Functional equations for Mahler measures of genus-one curves

(joint with Mat Rogers) Matilde N. Lalín

University of Bristish Columbia and Pacific Institute for the Mathematical Sciences
mlalin@math.ubc.ca
http://www.math.ubc.ca/~mlalin

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Mahler measure of one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n-1|} = \begin{cases} |\alpha| & \text{if } |\alpha|>1 \\ 1 & \text{if } |\alpha|<1 \end{cases}$$

For

$$P(x) = a\prod_i (x-\alpha_i)$$

$$M(P) = |a|\prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+ |\alpha_i|$$

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Lehmer's Question (1933): Does there exist $C > 0$ such that $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$$= 0.162357612\ldots$$

the best possible?

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

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The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd 1998

$$m(k) \stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

E_k determined by $x + \frac{1}{x} + y + \frac{1}{y} + k = 0$.

Deninger 1997

L-functions $\leftarrow$ Beilinson's conjectures
Kronecker-Eisenstein series for $k = 1$

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Rodriguez-Villegas 1997

$k = 4\sqrt{2}$ (CM case)

$$m\left(4\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\sqrt{2}\right) = \mathrm{L}'(E_{4\sqrt{2}}, 0)$$

$k = 3\sqrt{2}$ (modular curve $X_0(24)$)

$$m\left(3\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2}\right) = q\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

Theorem

(Rodriguez-Villegas) $E_k \sim$ modular elliptic surface assoc $\Gamma_0(4)$.

$$m(k) = \mathrm{Re}\left(\frac{16 y_\mu}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})}\right)$$

$$= \mathrm{Re}\left(-\pi \mathrm{i}\mu + 2\sum_{n=1}^{\infty}\sum_{d|n} \chi_{-4}(d) d^2 \frac{q^n}{n}\right)$$

where $j(E_k) = j\left(-\frac{1}{4\mu}\right)$

$$q = \mathrm{e}^{2\pi \mathrm{i}\mu} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)}\right)$$

and y_μ is the imaginary part of μ.

Theorem

(also Kurokawa & Ochiai 2005)
For $h \in \mathbb{R}^*$,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(\mathrm{i}h + \frac{1}{\mathrm{i}h}\right)\right) = m\left(\frac{4}{h^2}\right).$$

$$m\left(2\left(h + \frac{1}{h}\right)\right) - m\left(2\left(\mathrm{i}h + \frac{1}{\mathrm{i}h}\right)\right) = m\left(4h^2\right).$$

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Corollary

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

$$m\left(3\sqrt{2}\right) = q\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

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Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \,\text{rank}\, 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

The relation with Mahler measures

In the example,

$$yP_k(x,y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} (\log^+ |y_{(1)}(x)| + \log^+ |y_{(2)}(x)|) \frac{\mathrm{d}x}{x}.$$

By Jensen's formula respect to y.

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_{\mathbb{T}^1} \eta(x,y),$$

$$\eta(x,y) := \log |x| \mathrm{d} \arg y - \log |y| \mathrm{d} \arg x$$

1-form on $E(\mathbb{C}) \setminus S$

The elliptic regulator

The regulator map (Beilinson, Bloch):

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R})$$

$$\{x, y\} \to \left\{ \gamma \to \int_\gamma \eta(x, y) \right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$. ($H^1(E, \mathbb{R})$ dual of $H_1(E, \mathbb{Z})$)

In our case, $\mathbb{T}^1 \in H_1(E, \mathbb{Z})$.

Computing the regulator

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$$

$z \bmod \Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$.
Bloch regulator function

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right) = \frac{y_\tau^2}{\pi} \sum_{m,n\in\mathbb{Z}}{}' \frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}$$

y_τ is the imaginary part of τ.
Regulator function given by

$$R_\tau = D_\tau - \mathrm{i}J_\tau$$

$\mathbb{Z}[E(\mathbb{C})]^- = \mathbb{Z}[E(\mathbb{C})]/\sim \quad [-P] \sim -[P]$.

R_τ is an odd function,

$$\mathbb{Z}[E(\mathbb{C})]^- \to \mathbb{C}.$$

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

$$\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$$

$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$

Proposition

$E/\mathbb{R}$ elliptic curve, x, y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, $\omega \in \Omega^1$

$$-r\{x,y\} = -\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x) \diamond (y)\right)\right)$$

where Ω_0 is the real period and $\Omega = \int_\gamma \omega$.

Use results of Beilinson, Bloch, idea of Deninger

Recovering the identities

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

Weierstrass form:

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

$P = \left(1, \frac{k}{2}\right)$, torsion point of order 4.

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$

$$P \equiv -\frac{1}{4} \mod \mathbb{Z} + \tau\mathbb{Z} \qquad k \in \mathbb{R}$$

$$\tau = \mathrm{i} y_\tau \qquad k \in \mathbb{R}, |k| > 4,$$

$$\tau = \frac{1}{2} + \mathrm{i} y_\tau \qquad k \in \mathbb{R}, |k| < 4$$

Understand cycle $[|x| = 1] \in H_1(E, \mathbb{Z})$

$$\Omega = \tau \Omega_0 \quad k \in \mathbb{R}$$

$$-r\{x,y\} = -\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x) \diamond (y)\right)\right)$$

$$m(k) = \frac{4}{\pi}\,\mathrm{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right), \quad k \in \mathbb{R}$$

Modularity for the regulator

Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ and let $\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$, such that

$$\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}$$

Then:

$$R_{\tau'}\left(\mathrm{e}^{2\pi \mathrm{i}(a'+b'\tau')}\right) = \frac{1}{\gamma\bar{\tau}+\delta} R_{\tau}\left(\mathrm{e}^{2\pi \mathrm{i}(a+b\tau)}\right).$$

$$m(k) = \frac{4}{\pi} \mathrm{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right), \quad k \in \mathbb{R}$$

Take $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$.

$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi \mathrm{i}}{4\tau}}\right)$$

If we let $\mu = -\frac{1}{4\tau}$, then

$$m(k) = -\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi \mathrm{i}\mu}\right)$$

$$= \mathrm{Re}\left(\frac{16 y_\mu}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})}\right)$$

Functional equations

- Functional equations of the regulator

$$J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right) = 2J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) + 2J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}(\mu+1)}{2}}\right)$$

$$\frac{1}{y_{4\mu}}J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right) = \frac{1}{y_{2\mu}}J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) + \frac{1}{y_{2\mu}}J_{2\mu}\left(-\mathrm{e}^{\pi\mathrm{i}\mu}\right)$$

- Hecke operators approach

$$m(k) = \mathrm{Re}\left(-\pi\mathrm{i}\mu + 2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2\frac{q^n}{n}\right)$$

$$= \mathrm{Re}\left(-\pi\mathrm{i}\mu - \pi\mathrm{i}\int_{\mathrm{i}\infty}^{\mu}(e(z)-1)\mathrm{d}z\right)$$

$$e(\mu) = 1 - 4\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2q^n$$

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$$q = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

Second degree modular equation, $|h| < 1$, $h \in \mathbb{R}$,

$$q^2\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(h^4\right).$$

$h \rightarrow \mathrm{i}h$

$$-q\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right).$$

Then the equation with J becomes

$$m\left(q\left(\left(\frac{2h}{1+h^2}\right)^2\right)\right)+m\left(q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right)\right)=m\left(q\left(h^4\right)\right).$$

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

Direct approach

Also some equations can be proved directly using isogenies:

$$\phi_1 : E_{2\left(h+\frac{1}{h}\right)} \to E_{4h^2}, \qquad \phi_2 : E_{2\left(h+\frac{1}{h}\right)} \to E_{\frac{4}{h^2}}.$$

$$\phi_1 : (X,Y) \to \left(\frac{X(h^2X+1)}{X+h^2}, -\frac{h^3Y\left(X^2+2h^2X+1\right)}{(X+h^2)^2} \right)$$

$$m\left(4h^2\right) = r_1\left(\{x_1,y_1\}\right) = \frac{1}{2\pi}\int_{|X_1|=1} \eta(x_1,y_1)$$

$$= \frac{1}{4\pi}\int_{|X|=1} \eta(x_1\circ\phi_1, y_1\circ\phi_1) = \frac{1}{2} r\left(\{x_1\circ\phi_1, y_1\circ\phi_1\}\right)$$

The identity with $h = \frac{1}{\sqrt{2}}$

$$m(2) + m(8) = 2m\left(3\sqrt{2}\right)$$

$$m\left(3\sqrt{2}\right) + m\left(\mathrm{i}\sqrt{2}\right) = m(8)$$

$f = \frac{\sqrt{2}Y - X}{2}$ in $\mathbb{C}(E_{3\sqrt{2}})$.

$$(f) \diamond (1-f) = 6(P) - 10(P+Q) \Rightarrow 6(P) \sim 10(P+Q).$$

$Q = \left(-\frac{1}{h^2}, 0\right)$ has order 2.

$$\phi : E_{3\sqrt{2}} \to E_{\mathrm{i}\sqrt{2}} \qquad (X,Y) \to (-X, \mathrm{i}Y)$$

$$r_{\mathrm{i}\sqrt{2}}(\{x,y\}) = r_{3\sqrt{2}}(\{x\circ\phi, y\circ\phi\})$$

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But

$$(x \circ \phi) \diamond (y \circ \phi) = 8(P + Q)$$

$$(x) \diamond (y) = 8(P)$$

$$6 r_{3\sqrt{2}}(\{x, y\}) = 10 r_{\mathrm{i}\sqrt{2}}(\{x, y\})$$

and

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}).$$

Consequently,

$$m(8) = \frac{8}{5} m(3\sqrt{2})$$

$$m(2) = \frac{2}{5} m(3\sqrt{2})$$

Other families

- Hesse family

$$h(a^3) = m\left(x^3 + y^3 + 1 - \frac{3xy}{a}\right)$$

(studied by Rodriguez-Villegas 1997)

$$h(u^3) = \sum_{j=0}^{2} h\left(1 - \left(\frac{1-\xi_3^j u}{1+2\xi_3^j u}\right)^3\right) \qquad |u| \text{ small}$$

- More complicated equations for examples studied by Stienstra 2005:

$$m\left((x+1)(y+1)(x+y) - \frac{xy}{t}\right)$$

and Bertin 2004, Zagier $<$ 2005, and Stienstra 2005:

$$m\left((x+y+1)(x+1)(y+1) - \frac{xy}{t}\right)$$