2. *(*[Lal06a]*) We have the following identities. For $n \geq 1$,*

$$\mathrm{m}\left(S_{2n}\right)=\sum_{h=1}^{n}\frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!}\left(\frac{2}{\pi}\right)^{2h+2}\mathcal{C}_1(h),$$

where

$$\mathcal{C}_1(h):=\sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{4h}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3).$$

For $n \geq 0$,

$$\mathrm{m}\left(S_{2n+1}\right)=\sum_{h=0}^{n}\frac{s_{n-h}(1^2,3^2\dots,(2n-1)^2)}{(2n)!}\left(\frac{2}{\pi}\right)^{2h+3}\mathcal{D}_1(h),$$

where

$$\mathcal{D}_1(h):=\sum_{\ell=0}^{h}\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2(2h+1)}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+3)!L(\chi_{-4},2\ell+4).$$

For $n \geq 1$,

$$\mathrm{m}\left(T_{2n}\right)=\frac{\log 2}{2}+\sum_{h=1}^{n}\frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!}\left(\frac{2}{\pi}\right)^{2h}\mathcal{E}_1(h),$$

where

$$\begin{aligned}\mathcal{E}_1(h):=&\frac{(2h)!}{2}\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)+\sum_{\ell=1}^{h}(2^{2(h-\ell)-1}-1)\binom{2h}{2\ell}\frac{(-1)^{h-\ell+1}}{2h}\\ &\times B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell)!\left(1-\frac{1}{2^{2\ell+1}}\right)\zeta(2\ell+1).\end{aligned}$$

For $n \geq 0$,

$$\mathrm{m}\left(T_{2n+1}\right)=\frac{\log 2}{2}+\sum_{h=1}^{n}\frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n+1)!}\left(\frac{2}{\pi}\right)^{2h+2}\mathcal{F}_1(h)$$

where

$$\begin{aligned}\mathcal{F}_1(h) :=& \frac{(2h+2)!}{2}\left(1-\frac{1}{2^{2h+3}}\right)\zeta(2h+3)\\ &+\frac{\pi^2 n^2}{2}(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1)\\ &+n(2n+1)\sum_{\ell=1}^{h}(2^{2(h-\ell)-1}-1)\binom{2h}{2\ell}\frac{(-1)^{h-\ell+1}}{4h}\\ &\times B_{2(h-\ell)}\pi^{2h+2-2\ell}(2\ell)!\left(1-\frac{1}{2^{2\ell+1}}\right)\zeta(2\ell+1).\end{aligned}$$

We remark that the presentation of some of the above formulas differ from those in [Lal06a] as they have been simplified by recent results by the authors [LL].

3. The general method

Here we describe the general structure of the proof of Theorem 1.1, based on the ideas of [Lal06a].

Let $P_a \in \mathbb{C}(z)$ be such that its coefficients are rational functions in a parameter $a \in \mathbb{C}$. By making the change of variables $a = \left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_n-1}{x_n+1}\right)$ we can see the rational function P_a as a new rational function in $n+1$ variables, namely $\tilde{P} \in \mathbb{C}(x_1,\dots,x_n,z)$. Thus, the k-higher Mahler measure of $\tilde{P}$ is given as follows.

$$\begin{aligned}\mathrm{m}_k(\tilde{P}) &= \frac{1}{(2\pi i)^{n+1}}\int_{\mathbb{T}^{n+1}}\log^k|\tilde{P}|\frac{dx}{x}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\ &= \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\left(\frac{1}{(2\pi i)}\int_{\mathbb{T}}\log^k|\tilde{P}|\frac{dx}{x}\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}\\ &= \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\mathrm{m}_k\left(P_{\left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_n-1}{x_n+1}\right)}\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n},\end{aligned}$$

where the term $\mathrm{m}_k\left(P_{\left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_n-1}{x_n+1}\right)}\right)$ inside the integral is a function on the variables $x_1,\dots,x_n$. By making the change of variables $x_j = e^{i\theta_j}$, followed by $y_j = \tan(\theta_j/2)$, the integral above equals

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathrm{m}_k\left(P_{i^n\tan\left(\frac{\theta_1}{2}\right)\cdots\tan\left(\frac{\theta_n}{2}\right)}\right)d\theta_1\cdots d\theta_n\\ &=\frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{m}_k\left(P_{i^n y_1\cdots y_n}\right)\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}.\end{aligned}$$

Assume that the function on the parameter a given by $\mathrm{m}_k(P_a)$ is even, namely, that it verifies $\mathrm{m}_k(P_a) = \mathrm{m}_k(P_{-a})$ (this happens if, for instance,

$m_k(P_a)$ only depends on $|a|$). Then the same can be said for the integrand and the previous expression equals

$$\begin{aligned}&\frac{2^n}{\pi^n}\int_0^\infty\cdots\int_0^\infty m_k\left(P_{i^n y_1\cdots y_n}\right)\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}\\&=\frac{2^n}{\pi^n}\int_0^\infty\cdots\int_0^\infty m_k\left(P_{i^n\hat{x}_n}\right)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1}d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2},\end{aligned}$$

where we have set $\hat{x}_j = \prod_{i=1}^j y_i$. This change of variables is motivated by the goal of recovering a term of the form $m_k(P_x)$ inside the integral. Indeed, if we further assume that $m_k(P_a)$ depends only on $|a|$, we have that $m_k\left(P_{i^n\hat{x}_n}\right) = m_k\left(P_{\hat{x}_n}\right)$.

Now choose $P_a = z + a$. Then $\tilde{P} = R_n$. As we will see in Section 4, $m_k(P_a)$ is a function of $|a|$. This implies

(3.1)

$$\begin{aligned}&m_k(R_n)\\&=\frac{2^n}{\pi^n}\int_0^\infty\cdots\int_0^\infty m_k(z+\hat{x}_n)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1}d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}.\end{aligned}$$

Thus, if we have a good expression for $m_k(z + x)$, then, under favorable circumstances we may obtain a good expression for $m_k(R_n)$.

4. Zeta Mahler measure

In this section we discuss the Zeta Mahler measure, an object that is closely related to higher Mahler measure and that willl allow us to compute $m_k(z + a)$ for any $a \in \mathbb{C}$.

Definition 4.1. Let $P \in \mathbb{C}(x_1, \ldots, x_n)$ be a non-zero, n-variable, rational function. Its Zeta Mahler measure is defined by

$$Z(s,P) := \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}|P(x_1,\ldots,x_n)|^s\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

Remark 4.2. It is not hard to see that

$$\left.\frac{d^k Z(s,P)}{ds^k}\right|_{s=0} = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\log^k|P(x_1,\ldots,x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} = m_k(P).$$

The simplest possible case of Zeta Mahler measure was computed by Akatsuka.

Theorem 4.3 (Akatsuka, [Aka09])**.** *Let $a \in \mathbb{C}$ such that $|a| \neq 1$. We then have*

$$Z(s,z+a) = \begin{cases} {}_2F_1\left(\frac{-s}{2},\frac{-s}{2};1;|a|^2\right) & \text{if } |a|<1,\\[2ex] |a|^s\cdot {}_2F_1\left(\frac{-s}{2},\frac{-s}{2};1;|a|^{-2}\right) & \text{if } |a|>1,\end{cases}$$

where, for $|t|<1$,

$$ {}_2F_1(\alpha,\beta;\gamma;t)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{t^n}{n!} $$

denotes the hypergeometric series, and

$$ (\alpha)_n:=\begin{cases}1 & n=0,\\ \alpha(\alpha+1)\cdots(\alpha+n-1) & n\geq 1,\end{cases} $$

is the Pochhammer symbol.

Formulas for $\mathrm{m}_k(z+a)$ can be derived from $Z(s,z+a)$ by means of Remark 4.2. We proceed to compute some derivatives of $Z(s,z+a)$.

Lemma 4.4. *Let* $t\in\mathbb{C}$ *such that* $|t|<1$ *and*

$$ G_t(s)={}_2F_1\left(\frac{-s}{2},\frac{-s}{2};1;t\right)=\sum_{n=0}^{\infty}\left(\frac{-s}{2}\right)_n^2\frac{t^n}{(n!)^2}. $$

Then

$$ G_t(0)=1, $$

$$ G_t'(0)=0. $$

Proof. Indeed, setting $s=0$ we obtain that $\left(\frac{-s}{2}\right)_n=0$ unless $n=0$ and in that case $\left(\frac{-s}{2}\right)_0=1$. This implies that $G_t(0)=1$.

Differentiating $G_t(s)$, we obtain

$$ G_t'(s)=\sum_{n=1}^{\infty}2\left(\frac{-s}{2}\right)_n\left(\frac{-s}{2}\right)_n'\frac{t^n}{(n!)^2}. $$

If we now set $s=0$, we see that each term in the sum equals 0 and therefore $G_t'(0)=0$. □

Akatsuka [Aka09] found a formula for $\mathrm{m}_k(z+a)$ for $|a|<1$. This formula can be easily adapted to the general case of $a\in\mathbb{C}$.

Theorem 4.5. *Let*

$$ L_{(n_1,\dots,n_m)}(w):=\sum_{0<j_1<\cdots<j_m}\frac{w^{j_m}}{j_1^{n_1}\cdots j_m^{n_m}}. $$

We have, for $|a|\leq 1$ *and* $k\geq 2$,

$$ \mathrm{m}_k(z+a)=(-1)^k k!\sum_{\frac{k}{2}-1\leq n\leq k-2}\frac{1}{2^{2(k-n-1)}}\sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}}L_{(\epsilon_1,\dots,\epsilon_n,2)}(|a|^2). $$

For $|a|\geq 1$ *and* $k\geq 2$,

$$\mathrm{m}_k(z+a) = \log^k|a| + \sum_{j=0}^{k-2}\binom{k}{j}(-1)^{k-j}(k-j)! \sum_{\frac{k-j}{2}-1\leq n\leq k-j-2}\frac{1}{2^{2(k-j-n-1)}} \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}} \log^j|a| L_{(\epsilon_1,\dots,\epsilon_n,2)}(|a|^{-2})$$

Proof. The case of $|a|<1$ is Theorem 7 in [Aka09]. It is proved by observing that

$$Z(s,z+a)=G_{|a|^2}(s)$$

and

$$\mathrm{m}_k(z+a) = \left.\frac{d^kZ(s,z+a)}{ds^k}\right|_{s=0} = G^{(k)}_{|a|^2}(0).$$

Akatsuka applies the eighth formula in page 485 of [OZ01] for $\alpha=\beta$ and $x=0$ in order to conclude the result.

For $|a|>1$, we have, by Theorem 4.3,

$$Z(s,z+a)=|a|^sG_{|a|^{-2}}(s).$$

Thus we get, by Lemma 4.4,

$$\begin{aligned}\mathrm{m}_k(z+a) &= \left.\frac{d^kZ(s,z+a)}{ds^k}\right|_{s=0}\\ &=\sum_{j=0}^{k}\binom{k}{j}\log^j|a|G^{(k-j)}_{|a|^{-2}}(0)\\ &=\log^k|a| + \sum_{j=0}^{k-2}\binom{k}{j}\log^j|a|(-1)^{k-j}(k-j)!\\ &\quad\times\sum_{\frac{k-j}{2}-1\leq n\leq k-j-2}\frac{1}{2^{2(k-j-n-1)}}\sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}} L_{(\epsilon_1,\dots,\epsilon_n,2)}(|a|^{-2})\end{aligned}$$

and the result follows.

Finally, the case $|a|=1$ is considered in the third formula in page 273 of [KLO08]. It is not hard to see that both formulas given above remain true for $|a|=1$.

□

5. Integral simplification

In this section we discuss how to simplify the integral in Equation (3.1). In order to achieve this, we define certain polynomials and prove a recurrence relation for them. We then use these polynomials to compute certain family of integrals.

Definition 5.1. Let $A_m(x) \in \mathbb{Q}[x]$ be defined by

$$R(T;x) = \frac{e^{xT}-1}{\sin T} = \sum_{m\geq 0} A_m(x)\frac{T^m}{m!}.$$

Thus, $A_0(x) = x$, $A_1(x) = \frac{x^2}{2}$, $A_2(x) = \frac{x^3}{3} + \frac{x}{3}$, etc.

Lemma 5.2. *The polynomials $A_m(x)$ satisfy the following recurrence.*

$$A_m(x) = \frac{x^{m+1}}{m+1} + \frac{1}{m+1}\sum_{\substack{j>1\\ \text{odd}}}^{m+1}(-1)^{\frac{j+1}{2}}\binom{m+1}{j}A_{m+1-j}(x). \tag{5.1}$$

Proof. By writing $\sin T = \frac{e^{iT}-e^{-iT}}{2i}$, we obtain,

$$e^{xT}-1 = \left(\frac{e^{iT}-e^{-iT}}{2i}\right)R(T;x).$$

In other words,

$$\sum_{m\geq 1}^{\infty}\frac{x^mT^m}{m!} = \sum_{\substack{j>0\\ \text{odd}}}\frac{(-1)^{\frac{j-1}{2}}T^j}{j!}\sum_{\ell\geq 0}A_\ell(x)\frac{T^\ell}{\ell!}.$$

The result is obtained by comparing the coefficient of T^{m+1} in both sides of the above equality. □

Remark 5.3. More properties of the polynomials $A_m(x)$ can be found in the Appendix to [Lal06a]. For instance, one can prove that

$$A_m(x) = -\frac{2}{m+1}\sum_{h=0}^{m}B_h\binom{m+1}{h}(2^{h-1}-1)i^hx^{m+1-h},$$

where the B_n are the Bernoulli numbers.

We will eventually compute certain integral. For this, we need the following auxiliary result.

Lemma 5.4. *For $0<\beta<1$, we have the following integral evaluation:*

$$\int_0^\infty \frac{x^\beta dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi(a^{\beta-1}-b^{\beta-1})}{2\cos\frac{\pi\beta}{2}(b^2-a^2)}. \tag{5.2}$$

Proof. We decompose the integrand by means of partial fractions in order to obtain the following.

$$\int_0^\infty \frac{x^\beta dx}{(x^2+a^2)(x^2+b^2)} = \int_0^\infty\left(\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right)\frac{x^\beta dx}{(b^2-a^2)}. \tag{5.3}$$

By integrating over a well-chosen contour (see page 159, Section 5.3 of [Ahl79]), we obtain,

$$\int_0^\infty \frac{x^\beta dx}{(x^2+a^2)} = \frac{1}{1-e^{2\pi i\beta}}2\pi i\sum_{x\neq 0}\operatorname{Res}\left\{\frac{x^\beta}{x^2+a^2}\right\} = \frac{\pi a^{\beta-1}}{2\cos\frac{\pi\beta}{2}}.$$

By replacing this in (5.3), the result follows. □

We will use the polynomials $A_m(x)$ to compute certain integrals.

Proposition 5.5. *For $m \geq 0$, we have the following equation.*

$$\int_0^\infty \frac{x \log^m x dx}{(x^2+a^2)(x^2+b^2)} = \left(\frac{\pi}{2}\right)^{m+1} \frac{A_m\left(\frac{2\log a}{\pi}\right) - A_m\left(\frac{2\log b}{\pi}\right)}{a^2-b^2}$$

Proof. Let

$$f(\beta) := \int_0^\infty \frac{x^\beta dx}{(x^2+a^2)(x^2+b^2)}.$$

As the integral converges for $0 < \beta < 3$, the function is well-defined and continuous in this interval. We differentiate m times and obtain

$$f^{(m)}(1) = \int_0^\infty \frac{x \log^m x dx}{(x^2+a^2)(x^2+b^2)}.$$

Lemma 5.4 implies, for $0 < \beta < 1$,

$$f(\beta) \cos \frac{\pi\beta}{2} = \frac{\pi(a^{\beta-1} - b^{\beta-1})}{2(b^2-a^2)}.$$

By differentiating m times, we obtain

$$\sum_{j=0}^m \binom{m}{j} f^{(m-j)}(\beta) \left(\cos\frac{\pi\beta}{2}\right)^{(j)} = \frac{\pi}{2(b^2-a^2)}(a^{\beta-1}\log^m a - b^{\beta-1}\log^m b).$$

We now take the limit of $\beta \to 1$ in order to obtain

$$\sum_{\substack{j=1\\ \text{odd}}}^m (-1)^{\frac{j+1}{2}} \binom{m}{j} f^{(m-j)}(1) \left(\frac{\pi}{2}\right)^j = \frac{\pi(\log^m a - \log^m b)}{2(b^2-a^2)}.$$

Changing m to $m+1$, isolating the term $f^{(m)}(1)$, and dividing by $\frac{\pi}{2}(m+1)$,

$$\begin{aligned} f^{(m)}(1) =& \frac{1}{m+1} \sum_{\substack{j>1\\ \text{odd}}}^{m+1} (-1)^{\frac{j+1}{2}} \binom{m+1}{j} f^{(m+1-j)}(1) \left(\frac{\pi}{2}\right)^{j-1} \\ &+ \frac{\log^{m+1} a - \log^{m+1} b}{(m+1)(a^2-b^2)}. \end{aligned}$$

For $m = 0$ the above equation becomes

$$f^{(0)}(1) = f(1) = \frac{\log^m a - \log^m b}{a^2-b^2} = \left(\frac{\pi}{2}\right) \frac{A_0\left(\frac{2\log a}{\pi}\right) - A_0\left(\frac{2\log b}{\pi}\right)}{a^2-b^2}.$$

The rest of the proof proceeds by induction, using the recurrence for $A_m(x)$ that was proved in Lemma 5.2. □

By Proposition 5.5, we can write the integral in Formula (3.1) as a sum of simpler integrals. For instance, for $n=2$, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \mathrm{m}_k(z+\hat{x}_2)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\\
&=\int_0^\infty \mathrm{m}_k(z+\hat{x}_2)\left(\int_0^\infty \frac{\hat{x}_1 d\hat{x}_1}{(\hat{x}_1^2+\hat{x}_2^2)(\hat{x}_1^2+1)}\right)d\hat{x}_2\\
&=\int_0^\infty \mathrm{m}_k(z+\hat{x}_2)\left(\left(\frac{\pi}{2}\right)\frac{A_0\left(\frac{2\log\hat{x}_2}{\pi}\right)-A_0\left(\frac{2\log 1}{\pi}\right)}{\hat{x}_2^2-1^2}\right)d\hat{x}_2\\
&=\int_0^\infty \mathrm{m}_k(z+\hat{x}_2)\left(\left(\frac{\pi}{2}\right)\frac{\frac{2\log\hat{x}_2}{\pi}}{\hat{x}_2^2-1}\right)d\hat{x}_2\\
&=\int_0^\infty \mathrm{m}_k(z+x)\log x\frac{dx}{x^2-1}.
\end{aligned}$$

Analogously, for $n=3$, we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{m}_k(z+\hat{x}_3)\frac{\hat{x}_1 d\hat{x}_1}{\hat{x}_1^2+1}\frac{\hat{x}_2 d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\frac{d\hat{x}_3}{\hat{x}_3^2+\hat{x}_2^2}\\
&=\frac{\pi^2}{8}\int_0^\infty \mathrm{m}_k(z+x)\frac{dx}{x^2+1}+\frac{1}{2}\int_0^\infty \mathrm{m}_k(z+x)\log^2 x\frac{dx}{x^2+1}.
\end{aligned}$$

More generally, we can always reduce the computation to a sum of integrals of the form

$$\begin{cases}\displaystyle\int_0^\infty \mathrm{m}_k(z+x)\log^{2h+1}x\frac{dx}{x^2-1} & \text{for } n \text{ even},\\[2ex] \displaystyle\int_0^\infty \mathrm{m}_k(z+x)\log^{2h}x\frac{dx}{x^2+1} & \text{for } n \text{ odd}.\end{cases} \tag{5.4}$$

6. Coefficient formulas

In this section, we find the coefficients that allow us to express the integral from (3.1) as a linear combination of the integrals from (5.4). Let $\phi(a)$ be a function that depends on $|a|$. Eventually we will have $\phi(a)=\mathrm{m}_k(P_a)$, where P_a is a rational function such that $\mathrm{m}_k(P_a)=\mathrm{m}_k(P_{|a|})$ (for instance, we could take $P_a=z+a$). In what follows, it is assumed that $\phi(a)$ is such that all the integrals converge.

Definition 6.1. For $n\geq 1$ and $0\leq h\leq n-1$, let $a_{n,h}\in\mathbb{Q}$ be defined by

$$\begin{aligned}
&\int_0^\infty\cdots\int_0^\infty \phi(x_1)\frac{x_{2n}dx_{2n}}{x_{2n}^2+1}\frac{x_{2n-1}dx_{2n-1}}{x_{2n-1}^2+x_{2n}^2}\cdots\frac{dx_1}{x_1^2+x_2^2}\\
&=\sum_{h=1}^n a_{n,h-1}\left(\frac{\pi}{2}\right)^{2n-2h}\int_0^\infty \phi(x)\log^{2h-1}x\frac{dx}{x^2-1}.
\end{aligned}\tag{6.1}$$

For $n \geq 0$ and $0 \leq h \leq n$, let $b_{n,h} \in \mathbb{Q}$ be defined by

$$\int_0^\infty \cdots \int_0^\infty \phi(x_1) \frac{x_{2n+1}dx_{2n+1}}{x_{2n+1}^2+1} \frac{x_{2n}dx_{2n}}{x_{2n}^2+x_{2n+1}^2} \cdots \frac{dx_1}{x_1^2+x_2^2}$$

$$= \sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \phi(x) \log^{2h} x \frac{dx}{x^2+1}. \tag{6.2}$$

Lemma 6.2. *We have the following identities:*

$$\sum_{h=0}^{n} b_{n,h} x^{2h} = \sum_{h=1}^{n} a_{n,h-1} \left(A_{2h-1}(x) - A_{2h-1}(i)\right), \tag{6.3}$$

$$\sum_{h=1}^{n+1} a_{n+1,h-1} x^{2h-1} = \sum_{h=0}^{n} b_{n,h} A_{2h}(x), \tag{6.4}$$

where the $A_m(x)$ are the polynomials given in Definition 5.1.

Proof. By making the change of variables $x_i = y_i x_{2n+1}$ for $i = 1, \dots, 2n$, we have,

$$\int_0^\infty \cdots \int_0^\infty \phi(x_1) \frac{x_{2n+1}dx_{2n+1}}{x_{2n+1}^2+1} \frac{x_{2n}dx_{2n}}{x_{2n}^2+x_{2n+1}^2} \cdots \frac{dx_1}{x_1^2+x_2^2}$$
$$= \int_0^\infty \cdots \int_0^\infty \phi(y_1 x_{2n+1}) \frac{dx_{2n+1}}{x_{2n+1}^2+1} \frac{y_{2n}dy_{2n}}{y_{2n}^2+1} \frac{y_{2n-1}dy_{2n-1}}{y_{2n-1}^2+y_{2n}^2} \cdots \frac{dy_1}{y_1^2+y_2^2}.$$

We now ignore the variable x_{2n+1} and apply Equation (6.1) where $\int_0^\infty \phi(y_1 x_{2n+1}) \frac{dx_{2n+1}}{x_{2n+1}^2+1}$ is the function depending on y_1. The above equals

$$\sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty \phi(y_1 x_{2n+1}) \frac{dx_{2n+1}}{x_{2n+1}^2+1} \log^{2h-1} y_1 \frac{dy_1}{y_1^2-1}.$$

We set $x = y_1 x_{2n+1}$ and rename $y = y_1$. We obtain

$$\sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty \phi(x) \frac{y dx}{x^2+y^2} \log^{2h-1} y \frac{dy}{y^2-1}.$$

By applying Equation (6.2), we conclude

$$\sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \phi(x) \log^{2h} x \frac{dx}{x^2+1}$$
$$= \sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty \phi(x) y \log^{2h-1} y \frac{dy}{y^2-1} \frac{dx}{x^2+y^2}. \tag{6.5}$$

By Proposition 5.5 with $a = x$ and $b = i$, we obtain

$$\int_0^\infty \frac{y \log^{2h-1} y dy}{(y^2+x^2)(y^2-1)} = \left(\frac{\pi}{2}\right)^{2h} \frac{A_{2h-1}\left(\frac{2\log x}{\pi}\right) - A_{2h-1}(i)}{x^2+1}.$$

Thus, (6.5) equals

$$\sum_{h=1}^{n} a_{n,h-1}\left(\frac{\pi}{2}\right)^{2n}\int_0^\infty \phi(x)\left(A_{2h-1}\left(\frac{2\log x}{\pi}\right)-A_{2h-1}(i)\right)\frac{dx}{x^2+1}.$$

This equality is true for any choice of $\phi(x)$ and therefore (6.3) must hold.

Analogously, we can prove that

$$\sum_{h=1}^{n+1} a_{n+1,h-1}\left(\frac{\pi}{2}\right)^{2n+2-2h}\int_0^\infty \phi(x)\log^{2h-1}x\frac{dx}{x^2-1} \tag{6.6}$$
$$=\sum_{h=0}^{n} b_{n,h}\left(\frac{\pi}{2}\right)^{2n-2h}\int_0^\infty\int_0^\infty \phi(x)y\log^{2h}y\frac{dy}{y^2+1}\frac{dx}{x^2+y^2}.$$

By Proposition 5.5 with $a=x$ and $b=1$,

$$\int_0^\infty \frac{y\log^{2h}ydy}{(y^2+x^2)(y^2+1)}=\left(\frac{\pi}{2}\right)^{2h+1}\frac{A_{2h}\left(\frac{2\log x}{\pi}\right)-A_{2h}(0)}{x^2-1}.$$

Thus, (6.6) becomes

$$\sum_{h=0}^{n} b_{n,h}\left(\frac{\pi}{2}\right)^{2n+1}\int_0^\infty \phi(x)A_{2h}\left(\frac{2\log x}{\pi}\right)\frac{dx}{x^2-1}.$$

By comparing the above equation with (6.6), we obtain Equation (6.4). □

We now prove a result that will be key in finding formulas for $a_{n,h}$ and $b_{n,h}$.

Lemma 6.3. *We have the following identitites:*

$$2n(-1)^\ell s_{n-\ell}(2^2,4^2,\dots,(2n-2)^2)$$
$$=\sum_{h=\ell}^{n}(-1)^h\binom{2h}{2\ell-1}s_{n-h}(1^2,3^2,\dots,(2n-1)^2) \tag{6.7}$$
$$(2n+1)(-1)^\ell s_{n-\ell}(1^2,3^2,\dots,(2n-1)^2)$$
$$=\sum_{h=\ell}^{n}(-1)^h\binom{2h+1}{2\ell}s_{n-h}(2^2,4^2,\dots,(2n)^2). \tag{6.8}$$

Proof. We multiply by $x^{2\ell}$ in both sides of (6.7) and we sum for $\ell=1,\dots n$.

$$2n\sum_{\ell=1}^{n} s_{n-\ell}(2^2,4^2,\dots,(2n-2)^2)(-1)^\ell x^{2\ell}$$
$$=\sum_{\ell=1}^{n}\sum_{h=\ell}^{n}(-1)^h\binom{2h}{2\ell-1}s_{n-h}(1^2,3^2,\dots,(2n-1)^2)x^{2\ell}.$$

By identifying the left-hand side with the corresponding polynomial and reversing the sums on the right-hand side, we conclude that it suffices to prove that

$$2n\prod_{j=0}^{n-1}((2j)^2-x^2)$$
$$=\sum_{h=1}^{n}(-1)^h s_{n-h}(1^2,3^2,\dots,(2n-1)^2)\sum_{\ell=1}^{h}\binom{2h}{2\ell-1}x^{2\ell}.$$

The right-hand side equals

$$\sum_{h=1}^{n}(-1)^h s_{n-h}(1^2,3^2,\dots,(2n-1)^2)\frac{x}{2}((x+1)^{2h}-(x-1)^{2h})$$
$$=\frac{x}{2}\left(\prod_{j=1}^{n}((2j-1)^2-(x+1)^2)-\prod_{j=1}^{n}((2j-1)^2-(x-1)^2)\right)$$
$$=\frac{x}{2}\left(\prod_{j=1}^{n}(2j+x)(2j-2-x)-\prod_{j=1}^{n}(2j+x-2)(2j-x)\right).$$

By inspecting the common zeros in both products, we obtain that the line above equals

$$((-x)(2n+x)-x(2n-x))\frac{x}{2}\prod_{j=1}^{n-1}((2j)^2-x^2)$$
$$=2n\prod_{j=0}^{n-1}((2j)^2-x^2).$$

This concludes the proof for Equation (6.7).

For Equation (6.8) we proceed analogously by multiplying each side by $x^{2\ell+1}$ and summing over $\ell=1,\dots,n$.

$$(2n+1)\sum_{\ell=1}^{n}s_{n-\ell}(1^2,3^2,\dots,(2n-1)^2)(-1)^\ell x^{2\ell+1}$$
$$=\sum_{\ell=1}^{n}\sum_{h=\ell}^{n}(-1)^h\binom{2h+1}{2\ell}x^{2\ell+1}.$$

Then, it suffices to prove that

$$(2n+1)x\prod_{j=1}^{n}((2j-1)-x^2)$$
$$=\sum_{h=1}^{n}(-1)^h s_{n-h}(2^2,4^2,\dots,(2n)^2)\sum_{l=1}^{h}\binom{2h+1}{2l}x^{2l+1}.$$

The right-hand side equals

$$\begin{aligned}
&\sum_{h=0}^{n}(-1)^h s_{n-h}(2^2,4^2,\dots,(2n)^2)\frac{x}{2}((x+1)^{2h+1}-(x-1)^{2h+1})\\
=&\frac{x}{2}\left((x+1)\prod_{j=1}^{n}((2j)^2-(x+1)^2)-(x-1)\prod_{j=1}^{n}((2j)^2-(x-1)^2)\right)\\
=&\frac{x}{2}\left((x+1)\prod_{j=1}^{n}(2j+1+x)(2j-1-x)\right.\\
&\left.-(x-1)\prod_{j=1}^{n}(2j-1+x)(2j+1-x)\right)\\
=&((2n+1+x)+(2n+1-x))\frac{x}{2}\prod_{j=1}^{n}((2j-1)^2-x^2)\\
=&(2n+1)x\prod_{j=1}^{n}((2j-1)^2-x^2)
\end{aligned}$$

and this concludes the proof for Equation (6.8). □

Remark 6.4. Mathew Rogers has remarked that the sequences under consideration are related with Stirling numbers of the first class in the following way.

$$s_{n-h}(2^2,4^2,\dots,(2n-2)^2)=2^{2n-2h}\sum_{m=0}^{2h}(-1)^{h-m}S_n^{(m)}S_n^{(2h-m)}$$

We are now ready to compute the coefficients $a_{n,h}$ and $b_{n,h}$ from Definition 6.1.

Theorem 6.5. *We have the following identities. For $n \geq 1$,*

$$\sum_{h=0}^{n-1} a_{n,h}x^{2h} = \frac{(x^2+2^2)\cdots(x^2+(2n-2)^2)}{(2n-1)!},$$

and for $n \geq 0$,

$$\sum_{h=0}^{n} b_{n,h}x^{2h} = \frac{(x^2+1^2)(x^2+3^2)\cdots(x^2+(2n-1)^2)}{(2n)!}.$$

In other words,

$$a_{n,h} = \frac{s_{n-h-1}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!}, \tag{6.9}$$

$$b_{n,h} = \frac{s_{n-h}(1^2,3^2,\dots,(2n-1)^2)}{(2n)!}. \tag{6.10}$$

Proof. We proceed by recurrence. By definition, when $2n+1=1$, we have that $n=0$ and

$$\int_0^\infty \phi(x)\frac{dx}{x^2+1} = b_{0,0}\int_0^\infty \phi(x)\frac{dx}{x^2+1}.$$

Therefore, $b_{0,0}=1$.

Analogously, when $2n=2$ we have that $n=1$. We have seen that

$$\begin{aligned}\int_0^\infty\int_0^\infty \phi(x)\frac{ydy}{y^2+1}\frac{dx}{x^2+y^2} =& \int_0^\infty \phi(x)\frac{\log x dx}{x^2-1}\\ =&a_{1,0}\int_0^\infty \phi(x)\frac{\log x dx}{x^2-1}.\end{aligned}$$

Thus $a_{1,0}=1$ and the result holds for the first two cases. Now assume that for a fixed $n\geq 1$ and all $0\leq h\leq n-1$ we have that

$$a_{n,h}=\frac{s_{n-h-1}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!}.$$

We will then prove that for all $0\leq h\leq n$,

$$b_{n,h}=\frac{s_{n-h}(1^2,3^2,\dots,(2n-1)^2)}{(2n)!}.$$

By Lemma 6.2, it suffices to show that

$$\begin{aligned}&\sum_{h=0}^n s_{n-h}(1^2,3^2,\dots,(2n-1)^2)x^{2h}\\ &=2n\sum_{h=1}^n s_{n-h}(2^2,\dots,(2n-2)^2)\left(A_{2h-1}(x)-A_{2h-1}(i)\right).\end{aligned}$$

Taking $m=2h-1$ in Equation (5.1), we obtain

$$x^{2h}=\sum_{j=0}^{h-1}(-1)^j\binom{2h}{2j+1}A_{2h-2j-1}(x).$$

Multiplying by $s_{n-h}(1^2,3^2,\dots,(2n-1)^2)$ and summing over h, we get

$$\begin{aligned}&\sum_{h=0}^n s_{n-h}(1^2,3^2,\dots,(2n-1)^2)x^{2h}\\ =&\sum_{h=1}^n s_{n-h}(1^2,3^2,\dots,(2n-1)^2)\sum_{j=0}^{h-1}(-1)^j\binom{2h}{2j+1}A_{2h-2j-1}(x)\\ &+s_n(1^2,3^2,\dots,(2n-1)^2).\end{aligned}$$

Evaluating this equation at $x = i$, we obtain,

$$\begin{aligned}&\sum_{h=0}^{n} s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)(-1)^h\\ =&\sum_{h=1}^{n} s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)\sum_{j=0}^{h-1}(-1)^j\binom{2h}{2j+1}A_{2h-2j-1}(i)\\ &+ s_n(1^2, 3^2, \dots, (2n-1)^2).\end{aligned}$$

Since

$$\sum_{h=0}^{n} s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)(-1)^h = (x+1^2)\cdots(x+(2n-1)^2)|_{x=-1} = 0,$$

we deduce that

$$\begin{aligned}&\sum_{h=0}^{n} s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)x^{2h}\\ =&\sum_{h=1}^{n} s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)\\ &\times\sum_{j=0}^{h-1}(-1)^j\binom{2h}{2j+1}(A_{2h-2j-1}(x) - A_{2h-2j-1}(i)).\end{aligned}$$

By setting $\ell = h - j$, the right-hand side becomes

$$\begin{aligned}&=\sum_{h=1}^{n} s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)\sum_{\ell=1}^{h}(-1)^{h-\ell}\binom{2h}{2\ell-1}(A_{2\ell-1}(x) - A_{2\ell-1}(i))\\ &=\sum_{\ell=1}^{n}\left(\sum_{h=\ell}^{n}(-1)^h\binom{2h}{2\ell-1}s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)\right)\\ &\times(-1)^\ell(A_{2\ell-1}(x) - A_{2\ell-1}(i)).\end{aligned}$$

Lemma 6.3 then implies (6.10).

Now suppose that for fixed $n \geq 1$ and all $0 \leq h \leq n$, we have

$$b_{n,h} = \frac{s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)}{(2n)!}.$$

We wish to show that all $0 \leq h \leq n$

$$a_{n+1,h} = \frac{s_{n-h}(2^2, 4^2, \dots, (2n)^2)}{(2n+1)!}.$$

By Lemma 6.2, it suffices to show that

$$\begin{aligned}&\sum_{h=0}^{n} s_{n-h}(2^2, 4^2, \dots, (2n)^2)x^{2h+1}\\ &= (2n+1)\sum_{h=0}^{n} s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)A_{2h}(x).\end{aligned}$$

By setting $m = 2h$ in Equation (5.1), we obtain

$$x^{2h+1} = \sum_{j=0}^{h} (-1)^j \binom{2h+1}{2j+1} A_{2h-2j}(x).$$

Therefore,

$$\begin{aligned}&\sum_{h=0}^{n} s_{n-h}(2^2, 4^2, \dots, (2n)^2) x^{2h+1}\\ =&\sum_{h=0}^{n} s_{n-h}(2^2, 4^2, \dots, (2n)^2) \sum_{j=0}^{h} (-1)^j \binom{2h+1}{2j+1} A_{2h-2j}(x).\end{aligned}$$

Setting $\ell = h - j$, we obtain

$$\begin{aligned}=&\sum_{h=0}^{n} s_{n-h}(2^2, 4^2, \dots, (2n)^2) \sum_{\ell=0}^{h} (-1)^{h-\ell} \binom{2h+1}{2\ell} A_{2\ell}(x)\\ =&\sum_{\ell=0}^{h} \left(\sum_{h=\ell}^{n} (-1)^h \binom{2h+1}{2\ell} s_{n-h}(2^2, 4^2, \dots, (2n)^2) \right) (-1)^\ell A_{2\ell}(x).\end{aligned}$$

Thus, Equation (6.9) follows from Lemma 6.3. □

7. Polylogarithms and hyperlogarithms

In order to complete the proof of Theorem 1.1, we need to compute the integrals of the form

$$\int_0^\infty \mathrm{m}_k(z+x) \log^j x \frac{dx}{x^2 \pm 1}.$$

These integrals are related to polylogarithms.

Definition 7.1. Let $w_1, \dots, w_m$ be complex variables and $n_1, \dots, n_m$ be positive integers. Define the multiple polylogarithm by the power series

$$\mathrm{Li}_{n_1 \cdots n_m}(w_1, \dots, w_m) := \sum_{0<j_1<\cdots<j_m} \frac{w_1^{j_1} \cdots w_m^{j_m}}{j_1^{n_1} \cdots j_m^{n_m}}.$$

We say that the above series has length m and weight $\omega = n_1 + \cdots + n_m$. It is absolutely convergent for $|w_i| \le 1$ and $n_m > 1$.

Remark 7.2. We remark that Akatsuka's polylogarithm from Theorem 4.5 is a particular case of multiple polylogarithms.

$$L_{(n_1,\dots,n_m)}(w) := \sum_{0<j_1<\cdots<j_m} \frac{w^{j_m}}{j_1^{n_1} \cdots j_m^{n_m}} = \mathrm{Li}_{n_1,\dots,n_m}(1, \dots, 1, w).$$

Multiple polylogarithms have meromorphic continuations to the complex plane.

Definition 7.3. Hyperlogarithms are defined by the following iterated integral.

$$\mathrm{I}_{n_1\cdots n_m}(a_1:\cdots:a_{m+1}):=$$
$$\int_0^{a_{m+1}}\underbrace{\frac{dt}{t-a_1}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n_1}\circ\underbrace{\frac{dt}{t-a_2}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n_2}\circ\cdots\circ\underbrace{\frac{dt}{t-a_m}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n_m}$$

where

$$\int_0^{b_{h+1}}\frac{dt}{t-b_1}\circ\cdots\circ\frac{dt}{t-b_h}=\int_{0\le t_1\le\cdots\le t_h\le b_{h+1}}\frac{dt_1}{t_1-b_1}\cdots\frac{dt_h}{t_h-b_h}.$$

Remark 7.4. The path of integration should be interpreted as any path connecting 0 and b_{h+1} in $\mathbb{C}\setminus\{b_1,\dots,b_h\}$. The integral depends on the homotopy class of this path. For our purposes, we will always integrate in the real line.

Multiple polylogarithms and hyperlogarithms are related by the following identities (see [Gon95]).

Lemma 7.5.

$$\mathrm{I}_{n_1\cdots n_m}(a_1:\cdots:a_{m+1})=(-1)^m\mathrm{Li}_{n_1\cdots n_m}\left(\frac{a_2}{a_1},\frac{a_3}{a_2}\cdots,\frac{a_{m+1}}{a_m}\right)$$
$$\mathrm{Li}_{n_1\cdots n_m}(w_1,\dots,w_m)=(-1)^m\mathrm{I}_{n_1\cdots n_m}((w_1\cdots w_m)^{-1}:\cdots:w_m^{-1}:1).$$

The following example will be useful later.

Example 7.6. The second equation in Lemma 7.5 implies, for $(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n$, and $\epsilon_1+\cdots+\epsilon_n=k-2$,

$$L_{(\epsilon_1,\dots,\epsilon_n,2)}(w^2)=(-1)^{n+1}\int_0^1\frac{dt}{t-\frac{1}{w^2}}\circ\cdots\circ\frac{dt}{t-\frac{1}{w^2}}\circ\frac{dt}{t},$$

where there is a term of the form $\frac{dt}{t}$ after each $\frac{dt}{t-\frac{1}{w^2}}$ term corresponding to $\epsilon_i=2$ and there is no term $\frac{dt}{t}$ after each $\frac{dt}{t-\frac{1}{w^2}}$ term corresponding to $\epsilon_i=1$, and the last two terms $\frac{dt}{t-\frac{1}{w^2}}\circ\frac{dt}{t}$ correspond to the subindex 2.

By setting $s^2=w^2t$, the above equals

$$(-1)^{n+1}\int_0^w\frac{2sds}{s^2-1}\circ\cdots\circ\frac{2sds}{s^2-1}\circ\frac{2ds}{s}$$
$$=(-1)^{n+1}2^{k-n-1}\int_0^w\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\cdots\circ\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\frac{ds}{s}.$$

In the previous formula, each $\frac{ds}{s}$ has contributed a factor of 2 to the leading coefficient.

In order to express our results more clearly, we recall the notation from Equation (1.1).

Definition 7.7. We will work with certain combination of polylogarithms given by

$$\begin{aligned}&\mathcal{L}_{n_1,\dots,n_m}(w_1,\dots,w_m)\\&=\sum_{(r_1,\dots,r_m)\in\{0,1\}^m}(-1)^{r_m}\mathrm{Li}_{n_1,\dots,n_m}((-1)^{r_1}w_1,\dots,(-1)^{r_m}w_m),\end{aligned}$$

The following result relates the integrals that we need to evaluate in terms of polylogarithms.

Lemma 7.8. *We have the following identitites.*

$$\int_0^1 \log^j x\frac{dx}{x^2-1} = (-1)^{j+1}j!\left(1-\frac{1}{2^{j+1}}\right)\zeta(j+1)$$

$$\int_0^1 \log^j x\frac{dx}{x^2+1} = (-1)^{j}j!L(\chi_{-4},j+1)$$

For $(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n$, *and* $\epsilon_1+\dots+\epsilon_n=k-2$, *we have the following identities.*

(7.1)
$$\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^j x\frac{dx}{x^2-1} = (-1)^{j+1}2^{k-n-2}j!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,j+1}(1,\dots,1)$$

(7.2)
$$\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^j x\frac{dx}{x^2+1} = i(-1)^{j+1}2^{k-n-2}j!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,j+1}(1,\dots,1,i,i)$$

Proof. The first two identities are proven in Lemma 9 of [Lal06a]. By applying Example 7.6,

$$\begin{aligned}&\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^j x\frac{dx}{x^2-1}\\&=\int_0^1\left((-1)^{n+1}2^{k-n-1}\int_0^x\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\dots\circ\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\frac{ds}{s}\right)\\&\quad\times\log^j x\frac{dx}{x^2-1}.\end{aligned}$$

We now use the fact that $\int_x^1 \frac{dt}{t} = -\log x$ and obtain

$$\begin{aligned}
=&(-1)^{n+1+j}2^{k-n-2}j!\int_0^1 \left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\cdots\circ\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\frac{ds}{s}\\
&\circ\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx\circ\underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{j}\\
=&(-1)^{n+1+j}2^{k-n-2}j!\sum_{(r_1,\dots,r_{n+1},r)\in\{0,1\}^{n+2}}(-1)^r\\
&\times \mathrm{I}_{\epsilon_1,\dots,\epsilon_n,2,j+1}((-1)^{r_1}:\cdots:(-1)^{r_n}:(-1)^{r_{n+1}}:(-1)^r:1)\\
=&(-1)^{j+1}2^{k-n-2}j!\sum_{(r_1,\dots,r_{n+1},r)\in\{0,1\}^{n+2}}(-1)^r\\
&\times \mathrm{Li}_{\epsilon_1,\dots,\epsilon_n,2,j+1}((-1)^{r_1+r_2},\dots,(-1)^{r_n+r_{n+1}},(-1)^{r_{n+1}+r},(-1)^r)\\
=&(-1)^{j+1}2^{k-n-2}j!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,j+1}(1,\dots,1).
\end{aligned}$$

This yields Equation (7.1).

We proceed similarly to prove Equation (7.2).

$$\begin{aligned}
&\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^j x\frac{dx}{x^2+1}\\
=&\int_0^1\left((-1)^{n+1}2^{k-n-1}\int_0^x\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\cdots\circ\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\frac{ds}{s}\right)\\
&\times\log^j x\frac{dx}{x^2+1}\\
=&i(-1)^{n+j}2^{k-n-2}j!\int_0^1\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\cdots\circ\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\circ\frac{ds}{s}\\
&\circ\left(\frac{1}{x-i}-\frac{1}{x+i}\right)dx\circ\underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{j}\\
=&i(-1)^{n+j}2^{k-n-2}j!\sum_{(r_1,\dots,r_{n+1},r)\in\{0,1\}^{n+2}}(-1)^r\\
&\times \mathrm{I}_{\epsilon_1,\dots,\epsilon_n,2,j+1}((-1)^{r_1}:\cdots:(-1)^{r_n}:(-1)^{r_{n+1}}:(-1)^r i:1)\\
=&i(-1)^{j}2^{k-n-2}j!\sum_{(r_1,\dots,r_{n+1},r)\in\{0,1\}^{n+2}}(-1)^r\\
&\times \mathrm{Li}_{\epsilon_1,\dots,\epsilon_n,2,j+1}((-1)^{r_1+r_2},\dots,(-1)^{r_n+r_{n+1}},(-1)^{r_{n+1}+r}i,(-1)^{r+1}i)\\
=&i(-1)^{j+1}2^{k-n-2}j!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,j+1}(1,\dots,1,i,i).
\end{aligned}$$

□

8. The final steps in the proof of Theorem 1.1

By combining Equation (3.1), Definition 6.1, and Theorem 6.5, we obtain

$$\begin{aligned}&\pi^{2n}\mathrm{m}_k(R_{2n})\\ &=\sum_{h=1}^{n}\frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!}2^{2h}\pi^{2n-2h}\int_0^\infty \mathrm{m}_k(z+x)\log^{2h-1}x\frac{dx}{x^2-1}\end{aligned}$$

and

$$\begin{aligned}&\pi^{2n+1}\mathrm{m}_k(R_{2n+1})\\ &=\sum_{h=0}^{n}\frac{s_{n-h}(1^2,\dots,(2n-1)^2)}{(2n)!}2^{2h+1}\pi^{2n-2h}\int_0^\infty \mathrm{m}_k(z+x)\log^{2h}x\frac{dx}{x^2+1}.\end{aligned}$$

We are now ready to express these two integrals in terms of polylogarithms.

For the case of $2h-1$, we have, by Theorem 4.5,

$$\begin{aligned}&\int_0^\infty \mathrm{m}_k(z+x)\log^{2h-1}x\frac{dx}{x^2-1}\\ &=(-1)^k k!\sum_{\frac{k}{2}-1\le n\le k-2}\frac{1}{2^{2(k-n-1)}}\\ &\times\sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}}\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h-1}x\frac{dx}{x^2-1}\\ &+\int_1^\infty \log^{2h+k-1}x\frac{dx}{x^2-1}\\ &+\sum_{j=0}^{k-2}\binom{k}{j}(-1)^{k-j}(k-j)!\sum_{\frac{k-j}{2}-1\le n\le k-j-2}\frac{1}{2^{2(k-j-n-1)}}\\ &\times\sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}}\int_1^\infty \log^j x L_{(\epsilon_1,\dots,\epsilon_n,2)}\left(\frac{1}{x^2}\right)\log^{2h-1}x\frac{dx}{x^2-1}.\end{aligned}$$

By making the change of variables $y = \frac{1}{x}$ in the integrals over $1 \leq x$, and then replacing y by x again, the above expression becomes

$$\begin{aligned}
&(-1)^k k! \sum_{\frac{k}{2}-1\leq n\leq k-2} \frac{2}{2^{2(k-n-1)}}\\
&\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h-1} x\frac{dx}{x^2-1}\\
&+\int_0^1 (-1)^k \log^{2h+k-1} x\frac{dx}{x^2-1}\\
&+\sum_{j=1}^{k-2}\binom{k}{j}(-1)^k(k-j)! \sum_{\frac{k-j}{2}-1\leq n\leq k-j-2} \frac{1}{2^{2(k-j-n-1)}}\\
&\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}\left(x^2\right)\log^{2h+j-1} x\frac{dx}{x^2-1}.
\end{aligned}$$

Finally, by Lemma 7.8,

$$\begin{aligned}
&\int_0^\infty \mathrm{m}_k(z+x)\log^{2h-1} x\frac{dx}{x^2-1}\\
=&(2h+k-1)!\left(1-\frac{1}{2^{2h+k}}\right)\zeta(2h+k)\\
&+(-1)^k k! \sum_{\frac{k}{2}-1\leq n\leq k-2} \frac{1}{2^{k-n-1}}\\
&\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}} (2h-1)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h}(1,\dots,1)\\
&+\sum_{j=1}^{k-2}\binom{k}{j}(-1)^k(k-j)! \sum_{\frac{k-j}{2}-1\leq n\leq k-j-2} \frac{1}{2^{k-n-j}}\\
&\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}} (-1)^j(2h+j-1)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+j}(1,\dots,1).
\end{aligned}$$

The case of $2h$ proceeds similarly. First, by Theorem 4.5,

$$\begin{aligned}
&\int_0^\infty \mathrm{m}_k(z+x)\log^{2h} x \frac{dx}{x^2+1}\\
=&(-1)^k k! \sum_{\frac{k}{2}-1\leq n\leq k-2} \frac{1}{2^{2(k-n-1)}}\\
&\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h} x \frac{dx}{x^2+1}\\
&+\int_1^\infty \log^{2h+k} x\frac{dx}{x^2+1}\\
&+\sum_{j=0}^{k-2}\binom{k}{j}(-1)^{k-j}(k-j)! \sum_{\frac{k-j}{2}-1\leq n\leq k-j-2} \frac{1}{2^{2(k-j-n-1)}}\\
&\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}} \int_1^\infty \log^j x L_{(\epsilon_1,\dots,\epsilon_n,2)}\left(\frac{1}{x^2}\right)\log^{2h} x\frac{dx}{x^2+1}.
\end{aligned}$$

Then, by making the change of variables $y=\frac{1}{x}$ in the integrals over $1\leq x$, and then replacing y by x again,

$$\begin{aligned}
=&(-1)^k k! \sum_{\frac{k}{2}-1\leq n\leq k-2} \frac{2}{2^{2(k-n-1)}}\\
&\sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h} x \frac{dx}{x^2+1}\\
&+\int_0^1 (-1)^k\log^{2h+k} x\frac{dx}{x^2+1}\\
&+\sum_{j=1}^{k-2}\binom{k}{j}(-1)^{k}(k-j)! \sum_{\frac{k-j}{2}-1\leq n\leq k-j-2} \frac{1}{2^{2(k-j-n-1)}}\\
&\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}\left(x^2\right)\log^{2h+j} x\frac{dx}{x^2+1}.
\end{aligned}$$

Finally, by Lemma 7.8,

$$\begin{aligned}
&\int_0^\infty \mathrm{m}_k(z+x)\log^{2h} x\frac{dx}{x^2+1}\\
=&(2h+k)!L(\chi_{-4},2h+k+1)\\
&+(-1)^{k+1}k!\sum_{\frac{k}{2}-1\leq n\leq k-2}\frac{1}{2^{k-n-1}}\\
&\times\sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-n-2}} i(2h)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+1}(1,\dots,1,i,i)\\
&+\sum_{j=1}^{k-2}\binom{k}{j}(-1)^k(k-j)!\sum_{\frac{k-j}{2}-1\leq n\leq k-j-2}\frac{1}{2^{k-n-j}}\\
&\times\sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\epsilon_i=2\}=k-j-n-2}} i(-1)^{j+1}(2h+j)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+j+1}(1,\dots,1,i,i)
\end{aligned}$$

This concludes the proof of Theorem 1.1.

8.1. **The reduction to Corollary 1.2.** We now proceed to deduce Corollary 1.2 from Theorem 1.1.

Recall that Theorem 1.1 implies in particular that

$$\mathrm{m}_2(R_2)=\frac{\pi^2}{4}+\frac{4}{\pi^2}\mathcal{L}_{2,2}(1,1).$$

By using the following formulas that can be found in [BJOPL02],

$$\begin{aligned}
\mathrm{Li}_{2,2}(1,1)=&\frac{3}{10}\zeta(2)^2,\\
\mathrm{Li}_{2,2}(1,-1)=&\frac{1}{8}\zeta(2)^2-2\mathrm{Li}_{1,3}(1,-1),\\
\mathrm{Li}_{2,2}(-1,1)=&\frac{-11}{40}\zeta(2)^2+2\mathrm{Li}_{1,3}(1,-1),\\
\mathrm{Li}_{2,2}(-1,-1)=&\frac{-3}{40}\zeta(2)^2,
\end{aligned}$$

one can reduce $\mathcal{L}_{2,2}(1,1)$ in order to obtain

$$\mathrm{m}_2(R_2)=\frac{89\pi^2}{360}+\frac{16}{\pi^2}\mathrm{Li}_{1,3}(1,-1).$$

By combining the following identity from [BJOPL02]

$$\mathrm{Li}_{1,3}(-1,-1)=-\frac{11}{20}\zeta(2)^2+\frac{7}{4}\log 2\zeta(3)-\mathrm{Li}_{1,3}(1,-1)$$

with this result from [BBG95]

$$\mathrm{Li}_{1,3}(-1,-1)=\frac{\zeta(4)}{2}-2\left(\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{1}{24}\log^2 2(\log^2 2-\pi^2)\right),$$

one finally obtains Equation (1.2).

8.2. **The simpler case of Proposition 1.3.** In order to find formulas for $\mathrm{m}_k(Q_m)$, we start by taking $P_a = az$. Indeed, replacing a by $\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)$ yields zQ_m which has the same higher Mahler measures as Q_m. This computation is particularly easy to do because $\mathrm{m}_k(az) = \log^k |a|$. Therefore we obtain

$$\pi^{2n}\mathrm{m}_k(Q_{2n}) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2, 4^2, \dots, (2n-2)^2)}{(2n-1)!} 2^{2h}\pi^{2n-2h} \int_0^\infty \log^{k+2h-1} x \frac{dx}{x^2-1}$$

and

$$\pi^{2n+1}\mathrm{m}_k(Q_{2n+1}) = \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \dots, (2n-1)^2)}{(2n)!} 2^{2h+1}\pi^{2n-2h} \int_0^\infty \log^{k+2h} x \frac{dx}{x^2+1}.$$

Proposition 1.3 follows with the same arguments that we used in the final steps of the proof of Theorem 1.1 and the combination of the well-known formulas

$$\zeta(2j) = \frac{(-1)^{j+1}B_{2j}(2\pi)^{2j}}{2(2j)!}$$

and

$$L(\chi_{-4}, 2j+1) = \frac{(-1)^j E_{2j}\pi^{2j+1}}{2^{2j+2}(2j)!}.$$

Finally, Corollary 1.4 follows from the simple observation that

$$\mathrm{m}_k(Q_m) = \sum_{j_1+\cdots+j_m=k} \binom{k}{j_1, \dots, j_m} \mathrm{m}_{j_1}\left(\frac{1-x_1}{1+x_1}\right)\cdots \mathrm{m}_{j_m}\left(\frac{1-x_m}{1+x_m}\right),$$

and the fact that

$$\mathrm{m}_j\left(\frac{1-x}{1+x}\right) = \mathrm{m}_j(Q_1) = \begin{cases} (-1)^{j/2}\left(\frac{\pi}{2}\right)^j E_j & j \text{ even,} \\ 0 & j \text{ odd.} \end{cases}$$

9. Concluding remarks

We have proved exact formulas for m_k of a particular family of rational functions with an arbitrary number of variables. Much like in the case of the classical Mahler measure for this family, we have obtained formulas involving multiple polylogarithms evaluated in roots of the unity. It is expected that many, if not all, of the formulas from Theorem 1.1 could be reduced to expressions solely involving terms of length 1. For example, $\mathcal{L}_{2,2h+1}(1,1)$ can be reduced to combinations of products of $\zeta(n)$ by means of a result in [BBB97] that generalizes a result of Euler in the reduction of multizeta values of length 2. Additional efforts in this direction may be found in [Lal06b, LL], but they are currently insuffient to reduce all the terms involved in such expressions. Another direction for future exploration is the search for formulas for $\mathrm{m}_k(S_m)$ and $\mathrm{m}_k(T_m)$.

Acknowledgements. The authors would like to thank Andrew Granville and Mathew Rogers for interesting discussions and useful observations. In particular, they are grateful to Granville for the suggestion to use the generating function for the polynomials $A_k(x)$ and to Rogers for remarking that $s_{n-h}(2^2, 4^2, \ldots, (2n-2)^2)$ is related to Stirling numbers of the first kind. The authors would like to express their gratitude to the anonymous referee for several corrections and helpful suggestions.

This research was supported by the Natural Sciences and Engineering Research Council of Canada [Discovery Grant 355412-2013] and the Fonds de recherche du Québec - Nature et technologies [Établissement de nouveaux chercheurs 144987 to ML, Bourse de doctorat en recherche (B2) 175957 to JSL]

References

[Ahl79] Ahlfors, Lars V. Complex analysis. An introduction to the theory of analytic functions of one complex variable. Third edition. International Series in Pure and Applied Mathematics. *McGraw-Hill Book Co., New York*, 1978. xi+331.

[Aka09] Akatsuka, Hirotaka. Zeta Mahler measures. *J. Number Theory* 129 (2009), no. 11, 2713–2734.

[BJOPL02] Bigotte, M.; Jacob, G.; Oussous, N. E.; Petitot, M. Lyndon words and shuffle algebras for generating the coloured multiple zeta values relations tables. WORDS (Rouen, 1999). *Theoret. Comput. Sci.* 273 (2002), no. 1-2, 271–282.

[Bis14] Biswas, Arunabha. Asymptotic nature of higher Mahler measure. *Acta Arith.* 166 (2014), no. 1, 15–22.

[BM14] Biswas, Arunabha; Monico, Chris. Limiting value of higher Mahler measure. *J. Number Theory* 143 (2014), 357–362.

[BBG95] Borwein, David; Borwein, Jonathan M.; Girgensohn, Roland. Explicit evaluation of Euler sums. *Proc. Edinburgh Math. Soc. (2)* 38 (1995), no. 2, 277–294.

[BBSW12] D. Borwein, J. Borwein, A. Straub, J. Wan, Log-sine evaluations of Mahler measures, II. *Integers* 12 (2012), no. 6, 1179–1212.

[BBB97] Borwein, J. M.; Bradley, D. M.; Broadhurst, D. J. Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k. The Wilf Festschrift (Philadelphia, PA, 1996). *Electron. J. Combin.* 4 (1997), no. 2, Research Paper 5, approx. 21 pp.

[BS11] J. Borwein, Jonathan, A. Straub, Special values of generalized log-sine integrals. *ISSAC 2011–Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation*, 43–50, ACM, New York, 2011.

[BS12] J. Borwein, A. Straub, Log-sine evaluations of Mahler measures. *J. Aust. Math. Soc.* 92 (2012), no. 1, 15–36.

[Boy97] Boyd, David W. Mahler's measure and special values of L-functions. *Experiment. Math.* 7 (1998), no. 1, 37–82.

[Den97] Deninger, Christopher. Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions. *J. Amer. Math. Soc.* 10 (1997), no. 2, 259–281.

[Gon95] Goncharov, Alexander B. Polylogarithms in arithmetic and geometry. *Proceedings of the International Congress of Mathematicians,* Vol. 1, 2 (Zürich, 1994), 374–387, Birkhäuser, Basel, 1995.

[KLO08] Kurokawa, N.; Lalín, M.; Ochiai, H. Higher Mahler measures and zeta functions. *Acta Arith.* 135 (2008), no. 3, 269–297.

[Lal06a] Lalín, Matilde N. Mahler measure of some n-variable polynomial families. *J. Number Theory* 116 (2006), no. 1, 102–139.

[Lal06b] Lalín, Matilde N. On certain combination of colored multizeta values. *J. Ramanujan Math. Soc.* 21 (2006), no. 1, 11–127.

[Lal10] Lalín, M. Higher Mahler measure as a Massey product in Deligne Cohomology. Low-dimensional Topology and Number Theory. Abstracts from the workshop held August 15-August 21, 2010. Organized by Paul E. Gunnells, Walter Neumann, Adam S. Sikora and Don Zagier. Oberwolfach Reports. Vol. 7, no. 3. *Oberwolfach Rep.* 7 (2010), no. 3, 2101–2163.

[LS11] Lalín, Matilde; Sinha, Kaneenika. Higher Mahler measure for cyclotomic polynomials and Lehmer's question. *Ramanujan J.* 26 (2011), no. 2, 257–294.

[LL] Lalín, Matilde N.; Lechasseur, Jean-Sébastien. On certain multizeta values twisted by roots of unity. *Preprint* 2016.

[OZ01] Ohno, Yasuo; Zagier, Don. Multiple zeta values of fixed weight, depth, and height. *Indag. Math. (N.S.)* 12 (2001), no. 4, 483–487.

[R-V97] Rodriguez-Villegas, F. Modular Mahler measures. I. *Topics in number theory (University Park, PA, 1997)*, 17–48, Math. Appl., 467, *Kluwer Acad. Publ.*, Dordrecht, 1999.

[Sas10] Sasaki, Yoshitaka. On multiple higher Mahler measures and multiple L values. *Acta Arith.* 144 (2010), no. 2, 159–165.

[Sas12] Sasaki, Yoshitaka. On multiple higher Mahler measures and Witten zeta values associated with semisimple Lie algebras. *Commun. Number Theory Phys.* 6 (2012), no. 4, 771–784.

Département de mathématiques et de statistique, Université de Montréal, CP 6128, succ. Centre-ville, Montreal, QC H3C 3J7, Canada

E-mail address: `mlalin@dms.umontreal.ca`

Département de mathématiques et de statistique, Université de Montréal, CP 6128, succ. Centre-ville, Montreal, QC H3C 3J7, Canada

E-mail address: `jeansebl777@gmail.com`