Multivariable Mahler Measure and Regulators

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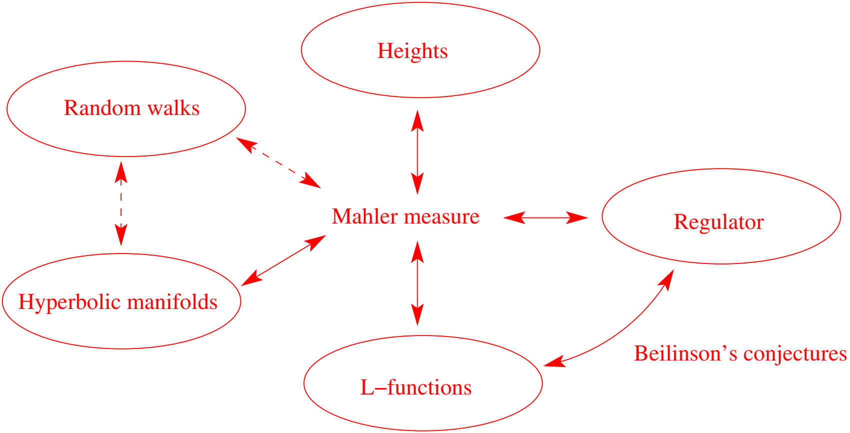
Heights
Random walks
Mahler measure
Regulator
Hyperbolic manifolds
Beilinson's conjectures
L–functions

Mahler measure for one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n-1|} = \begin{cases} |\alpha| & \text{if } |\alpha|>1 \\ 1 & \text{if } |\alpha|<1 \end{cases}$$

For

$$P(x) = a\prod_i (x-\alpha_i)$$

$$M(P) = |a|\prod_i \max\{1,|\alpha_i|\}$$

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+|\alpha_i|$$

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Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

Lehmer's question

Lehmer (1933)

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$$

$= \log(1.176280818\dots) = 0.162357612\dots$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

Does there exist $C>0$, for all $P(x)\in\mathbb{Z}[x]$

$$m(P)=0 \quad \text{or} \quad m(P)>C??$$

Is the above polynomial the best possible?

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Is the above polynomial the best possible?

- Dobrowolski (1979): P monic, irreducible, noncyclotomic, of degree d

$$M(P) > 1 + c\left(\frac{\log\log d}{\log d}\right)^3.$$

- Breusch (1951): P monic, irreducible, nonreciprocal,

$$M(P) > 1.324717... = \text{real root of} \quad x^3 - x - 1$$

(rediscovered by Smyth (1971))

- Bombieri & Vaaler (1983): $M(P) < 2$, then P divides a $Q \in \mathbb{Z}[x]$ whose coefficients belong to $\{-1, 0, 1\}$.

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Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| d\theta_1 \ldots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |e^{2\pi i \theta} - \alpha| d\theta = \log^+ |\alpha|$$

recovers one-variable case.

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Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

where h is the logarithmic Weil height.

Boyd & Lawton Theorem

$P \in \mathbb{C}[x_1, \ldots, x_n]$

$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots, x^{k_n})) = m(P(x_1, x_2, \ldots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

- $$m(1+x+y)=\frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2)=\mathrm{L}'(\chi_{-3},-1)$$

- $$m(1+x+y+z)=\frac{7}{2\pi^2}\zeta(3)$$

$$\mathrm{L}(\chi_{-3},s)=\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s}\qquad \chi_{-3}(n)=\begin{cases}1 & n\equiv 1 \bmod 3\\ -1 & n\equiv -1 \bmod 3\\ 0 & n\equiv 0 \bmod 3\end{cases}$$

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathrm{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathrm{L}'(A,0)$$

$$A: Y^2 = X^3 - 44X + 112$$

Examples in three variables

- Condon (2003):

$$\pi^2 m\left(z-\left(\frac{1-x}{1+x}\right)(1+y)\right)=\frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m\left(z(1-xy)^2-(1-x)(1-y)\right)=\frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005):

$$\pi^2 m(x^2+1+(x+1)y+(x-1)z)=\pi \mathrm{L}(\chi_{-4},2)+\frac{21}{8}\zeta(3)$$

Examples with more than three variables

L.(2003):

- $$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 24\mathrm{L}(\chi_{-4},4)$$

- $$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right) = 62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

- $$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = 93\zeta(5)$$

Known fromulas for n.

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$\mathcal{L}_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

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$\mathcal{L}_k$ satisfies lots of functional equations

$$\mathcal{L}_k\left(\frac{1}{x}\right) = (-1)^{k-1}\mathcal{L}_k(x) \qquad \mathcal{L}_k(\bar{x}) = (-1)^{k-1}\mathcal{L}_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \,\mathrm{rank}\, 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

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An algebraic integration for Mahler measure

Deninger (1997): General framework

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x,y) = \log|x|\,\mathrm{d}\arg y - \log|y|\,\mathrm{d}\arg x$$

$$\eta(x,1-x) = \mathrm{d}D(x) \qquad \mathrm{d}\eta(x,y) = \operatorname{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$$

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The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi i)^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{dx}{x} \frac{dy}{y} \frac{dz}{z}$$

$$= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{dx}{x} \frac{dy}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{dx}{x} \frac{dy}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x,y,z)$$

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$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \eta(x,y,z)$$

$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y| \wedge \mathrm{d}\log|z| - \mathrm{d}\arg y \wedge \mathrm{d}\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z| \wedge \mathrm{d}\log|x| - \mathrm{d}\arg z \wedge \mathrm{d}\arg x\right)$$

$$+\log|z|\left(\frac{1}{3}\mathrm{d}\log|x| \wedge \mathrm{d}\log|y| - \mathrm{d}\arg x \wedge \mathrm{d}\arg y\right)$$

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} \wedge \frac{\mathrm{d}z}{z}\right)$$

$$\eta(x,1-x,y) = \mathrm{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$z = \frac{1-x}{1-y}$$

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

$$m(P) = \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,1-x,y)+\eta(y,1-y,x) = \frac{1}{(2\pi)^2}\int_{\partial\Gamma}\omega(x,y)+\omega(y,x)$$

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$$m(P) = \frac{1}{(2\pi)^2}\int_\Gamma \eta(x, 1-x, y) + \eta(y, 1-y, x) = \frac{1}{(2\pi)^2}\int_{\partial\Gamma} \omega(x, y) + \omega(y, x)$$

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

Maillot: if $P \in \mathbb{R}[x,y,z]$,

$$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

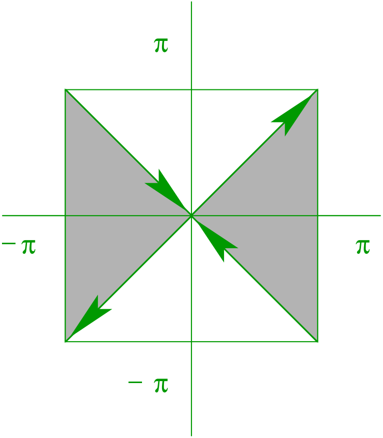
ω defined in

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)-(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$= \frac{1}{4\pi^2} 8(\mathcal{L}_3(1)-\mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_\Gamma \eta(x,y,z)$$

Need

$$x \wedge y \wedge z = \sum r_i \;\; x_i \wedge (1-x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(X)^*) \otimes \mathbb{Q}$,

($\{x,y,z\} = 0$ in $K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$) then

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_\Gamma \eta(x_i, 1-x_i, y_i)$$

$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Let

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] = 0$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.
F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F] / \langle [0], [\infty], R_2(x,y) \rangle$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.
Then

$$\int_\gamma \omega(x,y) = \sum r_i \, \mathcal{L}_3(x_i)|_{\partial\gamma}$$

Big picture in three variables

$$\cdots \to K_4(\partial\Gamma) \to K_3(X, \partial\Gamma) \to K_3(X) \to \dots$$

$$\partial\Gamma = X \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Polylogarithmic motivic complexes

Beilinson (after work of Bloch, Deligne, Beilinson, etc)
$r_{\mathcal{D}} : gr_j^{\gamma} K_{2j-i}(X)_{\mathbb{Q}} \to H^i_{\mathcal{D}}(X, \mathbb{R}(j))$

Goncharov: possible "Explicit construction" of $r_{\mathcal{D}}$ and $gr_j^{\gamma} K_{2j-i}(X)_{\mathbb{Q}}$.
F field, define $\mathcal{R}_n(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ and

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{R}_n(F)$$

$$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$$

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{R}_n(F)$$

$$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$$

$$\mathbb{Z}[\mathbb{P}^1_F] \xrightarrow{\delta_n} \begin{cases} \mathbf{B}_{n-1}(F) \otimes F^* & \text{if } n \geq 3 \\ \bigwedge^2 F^* & \text{if } n = 2 \end{cases}$$

$$\delta_n(\{x\}) = \begin{cases} \{x\}_{n-1} \otimes x & \text{if } n \geq 3 \\ (1-x) \wedge x & \text{if } n = 2 \\ 0 & \text{if } \{x\} = \{0\}, \{1\}, \{\infty\} \end{cases}$$

$$\mathcal{A}_n(F) := \ker \delta_n$$

$$\mathcal{R}_n(F) := \langle \alpha(0) - \alpha(1), \alpha(t) \in \mathcal{A}_n(F(t)) \rangle$$

$$\delta_n(\mathcal{R}_n(F)) = 0$$

$\mathbf{B}_F(n)$:

$$\mathbf{B}_n(F) \xrightarrow{\delta} \mathbf{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathbf{B}_2(F) \otimes \wedge^{n-2} F^* \xrightarrow{\delta} \wedge^n F^*$$

$$\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \longrightarrow \delta_p(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i .$$

$$H^n(\mathbf{B}_F(n)) \quad \cong \quad K_n^M(F)$$

Conjecture

$$H^i(\mathbf{B}_F(n) \otimes \mathbb{Q}) \cong gr_n^\gamma K_{2n-i}(F)_\mathbb{Q}$$

(Goncharov) X complex algebraic variety. There exist $\eta_n(m)$ inducing a homomorphism of complexes

$$\begin{array}{ccccccc}
\mathbf{B}_n(\mathbb{C}(X)) & \stackrel{\delta}{\to} & \mathbf{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \stackrel{\delta}{\to} & \dots & \stackrel{\delta}{\to} & \bigwedge^n \mathbb{C}(X)^* \\
\downarrow \eta_n(1) & & \downarrow \eta_n(2) & & & & \downarrow \eta_n(n) \\
\mathcal{A}^0(X)(n-1) & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^1(X)(n-1) & \stackrel{\mathrm{d}}{\to} & \dots & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^{n-1}(X)(n-1)
\end{array}$$

($\mathcal{A}^i(X)(j)$ = smooth i-forms with values in $(2\pi \mathrm{i})^j\mathbb{R}$) such that

- $\eta_n(1)(\{x\}_n) = \widehat{\mathcal{L}}_n(x)$.
- $\mathrm{d}\eta_n(n)(x_1 \wedge \cdots \wedge x_n) = \pi_n \left(\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \right)$.
- $\eta_n(m)$ compatible with residues (residues are given by tame symbols).

Conjecture

"Image of $\eta_n(i)$" coincides with image of regulator

Deninger(1997)

$$m(P) = m(P^*) + \frac{1}{(-2i\pi)^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$$\pi^{2n} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right)$$

$$= \sum_{h=1}^{n} c_{n,h} \pi^{2n-2h} \zeta(2h+1)$$

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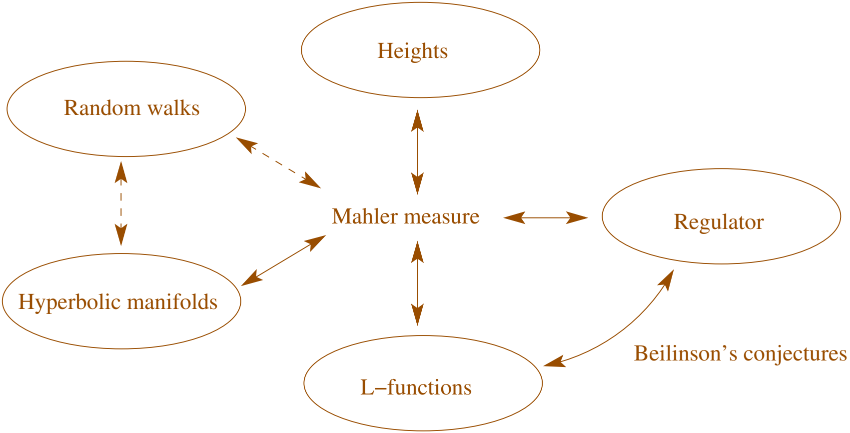
Heights
Random walks
Mahler measure
Regulator
Hyperbolic manifolds
Beilinson's conjectures
L–functions