An algebraic integration for Mahler Measure

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1. Mahler Measure

Definition 1 *For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic)* Mahler measure *is defined by*

$$m(P) \quad = \quad \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{1}$$

There is a simple expression for the Mahler measure in the one-variable case, as a function on the roots of the polynomial. The question is, what happens with several variables?

2. Polylogarithms

Definition 2 *The k*th polylogarithm *is the function defined by the power series*

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1 \tag{2}$$

This function can be continued analytically to $\mathbb{C} \setminus (1, \infty)$.

In order to avoid discontinuities, and to extend this function to the whole complex plane, several modifications have been proposed. Zagier [10] considers the following version:

$$P_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right) \tag{3}$$

where B_j is the jth Bernoulli number, $\mathrm{Li}_0(x) \equiv -\frac{1}{2}$ and Re_k denotes Re or Im depending on whether k is odd or even.

This function is one-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and continuous in $\mathbb{P}^1(\mathbb{C})$. Moreover, P_k satisfy very clean functional equations. The simplest ones are

$$P_k \left(\frac{1}{x} \right) = (-1)^{k-1} P_k(x) \qquad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

there are also lots of functional equations which depend on the index k. For instance, for $k = 2$, we have the well-known five-term relation[1]

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1 - y}{1 - xy} \right) + D\left(\frac{1 - x}{1 - xy} \right) = 0 \tag{4}$$

Polylogarithms appear in many examples as the Mahler measures of polynomials in several variables.

[1] Note that $P_2 = D$, the Bloch–Wigner dilogarithm

The most famous and basic examples in two and three variables are due to Smyth [9]:

$$m(1+x+y) = \frac{1}{\pi} D(\zeta_6) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) \tag{5}$$

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) \tag{6}$$

Note that $P_k(1) = \mathrm{Li}_k(1) = \zeta(k)$.

3. The two-variable case

In order to understand the formulas for the two variables polynomials, Rodriguez-Villegas [8] has proposed the following construction inspired in Deninger's work. This was later developed by Boyd and Rodriguez-Villegas [1], [2].

Given a smooth projective curve C and x, y rational functions ($x, y \in \mathbb{C}(C)^*$), define

$$\eta(x,y) = \log|x| \,\mathrm{d}\arg y - \log|y| \,\mathrm{d}\arg x \tag{7}$$

Here

$$\mathrm{d}\arg x = \mathrm{Im}\left(\frac{\mathrm{d}x}{x}\right) \tag{8}$$

is well defined in $\mathbb{C}$ in spite of the fact that arg is not. η is a 1-form in $C \setminus S$, where S is the set of zeros and poles of x and y. It is also closed, because of

$$\mathrm{d}\eta = \mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right) = 0$$

Let $P \in \mathbb{C}[x,y]$. Write

$$P(x,y) = a_d(x)y^d + \ldots + a_0(x)$$

$$P(x,y) = a_d(x)\prod_{n=1}^{d}(y - \alpha_n(x))$$

Then by Jensen's formula,

$$m(P) = m(a_d) + \frac{1}{2\pi \mathrm{i}} \sum_{n=1}^{d} \int_{\mathbb{T}^1} \log^+ |\alpha_n(x)| \frac{\mathrm{d}x}{x} = m(a_d) - \frac{1}{2\pi}\int_\gamma \eta(x,y) \tag{9}$$

Here γ is the union of paths in $C = \{P(x,y) = 0\}$ where $|x| = 1$ and $|y| \geq 1$. Also note that $\partial\gamma = \{(x,y) \in \mathbb{C}^2 \,|\, |x| = |y| = 1, P(x,y) = 0\}$

We want to arrive to one of these two situations:

1. η is exact, and $\partial\gamma \neq 0$. In this case we can integrate using Stokes Theorem.

2. η is not exact and $\partial\gamma = 0$. In this case we can compute the integral by using Residue's Theorem.

Here we will only care about the first case. Under certain conditions (see [8]) η can be extended to C and becomes a closed form there. We need it to be exact. In fact, we have:

Theorem 3

$$\eta(x, 1-x) = \mathrm{d}D(x) \tag{10}$$

We will associate η with an element in $H^1(C \setminus S, \mathbb{R})$ in the following way. Given $[\gamma] \in H_1(C \setminus S, \mathbb{Z})$,

$$[\gamma] \to \int_\gamma \eta \tag{11}$$

(we identify $H^1(C \setminus S, \mathbb{R})$ with $H_1(C \setminus S, \mathbb{Z})'$).

Note the following

Theorem 4 *η satisfies the following properties*

1. $\eta(x, y) = -\eta(y, x)$
2. $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$
3. $\eta(x, 1-x) = 0$ *in* $H^1(C, \mathbb{R})$

As a consequence, η is a symbol, and can be factored through $K_2(\mathbb{C}(C))$ (by Matsumoto's Theorem). Then we can guarantee that $\eta(x, y)$ is exact by having $\{x, y\}$ is trivial in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$. (Tensoring with $\mathbb{Q}$ kills roots of unity, which is fine, since η is trivial on them).

In general, if

$$x \wedge y = \sum_j r_j z_j \wedge (1 - z_j)$$

in $\bigwedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$, then

$$\eta(x, y) = \mathrm{d}\left(\sum_j r_j D(z_j)\right) = \mathrm{d}D\left(\sum_j r_j [z_j]\right)$$

We have $\gamma \subset C$ such that

$$\partial\gamma = \sum_k \epsilon_k [w_k] \quad \epsilon_k = \pm 1$$

where $w_k \in C(\mathbb{C})$, $|x(w_k)| = |y(w_k)| = 1$. Then

$$2\pi m(P) = D(\xi) \quad \text{for } \xi = \sum_k \sum_j r_j [z_j(w_k)]$$

In order to interpret Smyth's case, take $P(x, y) = x + y + 1$. Writing $x = \mathrm{e}^{2\pi \mathrm{i}\theta}$, the path of integration becomes

$$\gamma(\theta) = 1 + e^{2\pi \mathrm{i}\theta}, \quad \theta \in [1/6\,; 5/6] \quad \Rightarrow \quad \partial\gamma = [\xi_6] - [\bar{\xi}_6]$$

$$2\pi m(x + y + 1) = D(\zeta_6) - D(\bar{\zeta}_6) = 2D(\zeta_6)$$

4. The three-variable case

We are going to extend this situation to three variables. We will take

$$\eta(x,y,z) = \log|x| \left(\frac{1}{3} \,\mathrm{d}\log|y| \,\mathrm{d}\log|z| - \,\mathrm{d}\arg y \,\mathrm{d}\arg z\right)$$

$$+ \log|y| \left(\frac{1}{3} \,\mathrm{d}\log|z| \,\mathrm{d}\log|x| - \,\mathrm{d}\arg z \,\mathrm{d}\arg x\right) + \log|z| \left(\frac{1}{3} \,\mathrm{d}\log|x| \,\mathrm{d}\log|y| - \,\mathrm{d}\arg x \,\mathrm{d}\arg y\right)$$

Then η verifies

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} \wedge \frac{\mathrm{d}z}{z}\right)$$

We can express the Mahler measure of P

$$m(P) = -\frac{1}{(2\pi)^2} \int_\Gamma \eta(x,y,z)$$

(we are taking P monic to simplify notation).

Where $\Gamma = \{P(x,y,z) = 0\} \cap \{|x| = |y| = 1, |z| \geq 1\}$. We are integrating on a subset of the surface $\{P(x,y,z) = 0\}$. The differential form is defined in this surface.

Suppose we have

$$x \wedge y \wedge z = \sum r_i x_i \wedge (1 - x_i) \wedge y_i$$

Since we have

$$\eta(x, 1-x, y) = \,\mathrm{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x) \,\mathrm{d}\arg y + \frac{1}{3} \log|y| (\log|1-x| \,\mathrm{d}\log|x| - \log|x| \,\mathrm{d}\log|1-x|)$$

Then

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_\Gamma \eta(x_i, 1-x_i, y_i) = \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Where $\partial\Gamma = \{P(x,y,z) = 0\} \cap \{|x| = |y| = |z| = 1\} = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$. Note that we are integrating now on a path $\{|x| = |y| = 1\}$ inside the curve $\{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$. The differential form ω is defined in this new curve (this way of thinking the integral over a new curve has been proposed by Maillot). Now it makes sense to try to apply Stokes again. The condition is the following:

Suppose we have

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

Since we have

$$\omega(x,x) = \,\mathrm{d}P_3(x)$$

Then we have as before:

$$\int_\gamma \omega(x,y) = \sum r_i \; P_3(x_i)|_{\partial\gamma}$$

5. The K-theory context

We follow Goncharov, [5], [6]. Given a field F, we define subgroups $R_i(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ as

$$\begin{aligned}
R_1(F) &:= [x] + [y] - [xy] \\
R_2(F) &:= [x] + [y] + [1 - xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] \\
R_3(F) &:= \text{certain functional equation of the trilogarithm}
\end{aligned}$$

Define

$$B_i(F) := \mathbb{Z}[\mathbb{P}^1_F]/R_i(F) \tag{12}$$

The idea is that $B_i(F)$ is the place where P_i naturally acts. We have the following complexes:

$$\begin{aligned}
B_F(3) \otimes \mathbb{Q} &: B_3(F)_{\mathbb{Q}} \xrightarrow{\delta_1^3} (B_2(F) \otimes F^*)_{\mathbb{Q}} \xrightarrow{\delta_2^3} (\wedge^3 F^*)_{\mathbb{Q}} \\
B_F(2) \otimes \mathbb{Q} &: B_2(F)_{\mathbb{Q}} \xrightarrow{\delta_1^2} (\wedge^2 F^*)_{\mathbb{Q}} \\
B_F(1) \otimes \mathbb{Q} &: F^*_{\mathbb{Q}}
\end{aligned}$$

($B_i(F)$ is placed in degree 1).

$$\delta_1^3([x]_3) = [x]_2 \otimes x \qquad \delta_2^3([x]_2 \otimes x) = x \wedge (1-x) \wedge y \qquad \delta_1^2([x]_2) = x \wedge (1-x)$$

Proposition 5

$$\begin{aligned}
H^1(B_F(1)) &\cong K_1(F) \quad (13)\\
H^2(B_F(2)) &\cong K_2(F) \quad (14)\\
H^3(B_F(3)) &\cong K_3^M(F) \quad (15)
\end{aligned}$$

Goncharov [5] conjectures:

$$H^i(B_F(3) \otimes \mathbb{Q}) \cong K_{6-i}^{[3-i]}(F)$$

Where $K_n^{[i]}(F)$ is a certain quotient in a filtration of $K_n(F)$.

Note that our first condition is that $x \wedge y \wedge z = 0$ in $H^3(B_F(3) \otimes \mathbb{Q})$ and the second condition is $[x_i]_2 \otimes y_i = 0$ in $H^2(B_F(3) \otimes \mathbb{Q})$. Hence, everything can be translated as certain elements in different K-theories must be zero, which is analogous to the two-variable case.

6. Example

The importance of the following example is that in spite of being quite hard to compute the example by simple integrals, it can be solved with this algebraic method.

Consider

$$\mathrm{Res}(x + yt^m + t^{n+m}, z + wt^m + t^{n+m}) = (z - x)^{n+m} - (wx - yz)^m (y - w)^n$$

which reduces to compute the Mahler measure of:

$$z = \frac{(1-x)^m (1-y)^n}{(1-xy)^{m+n}}$$

$$\eta(x,y,z) = m\eta(x,y,1-x) + n\eta(x,y,1-y) - (m+n)\eta(x,y,1-xy)$$

$$= -m\eta(x,1-x,y) + n\eta(y,1-y,x) + m\eta(xy,1-xy,y) - n\eta(xy,1-xy,x)$$

$$\Delta = m([xy]_2 \otimes y - [x]_2 \otimes y) - n([xy]_2 \otimes x - [y]_2 \otimes x)$$

Now

$$[x]_2 + [y]_2 + [1-xy]_2 + \left[\frac{1-x}{1-xy}\right]_2 + \left[\frac{1-y}{1-xy}\right]_2 = 0$$

Let

$$x_1 = \frac{1-x}{1-xy} \qquad y_1 = \frac{1-y}{1-xy}$$

$$\hat{x}_1 = 1 - x_1 \qquad \hat{y}_1 = 1 - y_1$$

then

$$\Delta = m([y]_2 \otimes y + [x_1]_2 \otimes y + [y_1]_2 \otimes y) - n([x]_2 \otimes x + [x_1]_2 \otimes x + [y_1]_2 \otimes x)$$

Now note that $x = \frac{\hat{x}_1}{y_1}$, $y = \frac{\hat{y}_1}{x_1}$.

$$= m([y]_2 \otimes y + [x_1]_2 \otimes \hat{y}_1 - [x_1]_2 \otimes x_1 + [y_1]_2 \otimes \hat{y}_1 - [y_1]_2 \otimes x_1)$$

$$-n([x]_2 \otimes x + [x_1]_2 \otimes \hat{x}_1 - [x_1]_2 \otimes y_1 + [y_1]_2 \otimes \hat{x}_1 - [y_1]_2 \otimes y_1)$$

$$= m([y]_2 \otimes y + [x_1]_2 \otimes \hat{y}_1 - [x_1]_2 \otimes x_1 - [\hat{y}_1]_2 \otimes \hat{y}_1 - [y_1]_2 \otimes x_1)$$

$$-n([x]_2 \otimes x - [\hat{x}_1]_2 \otimes \hat{x}_1 - [x_1]_2 \otimes y_1 + [y_1]_2 \otimes \hat{x}_1 - [y_1]_2 \otimes y_1)$$

Recall that

$$x_1^m y_1^n \hat{x}_1^n \hat{y}_1^m = 1$$

in the curve.

Then

$$[x_1]_2 \otimes y_1^n \hat{y}_1^m - [y_1]_2 \otimes x_1^m \hat{x}_1^n = -[x_1]_2 \otimes x_1^m \hat{x}_1^n + [y_1]_2 \otimes y_1^n \hat{y}_1^m$$

$$= -m[x_1]_2 \otimes x_1 + n[\hat{x}_1]_2 \otimes \hat{x}_1 + n[y_1]_2 \otimes y_1 - m[\hat{y}_1]_2 \otimes \hat{y}_1$$

Then

$$\Delta = m([y]_2 \otimes y - [\hat{y}_1]_2 \otimes \hat{y}_1 - [x_1]_2 \otimes x_1 - [\hat{y}_1]_2 \otimes \hat{y}_1 - [x_1]_2 \otimes x_1)$$

$$-n([x]_2 \otimes x - [\hat{x}_1]_2 \otimes \hat{x}_1 - [y_1]_2 \otimes y_1 - [\hat{x}_1]_2 \otimes \hat{x}_1 - [y_1]_2 \otimes y_1)$$

$$\Delta = m([y]_2 \otimes y - 2[\hat{y}_1]_2 \otimes \hat{y}_1 - 2[x_1]_2 \otimes x_1) - n([x]_2 \otimes x - 2[\hat{x}_1]_2 \otimes \hat{x}_1 - 2[y_1]_2 \otimes y_1)$$

We now need to check the path of integration.

Let us write $x = \mathrm{e}^{2\mathrm{i}\alpha}$, $y = \mathrm{e}^{2i\beta}$, for $-\frac{\pi}{2} \leq \alpha, \beta \leq \frac{\pi}{2}$. Also, we can suppose $0 \leq \alpha \leq \frac{\pi}{2}$ and

$$m(P) = -\frac{1}{2\pi^2} \int_{-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2},\, 0 \leq \alpha \leq \frac{\pi}{2}} \eta(x,y,z)$$

Let $a = \frac{\sin\alpha}{\sin(\alpha+\beta)}$, $b = \frac{\sin\beta}{\sin(\alpha+\beta)}$. Then

$$x_1 = a\mathrm{e}^{-\mathrm{i}\beta} \quad y_1 = b\mathrm{e}^{-\mathrm{i}\alpha} \quad \hat{x}_1 = b\mathrm{e}^{\mathrm{i}\alpha} \quad \hat{y}_1 = a\mathrm{e}^{\mathrm{i}\beta}$$

Now, a, b and be though as the sides of a triangle with angles α, β opposing a and b respectively. See [4]. We know that we integrate in

$$0 \le b \le 1 \qquad 1-b \le a \le 1+b$$

$$1 \le b \qquad b-1 \le a \le 1+b$$

And also

$$a^m b^n = 1$$

Hence, if b_1 is a root of $b^{-\frac{n}{m}} = 1+b$ with $0 \le b_1 \le 1$. If b_2 is a root of $b^{-\frac{n}{m}} = b-1$ with $1 \le b_2$. Then, the integration path is

$$b_1 \le b \le b_2$$

$$1+b_1 \le a \le b_2 - 1$$

Then the limits are:

$$x_1 : 1+b_1 \sim 1-b_2$$

$$y_1 : -b_1 \sim b_2$$

$$\hat{x}_1 : -b_1 \sim b_2$$

$$\hat{y}_1 : 1+b_1 \sim 1-b_2$$

Then

$$m(P) = \frac{1}{2\pi^2}(2n(P_3(\hat{x}_1) + P_3(y_1))|_{-b_1}^{b_2} - 2m(P_3(\hat{y}_1) + P_3(x_1))|_{1+b_1}^{1-b_2})$$

$$= \frac{2n}{\pi^2}(P_3(b_2) - P_3(-b_1)) - \frac{2m}{\pi^2}(P_3(1-b_2) - P_3(1+b_1))$$

If we write $c_1 = b_1^{\frac{1}{m}}$ and $c_2 = b_2^{\frac{1}{m}}$, we have c_1 is a root of

$$x^{n+m} + x^n - 1 = 0$$

with $0 \le c_1 \le 1$. Also c_2 is a root of

$$x^{n+m} - x^n - 1 = 0$$

with $1 \le c_2$.

Then

$$m(P) = \frac{2n}{\pi^2}(P_3(c_2^m) - P_3(-c_1^m)) - \frac{2m}{\pi^2}\left(P_3\left(-\frac{1}{c_2^n}\right) - P_3\left(\frac{1}{c_1^n}\right)\right)$$

$$= \frac{2n}{\pi^2}(P_3(c_2^m) - P_3(-c_1^m)) + \frac{2m}{\pi^2}\left(P_3\left(c_1^n\right) - P_3\left(-c_2^n\right)\right)$$

7. What is next?

There are several open questions around this subject. We have been able to explain all the known examples in three variables, except Condon's example [3]

$$m(1+x+(1-x)(y+z)) = \frac{28}{5\pi^2}\zeta(3) \tag{16}$$

(John is working on it).

Also, we would like to generalize this construction to more variables. There are several examples that we would like to explain, including

$$m((1+x)(1+y)(1+w)+(1-x)(1-y)(1-w)z) = \frac{93}{\pi^4}\zeta(5) \tag{17}$$

in [7].

The algebraic computation, although easier to perform than the integral, requires some happy ideas that seem to be different for each case. it would be nice to have an algorithm to perform such decompositions. It would be nice being able to determine beforehand whether it is possible to achieve such decompositions or not in each example. (These questions are unknown for the two-variable case as well).

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