Problem 1:
Prove that $3^{2n} - 2^n$ is divisible by 7 whenever n is a positive integer.
Hint: Use Mathematical Induction.

Total: 6 points

Problem 3:
An element e of a ring R is said to be *idempotent* if $e^2 = e$.

(a) Find at least four different idempotent elements in $M_{2\times 2}$, the ring of 2×2-matrices with real entries. 4

(b) Find all idempotent elements in $\mathbb{Z}_{12}$. 3

(c) Prove that the zero-element and the identity element are idempotent in every ring R. 2

(d) Prove that the zero-element and the identity element are the only idempotent elements in an integral domain. 4

Total: 13 points

Problem 5:
Let $\sim$ be defined on the ring $\mathbb{Z}_7[t]$ of polynomials over $\mathbb{Z}_7$ by

$$f \sim g \iff t \mid f - g.$$

(a) Give at least three different elements that are equivalent to $[3]t^2 + [2]t + [5] \in \mathbb{Z}_7[t]$. 3

(b) Prove that $\sim$ defines an *equivalence relation* on $\mathbb{Z}_7[t]$. 6

(c) Give the *equivalence class* of $t \in \mathbb{Z}_7[t]$, i.e. describe the set of all elements in $\mathbb{Z}_7[t]$ that are equivalent to t using the equivalence relation defined above. 3

Total: 12 points

Problem 7:
Use the *Extended Euclidean Algorithm* to find the (monic) *greatest common divisor* of the polynomials

$$f(t) = 2t^4 + t^3 + t + 1 \in \mathbb{Z}_3[t] \text{ and } g(t) = t^3 + t^2 + 1 \in \mathbb{Z}_3[t]$$

and write it as a *polynomial linear combination* of f and g.

Total: 8 points

Problem 2:

In the ring $\mathbb{Z}_{100}$ of *integers modulo* 100 consider the subset

$$S = \{[a]_{100} : 20 \mid a\} = \{[0]_{100}, [20]_{100}, [40]_{100}, [60]_{100}, [80]_{100}\}$$

of all classes of the form $[a]_{100}$, where a is divisible by 20.

(a) Show that S together with the addition and multiplication from $\mathbb{Z}_{100}$ is a *subring* of $\mathbb{Z}_{100}$. [4]
Hint: To find (additive) inverses make a table.

(b) Decide whether S is an integral domain or a field. Justify your answer! [2]

(c) Decide whether the ring S is isomorphic to $\mathbb{Z}_5$, the ring of integers modulo 5 which also has 5 elements. If so, give an isomorphism and prove that it is one, if not explain why not. [4]

Total: 10 points

Problem 4:

Consider the set

$$I_2 = \{a_n t^n + \ldots + a_0 \in \mathbb{Z}[t] : a_0, \ldots, a_n \text{ are all even}\}$$

of all polynomials with even integer coefficients.

(a) Show that I_2 defines an *ideal* in $\mathbb{Z}[t]$. [6]

(b) Show that I is a *principal ideal* in $\mathbb{Z}[t]$ and give a *generator* for I_2. [4]
Hint: Find a polynomial $d(t)$ of smallest non-negative degree in I_2 and show that every polynomial in I_2 is a multiple of $d(t)$.

(c) Consider now the set [2]

$$I_p = \{a_n t^n + \ldots + a_0 \in \mathbb{Z}[t] : a_0, \ldots, a_n \text{ are all divisible by } p\}$$

of all polynomials whose coefficients are divisible by a prime number p. Is I_p still an ideal? And if so, can you give a generator? No need to justify your answer!

Total: 10 points

Problem 10:

Consider the polynomial

$$p(t) = t^4 + t^2 + [1] \in \mathbb{Z}_2[t]$$

in the ring $R := \mathbb{Z}_2[t]$.

(a) Decide whether the *quotient ring* $\overline{R} := R/p(t)$ is a field. [2]

(b) Solve the equation [2]

$$[t^3 + t^2 + [1]]_{p(t)} + x = [t^2 + t]_{p(t)}$$

in $\overline{R}$, i.e. find an x that satisfies this equation. Is the answer uniquely determined?

(c) Solve the equation [6]

$$[t^3 + t^2 + [1]]_{p(t)} \cdot x = [t^2 + t]_{p(t)}$$

in $\overline{R}$, i.e. find an x that satisfies this equation.
Hint: Find the inverse to $[t^3 + t^2 + [1]]_{p(t)}$ in $\overline{R}$ first.

Total: 10 points

Problem 6:

You are given a ring $R = \{A, B, C, D\}$ with four elements and know that there is a ringhomomorphism

$$T : \mathbb{Z} \longrightarrow R,$$

that maps 1 and 5 to A, 2 and 6 to B, 3 and 7 to C and 4 and 8 to D.

(a) Using the axioms for a ring homomorphism, find $T(9)$, $T(10)$, $T(11)$ and $T(12)$, e.g. $T(9) = T(1+8) = T(1) + T(8) = T(1) + D = T(1) + T(4) = T(5) = A$. [3]

(b) Write out the *addition* and *multiplication* table for R, e.g. $A + D = T(1) + T(4) = T(1+4) = T(5) = A$ and $A \cdot D = T(1) \cdot T(4) = T(1 \cdot 4) = T(4) = D$. [6]

(c) Give the zero-element in R. [2]

(d) Decide whether R is an integral domain and whether it is a field. [2]

Total: 13 points

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Problem 8:

Write the polynomial

$$\frac{1}{6}t^5 + \frac{2}{3}t^4 - \frac{1}{2}t^3 - 3t^2 \in \mathbb{Q}[t]$$

as a product of irreducible polynomials in $\mathbb{Q}[t]$

Total: 6 points

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Problem 9:

(a) Show that the polynomial $p(t) = 21t^3 - 6t + 8 \in \mathbb{Z}[t]$ has no rational root.
<u>Hint:</u> Use the Gauss-Lemma and reduction modulo 5. Don't use the Rational Root Test!!!
Can you conclude that $p(t)$ is irreducible over $\mathbb{Q}$? [5]

(b) You are given the polynomial $q(t) = 3t^{10} + 5 \cdot f(t) \in \mathbb{Z}[t]$ with $f(t) \in \mathbb{Z}[t]$ of degree < 10 and satisfying $f(0) = 17$.

i. Give such a polynomial. [1]

ii. Show that $q(t)$ is irreducible over $\mathbb{Q}$.
<u>Hint:</u> Use the *Eisenstein Criterion*. [4]

Total: 10 points