HYPERBOLIC HERON TRIANGLES AND ELLIPTIC CURVES

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ABSTRACT. We define hyperbolic Heron triangles (hyperbolic triangles with "rational" side-lengths and area) and parametrize them in two ways via rational points of certain elliptic curves. We show that there are infinitely many hyperbolic Heron triangles with one angle α and area A for any (admissible) choice of α and A; in particular, the congruent number problem has always infinitely many solutions in the hyperbolic setting. We also explore the question of hyperbolic triangles with a rational median and a rational area bisector (median splitting the triangle in half).

The problem of finding triangles with rational area and side lengths in the Euclidean plane goes at least as far back as $\sim$ 600 A.D with the Indian mathematician Brahmagupta (see [5]). If the triangle is assumed right, this is the classical congruent number problem (a number is congruent if it is the area of a right triangle with rational sides). Remarkably, this problem is equivalent to finding (non-torsion) rational solutions to the elliptic curve $y^2 = x^3 - n^2x$. For non-right triangles, it was shown by Goins and Maddox [5] that Heron triangles are in correspondence with rational points on the curve $y^2 = x(x - n\tau)(x + n\tau^{-1})$, where n denotes the area and τ is the tangent of half of an angle.

In this paper we investigate the analog problem in the hyperbolic plane[1]. The first concept we need to transport is that of rationality. Unlike the Euclidean case where trigonometric laws are polynomial in the area, side lengths and sine and cosine of the angles, their hyperbolic counter part involve the *hyperbolic* sine and cosine of the side lengths, and the sine and cosine of the area. For instance, in a triangle with side lengths a, b, c and a right angle opposing side a (see Figure 1), Pythagoras' Theorem and the area A are given by:

$$\cosh(a) = \cosh(b)\cosh(c) \text{ and } \sin(A) = \frac{\sinh(b)\sinh(c)}{\cosh(a) + 1}$$

respectively (see [9, §3.5]). It is thus natural to ask that the sine/cosine of the angles be rational, and similarly for the hyperbolic functions of the sides, instead of directly asking that these quantities be rational.

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[1]The unique complete, connected, simply connected Riemannian 2-manifold of constant sectional curvature -1.

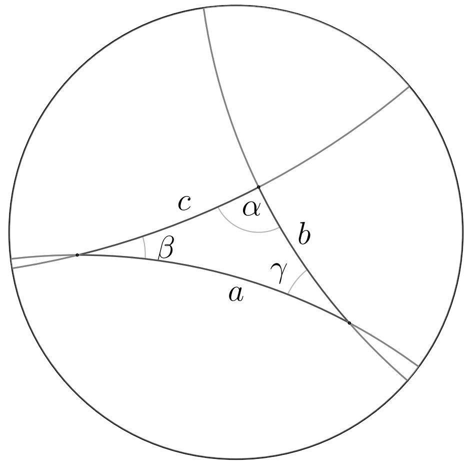

FIGURE 1. A hyperbolic triangle in the Poincaré disk model of the hyperbolic plane.

For the rationality of the hyperbolic functions on the side lengths, at least two conventions have been used in the literature. One such choice was made by Brody and Schettler [1]. They call a triangle *rational* if the hyperbolic tangent of its side lengths are rational, and they use this definition to prove a correspondence between rational triangles of semi-perimeter $\tanh^{-1}(\sigma)$ and inradius $\sinh^{-1}(\rho)$ and rational points on the curve $\sigma(x^2y^2 + xy + \rho^2(x^2 + xy + y^2 - 1)) = (1 + \rho^2)(x^2y + xy^2)$.

A second (stronger) choice is that of Hartshorne and van Luijk [7]. Here they call a length x *rational* if $e^x \in \mathbb{Q}$, and then study Pythagorean triples in this context. This is the notion we will adopt in this paper.

For the area, the Gauss–Bonnet theorem implies that a triangle with angles α, β, γ has area

$$A = \pi - \alpha - \beta - \gamma.$$

The previous discussion together with the above formula for the area suggest a common definition of rationality for angles and area: we call an area A and an angle α rational if the sine and cosine of these quantities are rational (or equivalently if $e^{iA}, e^{i\alpha} \in \mathbb{Q}[i]$). Thus a hyperbolic triangle with area A, angles α, β, γ and side lengths a, b, c is a *hyperbolic Heron triangle* if

$$e^a, e^b, e^c \in \mathbb{Q} \quad \text{and} \quad e^{i\alpha}, e^{i\beta}, e^{i\gamma}, e^{iA} \in \mathbb{Q}[i].$$

Recall that $e^{ix} \in \mathbb{Q}[i]$ if and only if $\cos(x) = \frac{1-t^2}{1+t^2}$ and $\sin(x) = \frac{2t}{1+t^2}$ for some $t \in \mathbb{Q}$. Indeed, we have

(1)
$$e^{ix} = \frac{i-t}{i+t} \in \mathbb{Q}[i] \iff t = \frac{\sin(x)}{1+\cos(x)} \in \mathbb{Q} \iff (\cos(x), \sin(x)) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right).$$

By abuse of terminology we will call t the *rational angle* (resp. *rational area*) of a hyperbolic triangle if its angle (resp. area) is x.

Our main result is the following:

Theorem 1. *There is a one-to-one correspondence between hyperbolic Heron triangles having rational area m and one rational angle u, and rational points on the curve:*

$$y^2 = x(x-n)(x-n(u^2+1)), \quad \text{where } n = m(m^2+1)(2u - m(u^2-1))$$

provided that $m, u > 0, mu < 1$ and x, y satisfy the following open condition:

$$\text{(A)} \quad 0 < \frac{-y + (m+u)(1-mu)(x-n)}{m(2u - m(u^2-1))(x-(m+u)^2(m^2+1))} < \frac{1-mu}{m+u}$$
$$\text{and} \qquad (x,y) \neq ((m^2+1)(m+u)^2, u^2(m^2+1)^2(m+u)(mu-1)).$$

The open condition (A) encodes the fact that the angles and area are positive and $< \pi$, and it also excludes one further point on the curve. Computing the rank of the above curve (and proving it is ≥ 1) we get:

Corollary 2. *For almost all $m, u \in \mathbb{Q}$ with $m, u > 0$ and $mu < 1$ there are infinitely many hyperbolic Heron triangles with rational area m and one rational angle u.*

Here "almost" means all values except possibly those lying on finitely many curves $\{f_i(m,u) = 0\}$ of with $\deg(f_i) \geq 8$. Setting $u = 1$ (which corresponds to the case of one right angle), it is easily verified that the $f_i(m,1)$ have no rational solution. Thus the congruent number problem always has infinitely many solutions in the hyperbolic setting.

One can also parametrize hyperbolic Heron triangles using side lengths:

Theorem 3. *There is a one-to-one correspondence between hyperbolic Heron triangles having two sides of lengths $\log(v)$ and $\log(w)$ with $v, w \in \mathbb{Q}$, and rational points on the curve:*

$$y^2 = x\left(x - (v - v^{-1})^2\right)\left(x - (w - w^{-1})^2\right).$$

provided that $v, w > 1$ and x, y satisfy the following open condition:

$$\text{(B)} \qquad \max\left(\frac{v}{w}, \frac{w}{v}\right) < \frac{v^2w^2x + (v^2-1)(w^2-1)}{vw\big(x + (v^2-1)(w^2-1)\big)} < vw \quad \text{and} \quad x > 0.$$

Here the condition (B) describes the fact that the lengths of the sides are positive and satisfy the triangle inequality. It turns out that (perhaps surprisingly) this curve has generically rank 0 over $\mathbb{Q}$, suggesting that it is harder to complete two sides to a rational triangle, than it is to find a triangle with a fixed angle and area.

In the Euclidean world, a further question one can ask is that of the rationality of the medians. This was first asked (and solved) by Euler in the case of one rational median [4]. The question for Heron triangles having 3 rational medians is problem D21 in Guy's book [6]; it is still open as of today. The two-median problem was solved by Buchholz and Rathbun [2, 3].

In the hyperbolic setting we have again two choices in our translation of median: the (hyperbolic) line from one vertex meeting the opposite edge in its midpoint, or the line from one vertex separating the triangle into two triangles of equal area. We will call the first one the *median* and the second one the *area bisector*. These two lines are not the same in general (one can easily be convinced by considering ideal triangles), but they coincide in an isosceles triangle (for the lines passing through the apex).

The first (negative) result along those lines concerns the simple case of equilateral triangles.

Proposition 4. *There are no equilateral hyperbolic Heron triangles. Moreover, equilateral triangles with rational sides or rational angles have no rational medians/area bisectors.*

With similar methods as those used for Theorem 1 and Theorem 3, we can parametrize hyperbolic triangles with one rational median and Heron triangles with one rational area bisector using elliptic curves. However, these curves are quite complicated, so we will present them in Section 5 and 6 respectively. We will only give one corollary of this parametrization, in the case of medians:

Theorem 5. *For almost all values $u, w \in \mathbb{Q}$ there are infinitely many hyperbolic triangles having rational side lengths, two of which given by $a = 2\log(u)$ and $b = \log(w)$, and one rational median (intersecting side a).*

Here almost all means all but those cut out by a curve in u, w.

The paper is organized as follows. Section 1 provides some basic formulas from hyperbolic trigonometry, including the hyperbolic laws of cosines and the hyperbolic law of sines. Section 2 and Section 3 are dual to each other, and cover the parametrization of Heron triangles in terms of angles (Theorem 1) and sides (Theorem 3) respectively. Section 4 is focused on medians and area bisectors in the simple case of equilateral triangles. Finally, in Section 5 we give the parametrization of hyperbolic triangles with rational side lengths and one rational median, while Section 6 is devoted to the dual computation of the parametrization of Heron triangles with one rational area bisector.

1. Some preliminaries on hyperbolic trigonometry

In this section we recall some basics of trigonometry in the hyperbolic setting. A basic reference is §3.5 of [9]. Consider a hyperbolic triangle with area A, angles α, β, γ and side lengths a, b, c, where a (resp. b, c) is opposite to α (resp. β, γ), as in Figure 1.

The hyperbolic law of cosines for the angles says

$$\sin(\beta)\sin(\gamma)\cosh(a) = \cos(\beta)\cos(\gamma) + \cos(\alpha) \tag{2}$$

and similarly for $\cosh(b)$, $\cosh(c)$.

The hyperbolic law of cosines for the side lengths says

$$\sinh(b)\sinh(c)\cos(\alpha) = \cosh(b)\cosh(c) - \cosh(a), \tag{3}$$

and similarly for $\cos(\beta)$, $\cos(\gamma)$.

The hyperbolic law of sines says

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)}. \tag{4}$$

It may be convenient to clear denominators. There are two ways of doing this. If we are working with angles and intend to deduce information about the hyperbolic sines of the sides, we may write

$$\Delta_1 := \sinh(a)\sin(\beta)\sin(\gamma) = \sinh(b)\sin(\alpha)\sin(\gamma) = \sinh(c)\sin(\alpha)\sin(\beta).$$

If, instead, we are working with sides and intend to deduce information about the sines of the angles, we may write

$$\Delta_2 := \sin(\alpha)\sinh(b)\sinh(c) = \sin(\beta)\sinh(a)\sinh(c) = \sin(\gamma)\sinh(a)\sinh(b).$$

As in (1), let

$$t = \frac{\sin(\alpha)}{1+\cos(\alpha)}, \quad u = \frac{\sin(\beta)}{1+\cos(\beta)}, \quad s = \frac{\sin(\gamma)}{1+\cos(\gamma)}, \quad m = \frac{\sin(A)}{1+\cos(A)}.$$

We can then express the trigonometric functions as follows,

$$\begin{array}{llll} \cos(\alpha) = \frac{1-t^2}{1+t^2}, & \cos(\beta) = \frac{1-u^2}{1+u^2}, & \cos(\gamma) = \frac{1-s^2}{1+s^2}, & \cos(A) = \frac{1-m^2}{1+m^2}, \\ \sin(\alpha) = \frac{2t}{1+t^2}, & \sin(\beta) = \frac{2u}{1+u^2}, & \sin(\gamma) = \frac{2s}{1+s^2}, & \sin(A) = \frac{2m}{1+m^2}. \end{array}$$

Thus $t \in \mathbb{Q}$ if and only if $\cos(\alpha), \sin(\alpha) \in \mathbb{Q}$, and similarly for the other parameters and angles/area. For the hyperbolic trigonometric functions of the sides, using the hyperbolic laws of cosines and sines, we get:

$$\begin{array}{lll} \cosh(a) = \frac{1-(u^2+s^2)t^2+u^2s^2}{2us(1+t^2)}, & \cosh(b) = \frac{1-(t^2+s^2)u^2+t^2s^2}{2ts(1+u^2)}, & \cosh(c) = \frac{1-(t^2+u^2)s^2+t^2u^2}{2tu(1+s^2)}, \\ \sinh(a) = \frac{\delta}{2us(1+t^2)}, & \sinh(b) = \frac{\delta}{2ts(1+u^2)}, & \sinh(c) = \frac{\delta}{2tu(1+s^2)}, \end{array}$$

where

$$\delta = \sqrt{(1-tu+us+st)(1+tu-us+st)(1+tu+us-st)(1-tu-us-st)}. \tag{5}$$

Using that $A = \pi - \alpha - \beta - \gamma$, one can then write

$$\Delta_1 = \frac{2\delta}{(1+t^2)(1+u^2)(1+s^2)}, \qquad m = \frac{1-tu-us-st}{t+u+s-tus}.$$

Thus, if the angles are rational (in the sense defined in the introduction), then the hyperbolic cosines of the side lengths are also rational (due to the law of cosines (2)), while the hyperbolic sines of the side lengths are rational if and only if Δ_1 is rational.

By contrast, if the sides are rational (again, in the sense defined in the introduction), the law of cosines (3) implies that, for a non-degenerate triangle, the cosines of the angles are also rational, while the sines of the angles are rational if and only if Δ_2 is rational. Furthermore, observe that if A denotes the area of the triangle, we have

$$\begin{aligned} \sin(A) = & -\sin(\alpha)\sin(\beta)\sin(\gamma) + \sin(\alpha)\cos(\beta)\cos(\gamma) \\ & + \sin(\beta)\cos(\alpha)\cos(\gamma) + \sin(\gamma)\cos(\alpha)\cos(\beta). \end{aligned}$$

When the side lengths are rational, $\frac{\Delta_2}{\sin(\alpha)} \in \mathbb{Q}$, and similarly for β, γ, and we conclude that $\sin(A) = r\Delta_2$ for some $r \in \mathbb{Q}$. Thus a hyperbolic triangle with rational side lengths has rational area exactly when $\Delta_2 \in \mathbb{Q}$, i.e., when all its angles are rational.

2. Hyperbolic Heron triangles – Angle parametrization

2.1. **Finding the angle parametrization.** In this section we give the parametrization of hyperbolic Heron triangles in terms of angles and area, and prove Theorem 1 and Corollary 2. Note that throughout this paper we only consider the case of non-degenerate bounded triangles (i.e., with no vertex at infinity), so that its angles and area are always positive and its side lengths finite.

Let $\alpha, \beta, \gamma > 0$ denote the angles of a hyperbolic triangle. Assume they are rational (as defined in the introduction) i.e., that $e^{i\alpha}, e^{i\beta}, e^{i\gamma} \in \mathbb{Q}[i]$. Since the area $A = \pi - \alpha - \beta - \gamma$ we get that A is also rational: $e^{iA} \in \mathbb{Q}[i]$. As before, we denote by a (resp. b, c) the side length opposite α (resp. β, γ), see Figure 1.

As seen in Section 1, the hyperbolic law of cosines for the angles (2) and similar formulas imply that $\cosh(a), \cosh(b), \cosh(c) \in \mathbb{Q}$. (Here we use that the triangles under consideration are non-degenerate and therefore the sines of α, β, γ are non-zero.) To get a hyperbolic Heron triangle, it thus only remains to find the condition that sinh of a, b, c are also rational. As explained in Section 1, this happens exactly when $\Delta_1 \in \mathbb{Q}$. Squaring equation (2), we obtain

$$\Delta_1^2 = (\cos(\alpha) + \cos(\beta)\cos(\gamma))^2 - \sin(\beta)^2 \sin(\gamma)^2 \in \mathbb{Q}. \tag{6}$$

Using trigonometric identities, we can rewrite this as a symmetric expression in α, β, γ:

$$\begin{aligned} 2\Delta_1^2 = &\cos(-\alpha+\beta+\gamma) + \cos(\alpha-\beta+\gamma) + \cos(\alpha+\beta-\gamma) \\ &+ \cos(\alpha+\beta+\gamma) + \cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) + 1. \end{aligned}$$

Substituting $\gamma = \pi - A - \alpha - \beta$ and expanding the cosines, we obtain (writing $c_A = \cos(A), s_A = \sin(A)$, etc...):

$$\begin{aligned} 2\Delta_1^2 = &(c_A^2 - s_A^2)\big[(c_\alpha c_\beta - s_\alpha s_\beta)^2 - (c_\alpha s_\beta + c_\beta s_\alpha)^2\big] \\ &-4c_A s_A\big[c_\alpha s_\alpha(c_\beta^2 - s_\beta^2) + c_\beta s_\beta(c_\alpha^2 - s_\alpha^2)\big] \\ &-c_A\big[(c_\alpha c_\beta - s_\alpha s_\beta)^2 - (c_\alpha s_\beta + c_\beta s_\alpha)^2 + 2c_\alpha^2 + 2c_\beta^2 - 1\big] \\ &+4s_A(c_\alpha s_\alpha c_\beta^2 + c_\beta s_\beta c_\alpha^2) + 2c_\alpha^2 + 2c_\beta^2 - 1. \end{aligned} \tag{7}$$

We wish to express this in terms of *rational* angles. Setting

$$t = \frac{\sin(\alpha)}{1+\cos(\alpha)}, \quad u = \frac{\sin(\beta)}{1+\cos(\beta)}, \quad m = \frac{\sin(A)}{1+\cos(A)},$$

as in Section 1, and $w = (m^2+1)(u^2+1)(t^2+1)\Delta_1$, equation (7) rewrites as:

$$\begin{aligned} w^2 = &4m(mu^2 - m - 2u)(mt^2 - 2t - m)\big[(mu^2 - m - 2u)t^2 \\ &+ (-4mu - 2u^2 + 2)t - mu^2 + m + 2u\big]. \end{aligned} \tag{8}$$

Condition	Translation
$m > 0$	$0 < A < \pi$
$t > 0$	$0 < \alpha < \pi$
$u > 0$	$0 < \beta < \pi$
$(tm + tu + mu - 1)(tmu - t - m - u) > 0$	$0 < \gamma < \pi$
$mu < 1$	$0 < A + \beta < \pi.$

TABLE 1. Conditions for Theorem 1

Finally, setting $n = m(m^2 + 1)(2u - m(u^2 - 1))$ and applying the change of variables

$$\begin{aligned} y =& -\frac{(2u - m(u^2-1))}{4}\big[2(2u - m(u^2-1))(mu-1)(m+u)m^2(3t^2-1) \\ &+ 2(m^2u^2 - m^2 - 6mu - 2u^2 + 2)(2u - m(u^2-1))m^2t + 2(2u - m(u^2-1))^2m^3t^3 \\ &+ (mu-1)(m+u)mw + (2u - m(u^2-1))m^2tw\big] \\ x =& \frac{(2u - m(u^2-1))}{4}\big[4(mu-1)(m+u)mt + 2(m^2u^2 + m^2 - 2mu + 2)m \\ &+ 2(2u - m(u^2-1))m^2t^2 + mw\big], \end{aligned}$$

with inverse

$$\begin{aligned} t =& \frac{-y + (m+u)(1-mu)(x-n)}{m(2u - m(u^2-1))(x - (m+u)^2(m^2+1))} \\ w =& 2(m(2u - m(u^2-1))(x - (m+u)^2(m^2+1))^2)^{-1} \\ &\times \big[x^3 - 3(m^2+1)(u+m)^2x^2 \\ &+ m(m^2+1)^2(2u - m(u^2-1))(m^2u^4 + 2u^4 + 2mu^3 + 2m^2u^2 + 4u^2 + 6mu + 3m^2)x \\ &- m^2(m^2+1)^3(u+m)^2(u^2+1)(2u - m(u^2-1))^2 - 2(m^2+1)^2u^2(u+m)(mu-1)y\big] \end{aligned}$$

we get the equation:

$$y^2 = x\big(x - n\big)\big(x - n(u^2+1)\big). \tag{9}$$

Its discriminant is $2^4u^4(u^2+1)^2n^6 = 2^4u^4m^6(u^2+1)^2(m^2+1)^6(2u - m(u^2-1))^6$.

We are ready to complete the proof of Theorem 1.

Proof of Theorem 1. It just remains to exhibit the open condition (A), which encodes the fact that the parameters give rise to an actual hyperbolic triangle. Assume first that the inverse of the change of variables is defined; this happens everywhere except when the expression $m(2u - m(u^2-1))(x - (m+u)^2(m^2+1))$ is 0.

We claim that the conditions on m, u, t, along with their meaning are given by Table 1. Indeed, the conditions follow immediately from inspecting the identities:

$$m = \frac{\sin(A)}{1+\cos(A)}, \qquad t = \frac{\sin(\alpha)}{1+\cos(\alpha)}, \qquad u = \frac{\sin(\beta)}{1+\cos(\beta)},$$

$$\frac{tm + tu + mu - 1}{tmu - t - m - u} = \frac{\sin(\gamma)}{1+\cos(\gamma)}, \qquad \frac{1-mu}{m+u} = \frac{s+t}{1-st} = \tan\left(\frac{A+\beta}{2}\right),$$

where the notation is the same as in Section 1. We now explain why we need the fifth condition on Table 1. We know that $0 < A, \alpha, \beta, \gamma < \pi$, and we must find a condition encoding $A + \alpha + \beta + \gamma = \pi$. By construction, any solution to (9) gives rise to a set of parameters satisfying this last equation *modulo* 2π; thus we have $A+\alpha+\beta+\gamma \in \{\pi, 3\pi\}$ and we wish to eliminate the 3π case. For a set of parameters satisfying the first four conditions of Table 1, the 3π case happens exactly when any two elements in A, α, β, γ sum up to $> \pi$. Hence we can just pick the condition $A + \beta < \pi$, which is equivalent to $mu < 1$.

Now if m, u are given parameters with $m, u > 0$ and $mu < 1$, the conditions from Table 1 can be simplified to the following open condition on the variable t:

$$0 < t < \frac{1 - mu}{m + u}. \tag{10}$$

Using the above change of variables, we get an open condition in terms of x, y encoding the desired properties. This is the first part of condition (A) in the statement of Theorem 1.

It remains to treat the case of points where the expression for t is not defined. First of all, the conditions $m, u > 0$ and $mu < 1$ imply that $m(2u-m(u^2-1)) > m(2u-u+m) > 0$. Hence the only points on (9) where the change of variables is not defined are those for which $x = (m+u)^2(m^2+1)$. There are actually two points satisfying this condition: the point

$$P = \big((m^2 + 1)(m + u)^2, u^2(m^2 + 1)^2(m + u)(mu - 1)\big)$$

and $-P$. On (8) however, there is only one point leading to a point on (9) with $x = (m + u)^2(m^2 + 1)$: the point (t, w) with

$$t = \frac{-u^4m^4 + 4u^3m^3 + u^2m^4 - 6u^2m^2 - 4um^3 - m^4 - u^2}{2m(m + u)(1 - mu)(2u - m(u^2 - 1))}.$$

To see this, one just applies the change of variable for x to the equation $x = (m + u)^2(m^2 + 1)$; one gets an equation that is linear in w, and whose solution gives a single point on the curve (8). One verifies that its image is the point $-P$. Hence P is the only point on (9) that does not correspond to a hyperbolic triangle, and condition (A) can thus be rewritten as

$$0 < \frac{-y + (m + u)(1 - mu)(x - n)}{m(2u - m(u^2 - 1))(x - (m + u)^2(m^2 + 1))} < \frac{1 - mu}{m + u} \quad \text{and } (x, y) \neq P. \qquad \square$$

2.2. **Rank computations.** In this section we investigate the rank of the above curve and prove Corollary 2. We will write $E_{m,u}$ for the curve given by (9). In this case, we think of m, u as fixed parameters. We may also think of fixing only one parameter (m or u) and studying the resulting family of $K3$-surfaces. When we fix the value of the parameter m (resp. u) we write E_m (resp. E_u), and we may study the points over $\mathbb{C}(u)$ (resp. $\mathbb{C}(m)$). Finally, we refer to the variety E when we keep both u and m free.

Lemma 2.1. *The ranks of the $K3$-surfaces $E_m(\mathbb{C}(u))$ and $E_u(\mathbb{C}(m))$ fulfill the inequalities:*

$$1 \leq \quad \mathrm{rk}(E_m(\mathbb{C}(u))), \mathrm{rk}(E_u(\mathbb{C}(m))) \quad \leq 2.$$

The torsion groups of both E_m and E_u are isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, with the points of order two given by $(0,0),(n,0)$, and $(n(u^2+1),0)$.

The point

$$P(m,u)=\big((m^2+1)(m+u)^2,u^2(m^2+1)^2(m+u)(mu-1)\big)$$

is a point of infinite order on E.

Proof. The lower bound follows from the fact that P is on the curve E and of infinite order, which is easily verified by inspection. There are several ways to prove this. One can consider ℓP for $\ell=1,\dots,12$ and verify that one obtains 12 points that are generically different and different from O. Upon specializing values for the two parameters, Mazur's theorem on the torsion of elliptic curves over $\mathbb{Q}$ guarantees that torsion points can not have order higher than 12, and we conclude that P can not be a torsion point. A much more concrete way to prove this is to observe that torsion injects into specialization, and therefore it suffices to check that P has infinite order for some specific values for m,u. For example, taking $m=-2$, $u=1$ yields the curve $y^2=x(x+20)(x+40)$ with the point $P=(5,75)$. Since $2P=(\frac{961}{36},-\frac{62279}{216})$ has non-integral coordinates, one concludes that P can not be torsion due to Nagell-Lutz theorem. A third way to prove that P is non-torsion is to compute the height pairing of the Mordell–Weil group on E_u (resp. on E_m), which gives $h(P)=2$, and we conclude that P is non-torsion since the height is non-zero (this last method could be exploited to show that P is a generator, but we will not pursue this direction).

We know that E_u is a $K3$-surface because the standard coefficients of the Weierstrass form satisfy $\deg_m(a_j)\le 2j$ with at least one coefficient satisfying $\deg_m(a_j)>j$. For the upper bound, we apply Tate's algorithm [12, IV.9]. E_u has singularities at $m=0,\pm i,\frac{2u}{u^2-1}$, which are all of type I_0^* in the Kodaira classification, and so the number of components of each singular fiber is 5. Now by the Shioda–Tate formula (see [11, Corollary 1.5], or alternatively [10, Corollary 6.7)]) we have

$$\rho(E_u)=\mathrm{rk}(E_u(\mathbb{C}(m)))+2+4\cdot(5-1)=\mathrm{rk}(E_u(\mathbb{C}(m)))+18$$

where $\rho(E_u)$ is the Picard number of E_u. Since E_u is a $K3$-surface, we have $\rho(E_u)\le 20$, and thus $\mathrm{rk}(E_u(\mathbb{C}(m)))\le 2$.

We proceed similarly with E_m, which is also a $K3$-surface. The singularities at $u=0$ and $u=\infty$ are of type I_4, the ones at $u=\pm i$ of type I_2 and the ones at the roots of mu^2-2u-m of type I_0^*. Applying the Shioda–Tate formula again,

$$\rho(E_m)=\mathrm{rk}(E_m(\mathbb{C}(u))+2+2\cdot(4-1)+2\cdot(2-1)+2\cdot(5-1)=\mathrm{rk}(E_m(\mathbb{C}(u))+18.$$

Thus $\mathrm{rk}(E_m(\mathbb{C}(u)))\le 2$ also in this case.

Finally, it is immediate to see that $(0,0),(n,0)$, and $(n(u^2+1),0)$ are points of order 2. Recall that the Euler characteristic satisfies $\chi=2$ for $K3$-surfaces. We combine this with the bound on the rank and Table (4.5) in [8] to conclude that E_u (resp. E_m) has torsion isomorphic to $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Then one can check directly that the points of order 2 cannot be written as twice a point in $E_u(\mathbb{C}(m))$ (resp. in $E_m(\mathbb{C}(u))$). This can be also achieved by considering an specialization such as $m=-2$, $u=1$ that yields $y^2=x(x+20)(x+40)$ and verifying that torsion is exactly $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ (which can be found by using again the theorem of Nagell—Lutz). $\square$

If we set $u = 1$ in the case of E_u, the singularities are at $m = 0, \pm i, \infty$ and the previous Lemma still applies. In this particular case, we can identify a second point of infinite order given by

$$Q(m) = \left(2m(m+1)^2, i4m^2(m^2-1)\right).$$

Since $P(m,1)$ is supported in $\mathbb{Q}(m)$, while $Q(m)$ is not, we conclude that $P(m,1)$ and $Q(m)$ are independent. In fact, by specialization again, we can set $m = -2$, which gives the Weierstrass form $y^2 = x(x+20)(x+40)$ and $Q = (-4, 48i)$. Since working on $\mathbb{Q}[i]$ may be challenging, we consider the quadratic twist $(x,y) \to (-x, -iy)$, which yields $y^2 = x(x-20)(x-40)$ and $Q \to (4,48)$. We see that $2(4,48) = (\frac{2401}{36}, -\frac{62279}{216})$, and therefore $(4,48)$ is non-torsion. From this we conclude that Q is non-torsion in E_1 and the rank of $E_1(\mathbb{C}(m))$ is exactly 2. One can also prove that the height pairing of the Mordell–Weil group on E_1 gives $h(Q) = 2$, and that $\langle P, Q\rangle = 0$, giving an alternative way of showing that $P(m,1)$ and $Q(m)$ are independent.

We move on to the proof of Corollary 2.

Proof of Corollary 2. We need to show that for each fixed $m, u > 0$ with $mu < 1$, there are infinitely many rational points on the curve $E_{m,u}$ satisfying the open condition (A). By a theorem of Poincaré and Hurwitz (see [13, Satz 11, p. 78]) the rational points $E_{m,u}(\mathbb{Q})$ of $E_{m,u}$ are dense in $E_{m,u}(\mathbb{R})$ provided $E_{m,u}(\mathbb{Q})$ is infinite and both connected components of $E_{m,u}(\mathbb{R})$ contain a rational point. Hence the corollary will follow by density, once these two conditions are proven.

Now by Lemma 2.1, $E_{m,u}(\mathbb{Q})$ has positive rank, so it is infinite. And since the three non-trivial torsion points are given by

$$\left(0,0\right), \quad \left(n,0\right), \quad \left(n(u^2+1),0\right)$$

and are rational, we conclude that both components have rational points. □

3. Hyperbolic Heron triangles – Side length parametrization

3.1. **Finding the side lengths parametrization.** This section is devoted to the parametrization of hyperbolic Heron triangles using side lengths. The arguments are quite similar to those of Section 2, but using the (dual) hyperbolic law of cosines for the side lengths.

Let a, b, c denote the side lengths of a (non-degenerate bounded) hyperbolic triangle, and assume that e^a, e^b, e^c are rational. Let α (resp. β, γ) be the angles opposing the sides of length a (resp. b, c), as in Figure 1. By the law of cosines (for the side lengths)

$$\sinh(b)\sinh(c)\cos(\alpha) = \cosh(b)\cosh(c) - \cosh(a) \tag{11}$$

and hence $\cos(\alpha)$ is rational (and thus also $\cos(\beta)$ and $\cos(\gamma)$ for similar reasons). Recall from Section 1 that the hyperbolic law of sines implies

$$\Delta_2 := \sin(\alpha)\sinh(b)\sinh(c) = \sin(\beta)\sinh(a)\sinh(c) = \sin(\gamma)\sinh(a)\sinh(b),$$

and we have that Δ_2 is rational if and only if $\sin(\alpha), \sin(\beta), \sin(\gamma)$ are rational.

As in Section 2, we square (11) to get

$$\Delta_2^2 = \sinh(b)^2\sinh(c)^2 - (\cosh(b)\cosh(c) - \cosh(a))^2 \in \mathbb{Q}.$$

Rewriting the equation for Δ_2^2 in a symmetric way, we get:

$$\Delta_2^2 = 1 - \cosh(a)^2 - \cosh(b)^2 - \cosh(c)^2 + 2\cosh(a)\cosh(b)\cosh(c).$$

Letting $u = e^a, v = e^b$ and $w = e^c$ (so that $u, v, w \in \mathbb{Q}$), this equation rewrites as:

$$4u^2v^2w^2\Delta_2^2 = (uv - w)(uw - v)(vw - u)(uvw - 1).$$

We introduce the following change of variables

$$y = \frac{2u(v^2-1)(w^2-1)(v^2w^2-1)\Delta_2}{vw(u-vw)^2}, \quad x = \frac{(v^2-1)(w^2-1)(uvw-1)}{vw(vw-u)},$$

with inverse

$$\Delta_2 = \frac{(v^2-1)(w^2-1)(v^2w^2-1)y}{2\big(x+(v^2-1)(w^2-1)\big)\big(v^2w^2x+(v^2-1)(w^2-1)\big)},$$
$$u = \frac{v^2w^2x+(v^2-1)(w^2-1)}{vw\big(x+(v^2-1)(w^2-1)\big)}.$$

This leads to the desired equation:

$$y^2 = x\left(x - (v - v^{-1})^2\right)\left(x - (w - w^{-1})^2\right). \tag{12}$$

Its discriminant is given by $2^4(v - v^{-1})^4(w - w^{-1})^4(v^{-1}w - vw^{-1})^2(vw - v^{-1}w^{-1})^2$.

We can now proceed towards the proof of Theorem 3.

Proof of Theorem 3. It suffices to exhibit the open condition (B) that ensures the parameters give rise to a hyperbolic triangle. First, all the side lengths must be positive, whence $u > 1$, $v > 1$ and $w > 1$. Second, the three triangle inequalities must be satisfied, i.e., $u < vw, v < uw$ and $w < uv$. Assuming v, w are fixed and > 1, all these conditions can be rewritten more compactly as

$$\max\left(\frac{v}{w}, \frac{w}{v}\right) < u < vw,$$

which is the first part of condition (B) from the statement of the theorem. Once this condition is fulfilled, the condition $x > 0$ follows at once by observing that the expression for x in the above change of variables is always positive. Finally, one sees that the change of variable and its inverse is always defined in that case, and thus the theorem is proven. □

3.2. **Rank Computations.** This section is similar to 2.2. Let $E_{v,w}$ denote the curve given by (12) (where, as before, we think of v, w as fixed parameters), and let E_v (resp. E_w) denote the corresponding $K3$-surfaces resulting from fixing v (resp. w), and we aim to study the points over $\mathbb{C}(w)$ (resp. $\mathbb{C}(v)$). We refer to E when we keep both v and w free. Our goal is to give bounds on the rank of E_v and E_w; since the equation for $E_{v,w}$ is symmetric in v and w, it is enough to consider the curve E_v.

Lemma 3.1. *The rank of the $K3$-surface $E_v(\mathbb{C}(w))$ satisfies*

$$1 \leq \quad \mathrm{rk}(E_v(\mathbb{C}(w))) \quad \leq 2.$$

In addition, the torsion group of E_v is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by

$$S_0(v,w) = \big((v - v^{-1})(w - w^{-1}), i(v - v^{-1})(w - w^{-1})(v^{-1} - w^{-1})(vw + 1)\big)$$

and

$$S_1(v,w) = \big((v-v^{-1})^2, 0\big).$$

Finally, the point

$$R(v,w) = \big(-vw(v-v^{-1})(w-w^{-1}), ivw(v-v^{-1})(w-w^{-1})(vw-v^{-1}w^{-1})\big)$$

has infinite order on E.

Proof. The proof of this result is very similar to that of Lemma 2.1. As before, the lower bound for the rank follows from the fact that R is on the curve and of infinite order, which is easily verified by various methods. If we specialize with $v=2, w=4$, then we get the curve $y^2 = x(x-\frac{9}{4})(x-\frac{225}{16})$ and $R = (-45, \frac{2835i}{8})$. To avoid working on $\mathbb{Q}[i]$, we introduce a quadratic twist composed with an isomorphism to make the coefficients in $\mathbb{Z}$, resulting in the change of coordinates $(x,y) \to (-16x, -i64y)$. This gives $y^2 = x(x+36)(x+225)$ and $R \to (720, 22680)$. One can see immediately that $2(720, 22680) = (\frac{2025}{16}, \frac{172125}{64})$ and therefore $(720, 22680)$ is non-torsion. From this we conclude that the original R is non-torsion. Alternatively one can see that the height pairing of the Mordell–Weil group on E_v gives $h(R) = \frac{1}{2}$, and R is non-torsion since the height is non-zero.

We observe that E_v has singularities at $w = 0, \infty, \pm 1$ of type I_4 in the Kodaira classification, while the singularities at $w = \pm v, \pm v^{-1}$ are of type I_2. By the Shioda–Tate formula, we have

$$\rho(E_v) = \mathrm{rk} E_v(\mathbb{C}(w)) + 2 + 4\cdot(4-1) + 4\cdot(2-1) = \mathrm{rk} E_v(\mathbb{C}(w)) + 18.$$

Since E_v is a $K3$-surface, we have $\rho(E_v) \leq 20$, and thus $\mathrm{rk}(E_v(\mathbb{C}(w))) \leq 2$.

One can directly check that S_0 and S_1 generate a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Combining the information about the rank and the Euler characteristic with Table (4.5) in [8] we conclude that the torsion group is given exactly by $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. □

Even if Lemmas 2.1 and 3.1 are very similar from the point of view of the arithmetic of the involved $K3$-surfaces, they contain a fundamental difference for the geometric problem. In the case of Lemma 2.1, the point $P(m,u)$ is defined over $\mathbb{Q}(m,u)$ and generates a nontrivial solution to the question of the Heron hyperbolic triangle with given area and angle. In contrast, the point $R(v,w)$ of Lemma 3.1 is certainly not defined over $\mathbb{Q}(v,w)$, and we speculate that there is no point of infinite order over $\mathbb{Q}(v,w)$. This results in the following: we do not know a priori whether there is a Heron hyperbolic triangle with given sides v and w. This will depend on the choice of v and w, much like the classical congruent number problem depends on the choice of the area.

4. Equilateral Triangles

This short section covers the specific case of (non-degenerate bounded) equilateral triangles.

Proposition 4.1. *There are no equilateral hyperbolic Heron triangles.*

Proof. Let α denote the angle of an equilateral hyperbolic triangle. The Heron condition (6) from Section 2 in this case is:

$$\Delta_1^2 = 2\cos(\alpha)^3 + 3\cos(\alpha)^2 - 1 = (2\cos(\alpha) - 1)(\cos(\alpha) + 1)^2.$$

Setting $u = \frac{\Delta_1}{\cos(\alpha)+1}$, this equation rewrites as $u^2 = 2\cos(\alpha) - 1$. Thus the solutions to the original equation are parametrized by

$$\cos(\alpha) = \frac{u^2+1}{2} \text{ and } \Delta_1 = \frac{u^3+3u}{2}, \tag{13}$$

for $u \in \mathbb{Q}$.

Now squaring the equation for $\cos(\alpha)$ in (13), writing $4\cos^2(\alpha) = 4 - 4\sin^2(\alpha)$, and setting $v = 2\sin(\alpha)$, we get

$$v^2 = -u^4 - 2u^2 + 3.$$

Making the change of variables $u = \frac{x-1}{x+1}, v = \frac{4y}{(x+1)^2}$ (with inverse $x = \frac{1+u}{1-u}, y = \frac{v}{(u-1)^2}$), we get the Weierstrass form

$$y^2 = x(x^2 + x + 1).$$

This has rank 0, and the only nontrivial torsion point is $(0,0)$ which does not give an actual triangle since $v = 0$. □

We proceed to prove the second part of Proposition 4:

Proposition 4.2. *If an equilateral hyperbolic triangle has either rational side lengths or rational angles, then it has no rational median/area bisector.*

Proof. First observe that for equilateral triangles, mediators, bisectors, medians, and area bisectors all coincide, so we are free to use any property of these we want. Consider an equilateral hyperbolic triangle of side lengths a and angles α. Let m denote the length of the median, and consider the half triangle defined by one median. This triangle has angles $\alpha, \frac{\alpha}{2}, \frac{\pi}{2}$ and sides $a, \frac{a}{2}, m$.

First, assume the length a is rational, i.e., that $e^a \in \mathbb{Q}$. By Pythagoras' theorem, $\cosh(m)\cosh(\frac{a}{2}) = \cosh(a)$, so that $\cosh(m) \in \mathbb{Q}$ if and only if $p = \cosh(\frac{a}{2}) \in \mathbb{Q}$. Let $t = \sinh(m)$. Squaring Pythagoras' formula, we get the following equation for t:

$$(1+t^2)p^2 = (2p^2-1)^2 \quad \text{i.e.} \quad s^2 = 4p^4 - 5p^2 + 1,$$

writing $s = pt$. Changing variables

$$s = \frac{9-x^2}{8x}, \quad p = \frac{y}{4x}, \quad y = 4p(-4s + 8p^2 - 5), \quad x = -4s + 8p^2 - 5,$$

we get the following elliptic curve:

$$y^2 = x(x^2 + 10x + 9).$$

The curve has rank 0, and the torsion is given by

$$E(\mathbb{Q})_{\text{tors}} = \langle(-3, 6), (-1, 0)\rangle \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Since we are looking for solutions with $p \neq 0$, we need $y \neq 0$. The only torsion points to consider are therefore $(-3, \pm 6)$ and $(3, \pm 12)$. However, we also need $t \neq 0$, leading to $s \neq 0$ and $x \neq \pm 3$. Therefore, there are no solutions.

Now assuming the angle α to be rational (i.e. $e^{i\alpha} \in \mathbb{Q}[i]$), the situation is entirely similar. From the law of cosines (for the angles) we get:

$$\sin\left(\frac{\alpha}{2}\right)\cosh(m) = \cos(\alpha), \tag{14}$$

whence $\cosh(m) \in \mathbb{Q}$ if and only if $p = \sin(\frac{\alpha}{2}) \in \mathbb{Q}$. We set $t = \sinh(m)$ and square equation (14) to get the same equation as before:

$$(1+t^2)p^2 = (2p^2-1)^2.$$

Thus also in this case, the median cannot be rational. □

5. Rational medians

The goal of this section is to study hyperbolic triangles with one rational median, in the same spirit as Euler's problem [4]. Consider a (non-degenerate bounded) hyperbolic triangle with sides a, b, c having opposite angles α, β, γ (by abuse of notation we will let a, b, c also denote the length of the sides). Let m denote the median at angle α, cutting side a into two equal parts. Denote by θ the angle at the intersection of m and a, on the side of β; the one on the side of γ is $\pi - \theta$. Applying the law of cosines in the two triangles, we get:

$$\begin{aligned}\cosh(b) &= \cosh(m)\cosh(a/2) - \sinh(m)\sinh(a/2)\cos(\pi-\theta),\\ \cosh(c) &= \cosh(m)\cosh(a/2) - \sinh(m)\sinh(a/2)\cos(\theta),\end{aligned}$$

and thus

$$2\cosh(m)\cosh(a/2) = \cosh(b) + \cosh(c). \tag{15}$$

Let us now assume that a, b, c are rational side lengths, i.e., that $e^a, e^b, e^c \in \mathbb{Q}$. In order for $\cosh(m)$ to be rational, it is thus necessary and sufficient that $\cosh(a/2)$ be rational. Since $e^a \in \mathbb{Q}$, this is equivalent to $e^{a/2} \in \mathbb{Q}$. For $\sinh(m) \in \mathbb{Q}$ we get the following condition from squaring equation (15):

$$(\cosh(b)+\cosh(c))^2 - 4\cosh(a/2)^2 = 4\sinh^2(m)\cosh^2(a/2) = \text{square}.$$

Setting $u = e^{a/2}, v = e^b, w = e^c$, we need to solve

$$(v^2w + w + vw^2 + v)^2u^2 - 4v^2w^2(u^2+1)^2 = t^2$$

for $t, u, v, w \in \mathbb{Q}$. Now applying the change of variables

$$x = 2w\left[u^2wv^2 + u^2(w^2+1)v - 2u^4w - 3u^2w + ut - 2w\right]$$

$$\begin{aligned}y = 2uw\big[&2u^2w^2v^3 + 3wu^2(w^2+1)v^2 + (-4u^4w^2 + u^2w^4 - 4u^2w^2 + 2uwt + u^2 - 4w^2)v\\ &+ u(w^2+1)(uw+t)\big]\end{aligned}$$

with inverse

$$v = -\frac{1}{2xuw}\left[4uw^2(w^2+1)(u^2+1)^2 + u(w^2+1)x - y\right]$$

$$t = -\frac{1}{4x^2uw}\big[-x^3 + 8w^2(u^2+1)^2(2w^2(u^2+1)^2 + (w^4+1)u^2)x \\ - 8uw^2(w^2+1)(u^2+1)^2y + 32u^2w^4(w^2+1)^2(u^2+1)^4\big],$$

we get the equation

$$y^2 = (x + 4(u^2+1)^2w^2)(x^2 + (u^2w^4 + 2(2u^4+3u^2+2)w^2 + u^2)x \\ + 4u^2(u^2+1)^2w^2(w^2+1)^2). \tag{16}$$

If, similarly as previously, we let $E_{u,w}$ denote the elliptic curve given by (16) seen over $\mathbb{Q}$, where $u, w \in \mathbb{Q}$ are parameters, we get:

Theorem 6. *A hyperbolic triangle with rational sides $a = 2\log(u)$, $b = \log(w)$ has a rational median (intersecting side a) if and only if it corresponds (using the above change of variables) to a rational point on the elliptic curve $E_{u,w}$.*

Let E_u denote the $K3$-surface when we fix u and leave w free. We are interested in the $\mathbb{C}(w)$-points of E_u. We also denote by E the variety where we keep u and w free. As previously, we have the following lemma, a weaker analogue to Lemmas 2.1 and 3.1.

Lemma 5.1. *The rank of the $K3$-surface E_u satisfies*

$$1 \le \mathrm{rk}(E_u(\mathbb{C}(w))) \le 4.$$

In addition, E_u has a torsion point of order 2 given by $(4(u^2+1)^2w^2, 0)$.

Finally, the point

$$P(u,w) = \big(0, 4u(u^2+1)^2w^2(w^2+1)\big)$$

has infinite order on E.

Proof. The proof is entirely similar to that of Lemmas 2.1 and 3.1. We can specialize in $u = 1, w = 2$ and we obtain the Weierstrass form $y^2 = (x+64)(x^2+73x+1600)$ and $P = (0, 320)$. But then $2P = (-\frac{1024}{25}, \frac{10176}{125})$, implying that P can not be torsion. The lower bound on the rank now follows from the fact that $P(u,w)$ is of infinite order.

The discriminant of $E_{u,w}$ is given by:

$$2^{12}u^4w^8(u^2+1)^4(uw^2 + u - 2(u^2+u+1)w)(uw^2 + u - 2(u^2-u+1)w) \\ \times(uw^2 + u + 2(u^2+u+1)w)(uw^2 + u + 2(u^2-u+1)w).$$

Looking at the Kodaira classification, we observe that E_u has singularities of type I_8 at $w = 0, \infty$, and of type I_1 for all the 8 others. Thus the Shioda–Tate formula gives

$$\rho(E_u) = \mathrm{rk}(E_u(\mathbb{C}(w))) + 2 + 2\cdot(8-1) + 8\cdot(1-1) = \mathrm{rk}(E_u(\mathbb{C}(w))) + 16.$$

Since E_u is a $K3$-surface, we have $\rho(E_u) \le 20$, and thus $\mathrm{rk}(E_u(\mathbb{C}(w))) \le 4$.

Finally, we see that $(4(u^2+1)^2w^2, 0)$ is a point of order 2. □

As a final note, we remark that we could include E_w in the statement of Lemma 5.1. In this case, we can only conclude that $1 \le \mathrm{rk}(E_w(\mathbb{C}(u))) \le 6$.

6. Area bisectors

This section is similar to Section 5, but focussing on hyperbolic triangles with rational area bisectors, instead of medians.

Consider a hyperbolic triangle with sides a, b, c having opposite angles α, β, γ. Let m denote the area bisector at angle α, cutting α into α_1 and $\alpha - \alpha_1$. Denote by θ the angle at the intersection of m and a, on the side of α_1, and (assume) on the side of β. Thus we have two triangles: one with angles α_1, β, θ and one with $\alpha - \alpha_1, \gamma, \pi - \theta$.

By the law of cosines (for the angles) we have

$$\sin(\alpha_1)\sin(\beta)\cosh(c) = \cos(\theta) + \cos(\alpha_1)\cos(\beta).$$

Combining this with the definition of area bisector

$$2(\pi - \alpha_1 - \theta - \beta) = A \quad \text{i.e.} \quad \theta = \pi - \frac{A}{2} - \alpha_1 - \beta,$$

we get

$$\sin(\alpha_1)\sin(\beta)\cosh(c) = -\cos\left(\frac{A}{2} + \alpha_1 + \beta\right) + \cos(\alpha_1)\cos(\beta). \tag{17}$$

Using trigonometric formulas, this rewrites as

$$\begin{aligned}\sin(\alpha_1)^{-2} &= 1 + \frac{1}{\tan(\alpha_1)^2}\\ &= \frac{2 + \sin(\beta)^2\sinh(c)^2 - 2\cos\left(\frac{A}{2}\right) + 2(1-\cosh(c))\sin(\beta)\sin\left(\frac{A}{2}+\beta\right)}{\left(\cos(\beta) - \cos\left(\frac{A}{2}+\beta\right)\right)^2}.\end{aligned}$$

Now using the law of cosines again:

$$\sin(\alpha)\sin(\beta)\cosh(c) = \cos(\gamma) + \cos(\alpha)\cos(\beta)$$

and setting $w_1 = \left(\cos(\beta) - \cos\left(\frac{A}{2}+\beta\right)\right)(\sin(\alpha_1))^{-1}\sin(\alpha)$, we get the equation

$$\begin{aligned}w_1^2 =& s_\alpha^2 + c_\beta^2 s_\alpha^2 + (c_\alpha c_\beta c_A - s_\alpha s_\beta c_A - s_\alpha c_\beta s_A - c_\alpha s_\beta s_A - c_\alpha c_\beta)^2\\ &- 2c_\beta s_\alpha^2\left(\cos\left(\frac{A}{2}\right)c_\beta - \sin\left(\frac{A}{2}\right)s_\beta\right)\\ &- 2s_\alpha(-(c_\alpha c_\beta c_A - s_\alpha s_\beta c_A - s_\alpha c_\beta s_A - c_\alpha s_\beta s_A) + c_\alpha c_\beta)\\ &\times\left(\sin\left(\frac{A}{2}\right)c_\beta + \cos\left(\frac{A}{2}\right)s_\beta\right),\end{aligned}$$

where, as in Section 2, $s_\alpha = \sin(\alpha)$, etc...

Assume now that our triangle has rational angles as well as rational *half-area*. We apply a similar change of variables as in Section 1, namely:

$$\begin{array}{lll}\cos(\frac{A}{2}) = \frac{1-n^2}{1+n^2}, & \cos(\beta) = \frac{1-u^2}{1+u^2}, & \cos(\alpha) = \frac{1-t^2}{1+t^2},\\ \sin(\frac{A}{2}) = \frac{2n}{1+n^2}, & \sin(\beta) = \frac{2u}{1+u^2}, & \sin(\alpha) = \frac{2t}{1+t^2}.\end{array}$$

Setting $w = \frac{w_1(n^2+1)^2(t^2+1)(u^2+1)}{4n}$, and clearing squares, we obtain:

$$\begin{aligned} w^2 =& 4(n+u)^2(nu-1)^2t^4 + 4(n+u)(nu-1)(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)t^3 \\ &+ (n^6u^4+2n^6u^2+8n^5u^3+11n^4u^4+n^6-8n^5u-50n^4u^2-64n^3u^3 \\ &- 13n^2u^4+11n^4+64n^3u+86n^2u^2+24nu^3+u^4-13n^2-24nu-6u^2+1)t^2 \\ &- 4(n+u)(nu-1)(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)t + 4(n+u)^2(nu-1)^2. \end{aligned}$$

We make the following final change of variables:

$$\begin{aligned} y =& \frac{4(nu-1)(n+u)}{t^3}\big[2(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)(nu-1)(n+u)t^3 \\ &+ (n^6u^4+2n^6u^2+8n^5u^3+11n^4u^4+n^6-8n^5u-50n^4u^2-64n^3u^3-13n^2u^4 \\ &+ 11n^4+64n^3u+86n^2u^2+24nu^3+u^4-13n^2-24nu-6u^2+1)t^2 \\ &+ 8(nu-1)^2(n+u)^2 - 6(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)(nu-1)(n+u)t \\ &+ 4(nu-1)(n+u)w - (2n^3u+3n^2u^2-3n^2-6nu-u^2+1)tw\big], \end{aligned}$$

$$\begin{aligned} x =& \frac{1}{t^2}\big[8(nu-1)^2(n+u)^2 - 4(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)(nu-1)(n+u)t \\ &+ (n^6u^4-2n^6u^2-4n^5u^3+2n^4u^4+n^6+4n^5u-8n^4u^2-24n^3u^3-7n^2u^4+2n^4+24n^3u \\ &+ 38n^2u^2+12nu^3-7n^2-12nu-4u^2)t^2 + 4(nu-1)(n+u)w\big], \end{aligned}$$

with inverse

$$\begin{aligned} t =& -\big[4(nu-1)x(n+u)\big] \cdot \big[(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)(nu^2-n-2u)^2(n^2+1)^2 \\ &- (2n^3u+3n^2u^2-3n^2-6nu-u^2+1)x - y\big]^{-1}, \end{aligned}$$

$$\begin{aligned} w =& -2(nu-1)(n+u)\big[(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)^2(nu^2-n-2u)^4(n^2+1)^4 \\ &- 2(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)(nu^2-n-2u)^2(n^2+1)^2y \\ &+ (2n^4u^4-8n^4u^2-20n^3u^3-7n^2u^4+2n^4+20n^3u+34n^2u^2+12nu^3-u^4-7n^2 \\ &- 12nu-6u^2-1)(n^2+1)x^2 - 2x^3 + y^2\big] \\ &\times \big[(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)^2(nu^2-n-2u)^4(n^2+1)^4 \\ &- 2(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)^2(nu^2-n-2u)^2(n^2+1)^2x \\ &- 2(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)(nu^2-n-2u)^2(n^2+1)^2y \\ &+ (2n^3u+3n^2u^2-3n^2-6nu-u^2+1)^2x^2 \\ &+ 2(2n^3u+3n^2u^2-3n^2-6nu-u^2+1)yx + y^2\big]^{-1}, \end{aligned}$$

and we get

$$\begin{aligned} y^2 =& (x-(n^2+1)^2(nu^2-2u-n)^2)(x^2-(n^2+1)(n^4u^4-8n^2u^4-u^4-16n^3u^3 \\ &+ 16nu^3-6n^4u^2+32n^2u^2-10u^2+16n^3u-16nu+n^4-8n^2-1)x \\ &- (n^2+1)^2(nu^2-2u-n)^2(3n^2u^2-u^2+2n^3u-6nu-3n^2+1)^2). \end{aligned} \tag{18}$$

Let $E_{n,u}$ denote this elliptic curve (18), where n, u are fixed parameters. The data it encodes is the following. By assumption, $\frac{A}{2}, \alpha, \beta, \gamma$ are all rational. Moreover, as in Section 1, it follows easily from the law of cosines that $\cosh(a)$, $\cosh(b)$ and $\cosh(c)$ are also rational. Now by construction, a rational solution to (18) corresponds to a triangle with $\sin(\alpha_1)$ rational. In addition, (17) implies that $\cos(\alpha_1) \in \mathbb{Q}$, and thus α_1 is rational. Since the area of the small triangle with angles α_1, β, θ is rational (it is $\frac{A}{2}$), it follows that θ is also a rational angle.

What is not encoded by the curve $E_{n,u}$ is the rationality of $\sinh(m)$. It is easy to see that this is actually equivalent to the original triangle being Heron: since all the angles and areas under consideration are rational, we have $\sinh(m) \in \mathbb{Q}$ if and only if the small triangle (with angles α_1, β, θ) is Heron (as explained in Section 2). Since c is a side of this triangle, this happens exactly when $\sinh(c) \in \mathbb{Q}$, which, in turn, is true if and only if the original triangle is Heron.

Therefore, we have proven:

Theorem 7. *A hyperbolic Heron triangle has one rational area bisector if and only if it corresponds (using the above change of variables) to a rational point of $E_{n,u}$.*

This curve is more complicated than the ones of Section 2 and 3. Yet we have the following lemma, analog to Lemma 5.1. As before, we let E_n denote the $K3$-surface that corresponds to fixing n and letting u free. (Remark that E_u is not a $K3$-surface, and we will not discuss its arithmetics.)

Lemma 6.1. *The rank of the $K3$-surface E_n satisfies*

$$1 \leq \mathrm{rk}(E_n(\mathbb{C}(u))) \leq 4.$$

Moreover, E_n has a torsion point of order 2 given by $((n^2+1)^2(nu^2-2u-n)^2, 0)$.

The point

$$Q(n,u) = \Big(0, (n^2+1)^2(nu^2-2u-n)^2(3n^2u^2-u^2+2n^3u-6nu-3n^2+1)\Big)$$

is of infinite order.

Proof. The lower bound follows from the fact that $Q(n,u)$ is of infinite order, which can be verified by the usual methods that we have previously discussed. For example, if we take $n=2, u=1$, then we must consider the point $Q=(0,400)$ in the Weierstrass form $y^2=(x-100)(x-20)(x+80)$. We can see that $2Q = (\frac{521}{4}, \frac{6699}{8})$, and from this we conclude that $Q(n,u)$ is non-torsion in E.

The discriminant of E_n is

$$\begin{aligned}
&2^{12}(n^2+1)^8(u+n)^4(nu-1)^4(u^2+1)^2((u^2-1)n-2u)^4\\
&\times\Big((n^4+18n^2+1)u^4+16n(n^2-3)u^3\\
&+2(n^4-30n^2+17)u^2-16n(n^2-3)u+n^4+18n^2+1\Big)
\end{aligned}$$

We have singularities at $u=-n$, $\frac{1}{n}$, and the roots of nu^2-2u-n of type I_4, $\pm i$ of type I_2, and the roots of the last factor of type I_1. By the Shioda–Tate formula,

$$\rho(E_n) = \mathrm{rk}(E_n(\mathbb{C}(u))) + 2 + 4\cdot(4-1) + 2\cdot(2-1) + 4\cdot(1-1) = \mathrm{rk}(E_n(\mathbb{C}(u))) + 16.$$

Since E_n is a $K3$-surface, the Picard number $\rho(E_n) \leq 20$ and the rank is ≤ 4.

Finally, it is immediate to see that $((n^2+1)^2(nu^2-2u-n)^2, 0)$ has order 2. $\square$

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