MATH 228 (A1)
Final Examination

10 December 2003

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Cross out anything which is not part of your final solution.

Q1. [10 pts]. (1.1) Find the greatest common divisor of $a = 726$ and $b = 275$.

(1.2) Write $\gcd(726, 275)$ as an integer combination of a and b.

(1.3) Let A be the set consisting of all integer combination of a and b:

$$A = \{ua + vb : \text{ for all integers } u, v\}.$$

Prove that A is an ideal of I.

(1.4) Is A a principal ideal of I? If so, find a positive integer n such that $A = (n)$.

Q2. [10 pts]. Do **only one** of (2.1) or (2.2):

(2.1) Let a, b, c be non-zero integers, and let $(b, c) = d$ and $(ab, c) = e$. Assume that $(a, c) = 1$. Prove that $d = e$.

(2.2) Let R be an integral domain, and let p be a prime element of R. Prove that p is irreducible in R.

Q3. [10 pts]. Prove, by induction on n, that

$$n^2 + (n+1)^2 + \ldots + (2n)^2 = \frac{n(n+1)(14n+1)}{6}$$

for all $n \geq 1$. (The left-hand side is the sum of the squares of all integers from n to $2n$, inclusive, i.e., $\sum_{i=n}^{2n} i^2$.)

Q4. [20 pts]. Let I be the ring of integers, and let $R = I/(3)$ be the quotient ring.

(4.1) Write down the distinct elements of R. (You do not have to use brackets around the elements of R if this does not confuse you.)

(4.2) Calculate $[2] + [2]$ and $[2]^2$ in R.

(4.3) Is R an integral domain? Why or why not. (Do not simply repeat the definition of an integral domain.)

Next, consider the polynomial ring $R[x]$, the polynomial $f(x) = x^2 + [1]$, the quotient ring $S = R[x]/(f)$, and $\alpha = [x]$.

(4.4) Show that no element of R is a root of f.

(4.5) Prove that f is an irreducible polynomial in $R[x]$.

(4.6) Prove that $\alpha^2 = [2]$ in the quotient ring S.

(4.7) How many elements does S have?

(4.8) Show that $\alpha^4 = [1]$.

(4.9) Put $\beta = [1] + \alpha$. Show that $\beta^2 = [2]\alpha$.

(4.10) Prove that $\beta^8 = [1]$.

(4.11) Is every non-zero element of S a unit? Give a reason for your answer.