Mahler Measure and values of Regulators

Matilde N. Lalín

Institute for Advanced Study
mlalin@math.ias.edu
http://www.math.ias.edu/~mlalin

October 5th, 2005

Mahler measure of multivariate polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}$$

$$\mathbb{T}^n = S^1 \times \cdots \times S^1$$

Mahler measure of multivariate polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n}$$

$$\mathbb{T}^n = S^1 \times \cdots \times S^1$$

$$P(x) = a_d \prod_{n=1}^{d} (x - \alpha_n)$$

$$m(P) = \log |a_d| + \sum_{n=1}^{d} \log^+ |\alpha_n|$$

Several-variable case?

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right)\stackrel{?}{=}\mathrm{L}'(E,0)$$

E elliptic curve, projective closure of

$$x+\frac{1}{x}+y+\frac{1}{y}+1=0$$

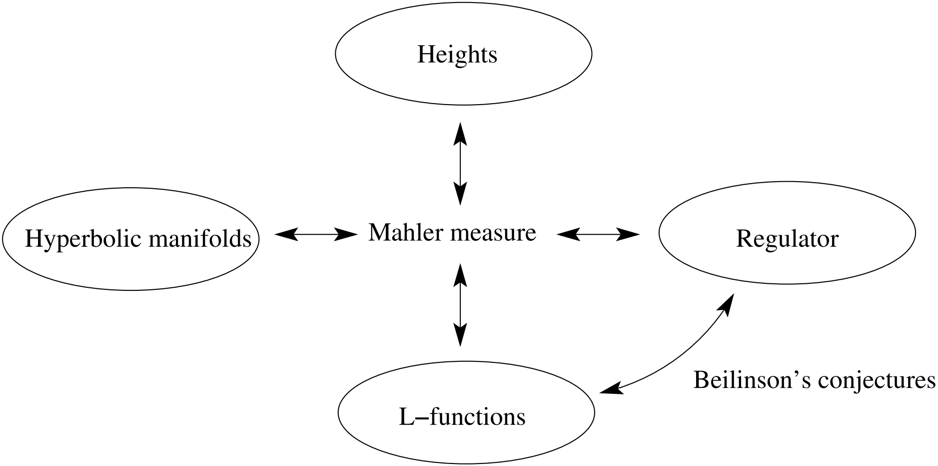
L.(2003)

$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=\frac{93}{\pi^4}\zeta(5)$$

L.(2003)

$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=\frac{93}{\pi^4}\zeta(5)$$

$$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)z\right)=\text{nice formula}$$


Heights
Hyperbolic manifolds
Mahler measure
Regulator
L–functions
Beilinson's conjectures

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

$$\text{special value of } \mathrm{L}_X \sim_{\mathbb{Q}^*} \int_\gamma r(\xi)$$

- X Arithmetic-geometric object
- $\xi \in K$ Finitely-generated abelian group
- $r : K \to$ smooth differential forms

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

$$\text{special value of } \mathrm{L}_X \sim_{\mathbb{Q}^*} \int_\gamma r(\xi)$$

- X Arithmetic-geometric object
- $\xi \in K$ Finitely-generated abelian group
- $r : K \to$ smooth differential forms

(E.g. Dirichlet class number formula, F real quadratic,
$\zeta_F'(0) \sim_{\mathbb{Q}^*} \log|\epsilon| \qquad \epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997)
$P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_\gamma \eta(x, y)$$

$$\eta(x, y) = \log |x| \mathrm{d} \arg y - \log |y| \mathrm{d} \arg x$$

The three-variable case

L.(2005): Smyth's example

$$P(x, y, z) = (1 - x) - (1 - y)z$$

The three-variable case

L.(2005): Smyth's example

$$P(x,y,z) = (1-x) - (1-y)z$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

The three-variable case

L.(2005): Smyth's example

$$P(x,y,z) = (1-x) - (1-y)z$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

The three-variable case

L.(2005): Smyth's example

$$P(x,y,z) = (1-x) - (1-y)z$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$\Gamma = S \cap \{|x| = |y| = 1, |z| \geq 1\}$ $\qquad$ $S = \{P(x,y,z) = 0\}$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma}\eta(x,y,z)$$

$$\eta(x,y,z) = \frac{1}{2}\mathrm{Alt}_3\left(\log|x|\left(\frac{1}{3}\mathrm{d}\log|y|\wedge\mathrm{d}\log|z| - \mathrm{d}\arg y\wedge\mathrm{d}\arg z\right)\right)$$

We want to apply Stokes' Theorem

$$\eta(x, 1 - x, y) = \mathrm{d}\,\omega(x, y)$$

$$\omega(x, y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1 - x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1 - x|)$$

We want to apply Stokes' Theorem

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x, y)$$

$$\omega(x, y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$D(x) = \mathrm{Im}(\mathrm{Li}_2(x)) + \log|x|\arg(1-x)$$

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2}$$

In Γ,

$$z = \frac{1-x}{1-y}$$

$$m((1-x)+(1-y)z) = -\frac{1}{4\pi^2}\int_{\Gamma} \eta(x,y,1-x) - \eta(x,y,1-y)$$

In Γ,

$$z = \frac{1-x}{1-y}$$

$$m((1-x)+(1-y)z) = -\frac{1}{4\pi^2}\int_\Gamma \eta(x,y,1-x) - \eta(x,y,1-y)$$

$$= \frac{1}{4\pi^2}\int_\gamma \omega(x,y) + \omega(y,x)$$

We want to apply Stokes' Theorem again.

$$\omega(x, x) = \mathrm{d}\mathcal{L}_3(x)$$

$$\mathcal{L}_3(x) = \mathrm{Re}\left(\mathrm{Li}_3(x) - \log|x|\mathrm{Li}_2(x) - \frac{1}{3}\log^2|x|\log(1-x)\right)$$

We want to apply Stokes' Theorem again.

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$\mathcal{L}_3(x) = \mathrm{Re}\left(\mathrm{Li}_3(x) - \log|x|\mathrm{Li}_2(x) - \frac{1}{3}\log^2|x|\log(1-x)\right)$$

Maillot: if $P \in \mathbb{R}[x,y,z]$,

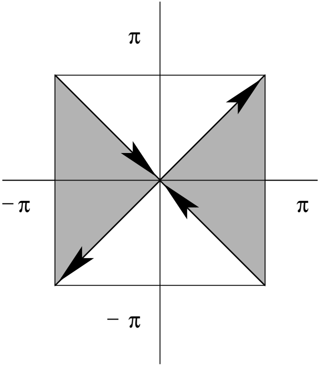
$$\Gamma = S \cap \{|x| = |y| = 1,\ |z| \geq 1\}$$

$$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

ω (re)defined in

$$C = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

We solved

$$x \wedge y \wedge z = \sum r_i \;\; x_i \wedge (1 - x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$.

We solved

$$x \wedge y \wedge z = \sum r_i \;\; x_i \wedge (1 - x_i) \wedge y_i$$

in $\bigwedge^3 (\mathbb{C}(S)^*) \otimes \mathbb{Q}$.
Same as

$$\{x, y, z\} = 0$$

in $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$.

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$
$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] = 0$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.
F field,
$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F] / \langle [0], [\infty], R_2(x,y) \rangle$$

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$
$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] = 0$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.
F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F] / \langle [0], [\infty], R_2(x,y) \rangle$$

We solved

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.
Goncharov: zero element in $\mathrm{gr}^{\gamma}_3 K_4(\mathbb{C}(C)) \otimes \mathbb{Q}$ (?).

Big picture

$$\cdots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \dots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Open: use method to explain/compute

- n-variable cases ($n > 3$)
- non-exact cases