MULTIPLE ZETA FUNCTIONS AND POLYLOGARITHMS OVER GLOBAL FUNCTION FIELDS

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Abstract. In [Tha04] Thakur defines function field analogs of the classical multiple zeta function, namely, $\zeta_d(\mathbb{F}_q[T]; s_1, \dots, s_d)$ and $\zeta_d(K; s_1, \dots, s_d)$, where K is a global function field. Star versions of these functions were further studied by Masri [Mas06]. We prove reduction formulas for these star functions, extend the construction to function field analogs of multiple polylogarithms, and exhibit some formulas for multiple zeta values.

1. Introduction

The multiple zeta function is defined by the infinite series

$$\zeta(s_1, \dots, s_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}},$$

and is absolutely convergent and analytic in the region

$$\operatorname{Re}(s_k + \cdots + s_d) > d - k + 1, \quad k = 1, \dots, d. \tag{1}$$

In the above formula, we say that d is the depth and that $s_1 + \cdots + s_d$ is the weight.

Multiple zeta values are given by $\zeta(a_1, \dots, a_d)$, with $a_k \in \mathbb{Z}_{\geq 1}$, $a_d > 1$. The formal definition is given by Zagier [Zag94], but they already appear as early as Euler [Eul75].

These numbers appear extensively in Number Theory, Geometry, and Physics. See for example, the works of Brown [Bro11, Bro12], Deligne [Del89], Drinfeld [Dri90], Hoffman [Hof92, Hof97], Goncharov and Manin [GM04], Kontsevich [Kon93, Kon99], Manin [Man06], Zagier [Zag12]. The survey [KZ01] by Kontsevich and Zagier discusses multiple zeta values in the context of periods and special values of L-functions.

A variant of the multiple zeta function is considered by Hoffman [Hof92]

$$\zeta^\star(s_1, \dots, s_d) = \sum_{1\leq n_1\leq\cdots\leq n_d} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}}.$$

Here the notation $\star$ indicates that the indexes are ordered with non-strict inequalities. Multiple zeta (star) values have been largely studied, see for example, [OW06, OO07, AKO08, Mun08, OZ08, KO10, Yam10, IKOO11, AOW11, KST12, TY13, Yam13, LZ15, HPHP15, Zha16, HPHPZ16, CC17, HPHPT17, LQ18, Mac19]. They generally yield simpler identities than multiple zeta values.

In this work, we consider a function field version of the multiple zeta function. Let $\mathbb{F}_q$ be the finite field with q elements, where q is a prime power. For $f \in \mathbb{F}_q[T]$, denote by $\deg(f)$ its degree and by $|f| = q^{\deg(f)}$ its norm. The zeta function of $\mathbb{F}_q[T]$ is defined by

$$\zeta(\mathbb{F}_q[T]; s) = \sum_{\substack{f\in\mathbb{F}_q[T] \\ f \text{ monic} \\ 0\leq\deg(f)}} \frac{1}{|f|^s},$$

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and further verifies

$$\zeta(\mathbb{F}_q[T];s) = \frac{1}{1-q^{1-s}}.$$

The multiple zeta function of depth d over $\mathbb{F}_q[T]$ is defined by Thakur ([Tha04], Section 5.10) and further developed by Masri [Mas06]. It is given by

$$\zeta_d^\star(\mathbb{F}_q[T];s_1,\dots,s_d) = \sum_{\substack{(f_1,\dots,f_d)\in(\mathbb{F}_q[T])^d\\ f_1,\dots,f_d \text{ monic}\\ 0\le\deg(f_1)\le\cdots\le\deg(f_d)}} \frac{1}{|f_1|^{s_1}\cdots|f_d|^{s_d}}.$$

Thakur considers the definition where the indexes are ordered with strict inequalities. Masri studies the version with non-strict inequalities given above and denotes it by $Z_d(\mathbb{F}_q[T];s_1,\dots,s_d)$. We have chosen a notation that better reflects the analogy with $\zeta^\star(s_1,\dots,s_d)$. For uniformity of notation we will also write $\zeta^\star(\mathbb{F}_q[T];s)$ in place of $\zeta(\mathbb{F}_q[T];s)$.

We remark that this complex function is different from the multizeta version of the Carlitz zeta function introduced by Thakur in [Tha04] (see also [AT09, Tha09, Tha10, Tha17, CM17, CM19]).

Masri proves that $\zeta_d^\star(\mathbb{F}_q[T];s_1,\dots,s_d)$ is absolutely convergent and analytic in the region given by (1). He further proves the existence of a rational meromorphic continuation, an Euler product, and a functional equation, and he proves that

$$\zeta_d^\star(\mathbb{F}_q[T];s_1,\dots,s_d) = \prod_{k=1}^{d}\zeta^\star(\mathbb{F}_q[T];s_k+\cdots+s_d-(d-k)), \tag{2}$$

which provides an exact formula for the function.

Masri further considers an extension to a global function field K,

$$\zeta_d^\star(K;s_1,\dots,s_d) = \sum_{\substack{(D_1,\dots,D_d)\in(\mathcal{D}_K^+)^d\\ 0\le\deg(D_1)\le\cdots\le\deg(D_d)}} \frac{1}{|D_1|^{s_1}\cdots|D_d|^{s_d}},$$

where $\mathcal{D}_K^+$ is the semigroup of effective divisors (see Section 2.)

Masri proves the following result.

Theorem 1 ([Mas06], Main Theorem)**.** *The multiple zeta function $\zeta_d^\star(K;s_1,\dots,s_d)$ has a meromorphic continuation to all s in $\mathbb{C}^d$ and is a rational function in each $q^{-s_1},\dots,q^{-s_d}$, with a specified denominator. Further, $\zeta_d^\star(K;s_1,\dots,s_d)$ has possible simple poles on the linear subvarieties*

$$s_k+\cdots+s_d = 0,1,\dots,d-k+1, \quad k=1,\dots,d.$$

A central question in the theory of multiple zeta functions has to do with reduction, that is, the property that a multiple zeta value can be written as a rational linear combination of products of lower depth multiple zeta values.

Masri proves the following.

Theorem 2 ([Mas06], Corollary 1.5)**.** *$\zeta_d^\star(\mathbb{F}_q(T);s_1,\dots,s_d)$ is a rational linear combination of products of zeta functions from the set*

$$\{\zeta^\star(\mathbb{F}_q[T];s_k+\cdots+s_d+\ell) : k=1,\dots,d, \quad \ell=-1,0,1\}.$$

A goal of this note is to give a precise formula for $\zeta_d^\star(K;s_1,\dots,s_d)$ when the genus of K satisfies $g\ge1$, which implies an analogous result to Theorem 2. More precisely, we prove the following.

Theorem 3. *We have*

$$\begin{aligned}\zeta_d^\star(K;s_1,\dots,s_d) =& R_d(K;s_1,\dots,s_d) + \sum_{\ell=1}^{d}\left(\frac{q^g h_K}{q-1}\right)^{\ell} R_{d-\ell}(K;s_1,\dots,s_{d-\ell})q^{-(2g-1)(s_{d+1-\ell}+\cdots+s_d)}\\ &\times \sum_{e_{d+1-\ell},\dots,e_d\in\{0,1\}} (-q^g)^{-(e_{d+1-\ell}+\cdots+e_d)}\prod_{j=0}^{\ell-1}\zeta^\star(\mathbb{F}_q[T];s_{d-j}+e_{d-j}+\cdots+s_d+e_d-j).\end{aligned}$$

where $R_m(K;s_1,\dots,s_m)$ is a certain polynomial on $q^{-s_1},\dots,q^{-s_m}$ given more precisely by equation (9).

The multiple polylogarithm is defined by

$$\mathrm{Li}_{k_1,\ldots,k_d}(z_1,\ldots,z_d) := \sum_{0<n_1<\cdots<n_d} \frac{z_1^{n_1}\cdots z_d^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}},$$

which is absolutely convergent for $|z_i| \leq 1$ for $i = 1,\ldots,d-1$ and $|z_d| \leq 1$ (respectively $|z_d| < 1$) if $k_d > 1$ (resp. $k_d = 1$). We then extend the construction of multiple zeta functions over global function fields to multiple polylogarithms:

$$\mathrm{Li}^{\star}_{k_1,\ldots,k_d}(\mathbb{F}_q[T]; z_1,\ldots,z_d) = \sum_{\substack{(f_1,\ldots,f_d)\in(\mathbb{F}_q[T])^d\\ f_1,\ldots,f_d \text{ monic}\\ 0\leq \deg(f_1)\leq\cdots\leq\deg(f_d)}} \frac{z_1^{\deg(f_1)}\cdots z_d^{\deg(f_d)}}{|f_1|^{k_1}\cdots|f_d|^{k_d}}$$

and

$$\mathrm{Li}^{\star}_{k_1,\ldots,k_d}(K; z_1,\ldots,z_d) = \sum_{\substack{(D_1,\ldots,D_d)\in(\mathcal{D}_K^+)^d\\ 0\leq \deg(D_1)\leq\cdots\leq\deg(D_d)}} \frac{z_1^{\deg(D_1)}\cdots z_d^{\deg(D_d)}}{|D_1|^{k_1}\cdots|D_d|^{k_d}}.$$

We prove the following.

Theorem 4. $\mathrm{Li}^{\star}_{k_1,\ldots,k_d}(K; z_1,\ldots,z_d)$ *is absolutely convergent and analytic in the complex region determined by*

$$|z_j\cdots z_d| < q^{k_j+\cdots+k_d-d+j-1}, \qquad j=1,\ldots,d,$$

and has an analytic continuation to the complex region determined by

$$z_j\cdots z_d \neq q^{k_j+\cdots+k_d},\ldots,q^{k_j+\cdots+k_d-d+j-1}, \qquad j=1,\ldots,d.$$

We prove a reduction formula that is analogous to Theorem 3 for $g \geq 1$.

Theorem 5. *We have*

$$\begin{aligned}\mathrm{Li}^{\star}_{k_1,\ldots,k_d}(K; z_1,\ldots,z_d) =& \mathcal{R}_{k_1,\ldots,k_d}(K; z_1,\ldots,z_d) + \sum_{\ell=1}^{d}\left(\frac{q^g h_K}{q-1}\right)^{\ell} \mathcal{R}_{k_1,\ldots,k_{d-\ell}}(K; z_1,\ldots,z_{d-\ell})\\ &\times q^{-(2g-1)(k_{d+1-\ell}+\cdots+k_d)}\left(\prod_{i=d+1-\ell}^{d} z_i\right)^{2g-1}\\ &\times \sum_{e_{d+1-\ell},\ldots,e_d\in\{0,1\}} (-q^g)^{-(e_{d+1-\ell}+\cdots+e_d)} \prod_{j=0}^{\ell-1} \mathrm{Li}^{\star}_{k_{d-j}+e_{d-j}+\cdots+k_d+e_d-j}\left(\mathbb{F}_q[T]; \prod_{i=d-j}^{d} z_i\right),\end{aligned}$$

where $\mathcal{R}_{k_1,\ldots,k_m}(K; z_1,\ldots,z_m)$ *is a certain polynomial on* $q^{-k_1},\ldots,q^{-k_m}$ *given more precisely by equation* (20).

It is also natural to consider multiple zeta values. Formula (2) implies that special values of $\zeta_d^{\star}(\mathbb{F}_q[T]; s_1,\ldots,s_d)$ are very easy to compute, and we exhibit several examples that are counterparts of results for $\zeta^{\star}(s_1,\ldots,s_d)$. Special values of $\zeta_d^{\star}(\mathbb{F}_q(T); s_1,\ldots,s_d)$ are harder to compute, and we focus on the case where $s_1 = \cdots = s_d$. For example, we prove the following.

Theorem 6. $\zeta_d^{\star}(\mathbb{F}_q(T); m,\ldots,m)$ *satisfies the recurrence*

$$\zeta_d^{\star}(\mathbb{F}_q(T); m,\ldots,m) = \frac{1}{d}\sum_{j=0}^{d-1} \frac{\zeta^{\star}_{d-1-j}(\mathbb{F}_q(T); m,\ldots,m)}{(q-1)^j} \sum_{\ell=0}^{j} (-1)^{\ell}\binom{j}{\ell} \zeta^{\star}(\mathbb{F}_q(T); (j+1)(m-1)+\ell+1) q^{j-\ell}.$$

The proofs of our results follow manipulations at the level of the series. It would be interesting to see if the special values considered here can be expressed in terms of integrals as in the case of $\zeta(k_1,\ldots,k_d)$, as this would open the door for even more relationships among these values. As Masri [Mas06] mentions, it would be also interesting to see if there is a particular interpretation for the numerators of $\zeta^{\star}(K; s_1,\ldots,s_d)$ in the same vein as the numerator of $\zeta(K; s)$ is the characteristic polynomial of the action of the Frobenius automorphism on the Tate module ([Ros02], pg. 275).

This article is structured as follows. In Section 2 we review some background on arithmetic of global function fields. In Section 3 we prove several results reducing the depth of $\zeta_d^\star(\mathbb{F}_q(T); s_1, \ldots, s_d)$ and $\zeta_d^\star(K; s_1, \ldots, s_d)$ including Theorem 3. The construction of multiple zeta functions is extended to multiple polylogarithms in Sections 4 and 5, where Theorems 4 and 5 are proven. Finally, we focus on multiple zeta values in Section 6.

2. Background on function fields

We follow the definitions of [Ros02] and [Mas06].

Let F be a field. A function field in one variable over F is a field K containing F and at least one element x transcendental over F such that $K/F(x)$ is a finite algebraic extension. We will work with a global function field, which is a function field in one variable with finite constant field $F = \mathbb{F}_q$.

A prime in K is a discrete valuation ring R with maximal ideal P such that $F \subset R$ and the quotient field of R equals K. Such prime is typically denoted by P. We define $\deg P := [R/P : F]$.

The group of divisors of K, denoted $\mathcal{D}_K$, is the free abelian group generated by the primes. A typical divisor is written as $D = \sum_P a(P)P$, where $a(P) = \mathrm{ord}_P(D)$ is uniquely determined by D. The degree of D is given by $\deg(D) = \sum_P a(P) \deg P$. For $a \in K^*$, the divisor of a is defined by $(a) = \sum_P \mathrm{ord}_P(a)P$. The map $a \mapsto (a)$ is a homomorphism $K^* \to \mathcal{D}_K$ whose image is the group of principal divisors $\mathcal{P}_K$. Two divisors D_1 and D_2 are said to be equivalent, written $D_1 \sim D_2$, if their difference is principal, that is, $D_1 - D_2 = (a)$ for some $a \in K^*$. The divisor class group is defined by $Cl_K = \mathcal{D}_K/\mathcal{P}_K$. Since the degree of principal divisors is zero (see [Ros02], Proposition 5.1), the degree map $\deg : Cl_K \to \mathbb{Z}$ is a homomorphism. Its kernel is the group of divisor classes of degree zero, denoted by Cl_K^0. The number of divisor classes of degree zero is finite ([Ros02], Lemma 5.6) and is denoted by h_K. Schmidt [Sch31] proved that a function field over a finite field always has divisors of degree 1. This gives an exact sequence

$$0 \to Cl_K^0 \to Cl_K \to \mathbb{Z} \to 0.$$

A divisor D is effective if $a(P) \geq 0$ for all P. We write $D \geq 0$ in this case. For any integer $n \geq 0$, the number of effective divisors of degree n is finite ([Ros02], Lemma 5.5).

Let $\mathcal{D}_K^+$ be the semi-group of effective divisors. Given a divisor, consider

$$L(D) = \{x \in K^* \mid (x) + D \geq 0\} \cup \{0\}.$$

It can be seen that $L(D)$ is a finite dimensional vector space over $\mathbb{F}_q$. Its dimension is denoted by $l(D)$. For any divisor D, the number of effective divisors in its class $[D]$ is $\frac{q^{l(D)}-1}{q-1}$ ([Ros02], Lemma 5.7).

Theorem 7 (Riemann–Roch)**.** *There is an integer $g = g(K) \geq 0$ and a divisor class $\mathcal{C}$ such that for $C \in \mathcal{C}$ and $D \in \mathcal{D}_K$ we have*

$$l(D) = \deg(D) - g + 1 + l(C - D).$$

The integer g is called the genus.

Corollary 8. *If $\deg(D) \geq 2g-2$, then $l(D) = \deg(D)-g+1$, unless $D \sim C$ and in that case $\deg(D) = 2g-2$ and $l(D) = g$.*

Let b_n denote the number of effective divisors of degree n.

Lemma 9 ([Ros02], Lemma 5.8)**.** *For every integer n there are h_K divisor classes of degree n. Suppose $n \geq 0$ and that $\{[D_1], \ldots, [D_{h_K}]\}$ are the divisor classes of degree n. Then*

$$b_n = \sum_{j=1}^{h_K} \frac{q^{l(D_j)} - 1}{q - 1}.$$

Combining the above statements, one gets

Lemma 10 ([Ros02], p.52, [Mas06] Proposition 3.1)**.** *For all non-negative integers $n > 2g - 2$,*

$$b_n = h_K \frac{q^{n-g+1} - 1}{q - 1}.$$

Similarly,

Lemma 11.

$$b_0 = 1, \tag{3}$$

and

$$b_{2g-2} = q^{g-1} + \frac{h_K(q^{g-1}-1)}{q-1}. \tag{4}$$

Proof. The first statement is a consequence that the only effective divisor of degree 0 is 0 itself.

We use Lemma 9. Let $\{[D_1], \dots, [D_{h_K-1}], [C]\}$ be the divisor classes of degree $2g-2$. Then

$$b_{2g-2} = \frac{q^{l(C)}-1}{q-1} + \sum_{j=1}^{h_K-1} \frac{q^{l(D_j)}-1}{q-1}.$$

By Corollary 8, the above equals

$$\frac{q^g-1}{q-1} + (h_K-1)\frac{q^{g-1}-1}{q-1}$$

and this simplifies to the formula in the statement. $\square$

For a divisor D, let $|D| = q^{\deg(D)}$ be its norm, which is a positive integer and satisfies that $|D_1 + D_2| = |D_1| \cdot |D_2|$. Recall that the zeta function of K is defined by

$$\zeta(K; s) = \sum_{D \in \mathcal{D}_K^+} \frac{1}{|D|^s}, \quad \mathrm{Re}(s) > 1.$$

It is immediate to see that

$$\zeta(K; s) = \sum_{n=0}^{\infty} \frac{b_n}{q^{ns}}.$$

Masri [Mas06] extends this to define the multiple zeta function of depth d over K as

$$\zeta_d^\star(K; s_1, \dots, s_d) = \sum_{\substack{(D_1,\dots,D_d) \in (\mathcal{D}_K^+)^d \\ 0 \le \deg(D_1) \le \dots \le \deg(D_d)}} \frac{1}{|D_1|^{s_1} \cdots |D_d|^{s_d}},$$

which can also be expressed as

$$\zeta_d^\star(K; s_1, \dots, s_d) = \sum_{0 \le n_1 \le \dots \le n_d} \frac{b_{n_1} \cdots b_{n_d}}{q^{n_1 s_1} \cdots q^{n_d s_d}}.$$

3. Some reduction results for multiple zeta functions

We start this section by considering the case $K = \mathbb{F}_q(T)$.

Theorem 12. *We have*

$$\begin{aligned}
\zeta_d^\star(\mathbb{F}_q(T); s_1, \dots, s_d) =& \frac{q^d}{(q-1)^d} \sum_{e_1,\dots,e_d \in \{0,1\}} (-1)^{e_1+\dots+e_d} q^{-(e_1+\dots+e_d)} \\
&\times \prod_{j=0}^{d-1} \zeta^\star(\mathbb{F}_q[T]; s_{d-j} + e_{d-j} + \dots + s_d + e_d - j) \qquad (5) \\
=& \frac{q^{s_1+\dots+s_d}}{(q-1)^d} \sum_{e_1,\dots,e_d \in \{0,1\}} (-1)^{e_1+\dots+e_d} \\
&\times \prod_{j=0}^{d-1} \zeta^\star(\mathbb{F}_q[T]; s_{d-j} + e_{d-j} + \dots + s_d + e_d - j). \qquad (6)
\end{aligned}$$

Proof. When $d=1$ we have

$$\begin{aligned}
\zeta^\star(\mathbb{F}_q(T);s) =& \frac{1}{q-1}\sum_{n=0}^{\infty}\frac{q^{n+1}-1}{q^{ns}} \\
=& \frac{1}{q-1}\left(q\zeta^\star(\mathbb{F}_q[T];s)-\zeta^\star(\mathbb{F}_q[T];s+1)\right) \\
=& \frac{q^s}{q-1}\left(\zeta^\star(\mathbb{F}_q[T];s)-\zeta^\star(\mathbb{F}_q[T];s+1)\right),
\end{aligned}$$

which satisfies both equations (5) and (6).

By extracting the last term in the sum and using the geometric summation, we have

$$\begin{aligned}
\zeta^\star_d(\mathbb{F}_q(T);s_1,\dots,s_d) =& \frac{1}{(q-1)^d}\sum_{0\le n_1\le\cdots\le n_{d-1}}\frac{(q^{n_1+1}-1)\cdots(q^{n_{d-1}+1}-1)}{q^{n_1s_1+\cdots+n_{d-1}s_{d-1}}}\sum_{n_d\ge n_{d-1}}\frac{q^{n_d+1}-1}{q^{n_ds_d}} \\
=& \frac{1}{(q-1)^d}\sum_{0\le n_1\le\cdots\le n_{d-1}}\frac{(q^{n_1+1}-1)\cdots(q^{n_{d-1}+1}-1)}{q^{n_1s_1+\cdots+n_{d-1}s_{d-1}}}\left(\frac{q^{n_{d-1}(1-s_d)+1}}{1-q^{1-s_d}}-\frac{q^{-n_{d-1}s_d}}{1-q^{-s_d}}\right) \\
=& \frac{q}{(q-1)^d(1-q^{1-s_d})}\sum_{0\le n_1\le\cdots\le n_{d-1}}\frac{(q^{n_1+1}-1)\cdots(q^{n_{d-1}+1}-1)}{q^{n_1s_1+\cdots+n_{d-2}s_{d-2}+n_{d-1}(s_{d-1}+s_d-1)}} \\
&-\frac{1}{(q-1)^d(1-q^{-s_d})}\sum_{0\le n_1\le\cdots\le n_{d-1}}\frac{(q^{n_1+1}-1)\cdots(q^{n_{d-1}+1}-1)}{q^{n_1s_1+\cdots+n_{d-2}s_{d-2}+n_{d-1}(s_{d-1}+s_d)}} \\
=& \frac{q}{q-1}\zeta^\star(\mathbb{F}_q[T];s_d)\zeta^\star_{d-1}(\mathbb{F}_q(T);s_1,\dots,s_{d-2},s_{d-1}+s_d-1) \\
&-\frac{1}{q-1}\zeta^\star(\mathbb{F}_q[T];s_d+1)\zeta^\star_{d-1}(\mathbb{F}_q(T);s_1,\dots,s_{d-2},s_{d-1}+s_d),
\end{aligned} \tag{7}$$

which gives a recurrence relation reducing the depth.

Proceeding by induction, we replace formula (5) for $d-1$ in (7) and make the change $h=j+1$ in order to obtain

$$\begin{aligned}
\zeta^\star_d(\mathbb{F}_q(T);s_1,\dots,s_d) =& \frac{q}{q-1}\zeta^\star(\mathbb{F}_q[T];s_d)\frac{q^{d-1}}{(q-1)^{d-1}}\sum_{e_1,\dots,e_{d-1}\in\{0,1\}}(-1)^{e_1+\cdots+e_{d-1}}q^{-(e_1+\cdots+e_{d-1})} \\
&\times\prod_{h=1}^{d-1}\zeta^\star(\mathbb{F}_q[T];s_{d-h}+e_{d-h}+\cdots+s_{d-1}+e_{d-1}+s_d-h) \\
&-\frac{1}{q-1}\zeta^\star(\mathbb{F}_q[T];s_d+1)\frac{q^{d-1}}{(q-1)^{d-1}}\sum_{e_1,\dots,e_{d-1}\in\{0,1\}}(-1)^{e_1+\cdots+e_{d-1}}q^{-(e_1+\cdots+e_{d-1})} \\
&\times\prod_{h=1}^{d-1}\zeta^\star(\mathbb{F}_q[T];s_{d-h}+e_{d-h}+\cdots+s_{d-1}+e_{d-1}+s_d+1-h)
\end{aligned}$$

and this proves formula (5) by induction.

Similarly, we can replace formula (6) for $d-1$ in (7) and make the change $h = j+1$ in order to obtain

$$\begin{aligned}\zeta_d^\star(\mathbb{F}_q(T); s_1, \ldots, s_d) =&\frac{q}{q-1}\zeta^\star(\mathbb{F}_q[T]; s_d)\frac{q^{s_1+\cdots+s_d-1}}{(q-1)^{d-1}}\sum_{e_1,\ldots,e_{d-1}\in\{0,1\}}(-1)^{e_1+\cdots+e_{d-1}}\\ &\times\prod_{h=1}^{d-1}\zeta^\star(\mathbb{F}_q[T]; s_{d-h}+e_{d-h}+\cdots+s_{d-1}+e_{d-1}+s_d-h)\\ &-\frac{1}{q-1}\zeta^\star(\mathbb{F}_q[T]; s_d+1)\frac{q^{s_1+\cdots+s_d}}{(q-1)^{d-1}}\sum_{e_1,\ldots,e_{d-1}\in\{0,1\}}(-1)^{e_1+\cdots+e_{d-1}}\\ &\times\prod_{h=1}^{d-1}\zeta^\star(\mathbb{F}_q[T]; s_{d-h}+e_{d-h}+\cdots+s_{d-1}+e_{d-1}+s_d+1-h)\end{aligned}$$

and this proves formula (6) by induction.

□

Corollary 13. *We have*

$$\begin{aligned}(8)\qquad \zeta_d^\star(\mathbb{F}_q(T); s_1, \ldots, s_d) =&\frac{q^d}{(q-1)^d}\sum_{e_1,\ldots,e_d\in\{0,1\}}(-1)^{e_1+\cdots+e_d}q^{-(e_1+\cdots+e_d)}\zeta_d^\star(\mathbb{F}_q[T]; s_1+e_1, \ldots, s_d+e_d)\\ =&\frac{q^{s_1+\cdots+s_d}}{(q-1)^d}\sum_{e_1,\ldots,e_d\in\{0,1\}}(-1)^{e_1+\cdots+e_d}\zeta_d^\star(\mathbb{F}_q[T]; s_1+e_1, \ldots, s_d+e_d)\end{aligned}$$

Proof. This is a consequence of equation (2).

□

We continue with the general case of $g \geq 1$.

Lemma 14. *Assume that $g \geq 1$ and let*

$$S_m(s_1, \ldots, s_m) = \sum_{2g-2<n_1\leq\cdots\leq n_m}\frac{(q^{n_1-g+1}-1)\cdots(q^{n_m-g+1}-1)}{q^{n_1s_1+\cdots+n_ms_m}}.$$

Then we have

$$\begin{aligned}S_m(s_1, \ldots, s_m) =&q^{gm-(2g-1)(s_1+\cdots+s_m)}\sum_{e_1,\ldots,e_m\in\{0,1\}}(-q^g)^{-(e_1+\cdots+e_m)}\\ &\times\prod_{j=0}^{m-1}\zeta^\star(\mathbb{F}_q[T]; s_{m-j}+e_{m-j}+\cdots+s_m+e_m-j).\end{aligned}$$

Proof. First notice that the case $m = 1$ is given by

$$S_1(s) = \sum_{2g-2<n}\frac{q^{n-g+1}-1}{q^{ns}} = q^{-(2g-1)s}\left(q^g\zeta^\star(\mathbb{F}_q[T]; s) - \zeta^\star(\mathbb{F}_q[T]; s+1)\right),$$

which satisfies the claim.

By extracting the last term in the sum, and using the geometric summation, we obtain,

$$\begin{aligned}
S_m(s_1,\dots,s_m) =& \sum_{2g-2<n_1\leq\cdots\leq n_{k-1}} \frac{(q^{n_1-g+1}-1)\cdots(q^{n_{m-1}-g+1}-1)}{q^{n_1s_1+\cdots+n_{m-1}s_{m-1}}} \sum_{n_m\geq n_{m-1}} \frac{q^{n_m-g+1}-1}{q^{n_ms_m}}\\
=& \sum_{2g-2<n_1\leq\cdots\leq n_{m-1}} \frac{(q^{n_1-g+1}-1)\cdots(q^{n_{m-1}-g+1}-1)}{q^{n_1s_1+\cdots+n_{m-1}s_{m-1}}} \left(\frac{q^{n_{m-1}(1-s_m)-g+1}}{1-q^{1-s_m}} - \frac{q^{-n_{m-1}s_m}}{1-q^{-s_m}}\right)\\
=&\frac{q^{-g+1}}{1-q^{1-s_m}} \sum_{2g-2<n_1\leq\cdots\leq n_{m-1}} \frac{(q^{n_1-g+1}-1)\cdots(q^{n_{m-1}-g+1}-1)}{q^{n_1s_1+\cdots+n_{m-1}(s_{m-1}+s_m-1)}}\\
&-\frac{1}{1-q^{-s_m}} \sum_{2g-2<n_1\leq\cdots\leq n_{m-1}} \frac{(q^{n_1-g+1}-1)\cdots(q^{n_{m-1}-g+1}-1)}{q^{n_1s_1+\cdots+n_{m-1}(s_{m-1}+s_m)}}\\
=&q^{-g+1}\zeta^\star(\mathbb{F}_q[T];s_m)S_{m-1}(s_1,\dots,s_{m-2},s_{m-1}+s_m-1)\\
&-\zeta^\star(\mathbb{F}_q[T];s_m+1)S_{m-1}(s_1,\dots,s_{m-2},s_{m-1}+s_m),
\end{aligned}$$

which gives a recurrence relation that reduces the depth.

Proceeding by induction, we replace the formula for $m-1$ and make the change $h=j+1$ in order to obtain

$$\begin{aligned}
S_m(s_1,\dots,s_m) =&q^{-g+1}\zeta^\star(\mathbb{F}_q[T];s_m)q^{g(m-1)-(2g-1)(s_1+\cdots+s_m-1)} \sum_{e_1,\dots,e_{m-1}\in\{0,1\}} (-q^g)^{-(e_1+\cdots+e_{m-1})}\\
&\times \prod_{h=1}^{m-1} \zeta^\star(\mathbb{F}_q[T];s_{m-h}+e_{m-h}+\cdots+s_{m-1}+e_{m-1}+s_m-h)\\
&-\zeta^\star(\mathbb{F}_q[T];s_m+1)q^{g(m-1)-(2g-1)(s_1+\cdots+s_m)} \sum_{e_1,\dots,e_{m-1}\in\{0,1\}} (-q^g)^{-(e_1+\cdots+e_{m-1})}\\
&\times \prod_{h=1}^{m-1} \zeta^\star(\mathbb{F}_q[T];s_{m-h}+e_{m-h}+\cdots+s_{m-1}+e_{m-1}+s_m-h+1),
\end{aligned}$$

and this simplifies to the formula for m.

□

Theorem 15. *Assume that $g\geq 1$ and let*

$$R_m(K;s_1,\dots,s_m) = \sum_{0\leq n_1\leq\cdots\leq n_m\leq 2g-2} \frac{b_{n_1}\cdots b_{n_m}}{q^{n_1s_1+\cdots+n_ms_m}}. \tag{9}$$

Also set $R_0=1$.

We have

$$\begin{aligned}
\zeta_d^\star(K;s_1,\dots,s_d) =&R_d(K;s_1,\dots,s_d) + \sum_{\ell=1}^{d} \left(\frac{q^gh_K}{q-1}\right)^\ell R_{d-\ell}(K;s_1,\dots,s_{d-\ell})q^{-(2g-1)(s_{d+1-\ell}+\cdots+s_d)}\\
&\times \sum_{e_{d+1-\ell},\dots,e_d\in\{0,1\}} (-q^g)^{-(e_{d+1-\ell}+\cdots+e_d)} \prod_{j=0}^{\ell-1} \zeta^\star(\mathbb{F}_q[T];s_{d-j}+e_{d-j}+\cdots+s_d+e_d-j).
\end{aligned}$$

This is Theorem 3 from the introduction.

Proof. By direct application of the definition and by replacing the value of $S_m(s_1,\dots,s_m)$ given in Lemma 14, we have

$$\begin{aligned}
\zeta_d^\star(K;s_1,\dots,s_d) = &\sum_{0\le n_1\le\cdots\le n_d} \frac{b_{n_1}\cdots b_{n_d}}{q^{n_1s_1+\cdots+n_ds_d}}\\
&=R_d(K;s_1,\dots,s_d)+\sum_{\ell=1}^{d}\frac{h_K^\ell}{(q-1)^\ell}R_{d-\ell}(K;s_1,\dots,s_{d-\ell})S_\ell(s_{d+1-\ell},\dots,s_d).\\
&=R_d(K;s_1,\dots,s_d)+\sum_{\ell=1}^{d}\frac{h_K^\ell}{(q-1)^\ell}R_{d-\ell}(K;s_1,\dots,s_{d-\ell})q^{g\ell-(2g-1)(s_{d+1-\ell}+\cdots+s_d)}\\
&\quad\times\sum_{e_{d+1-\ell},\dots,e_d\in\{0,1\}}(-q^g)^{-(e_{d+1-\ell}+\cdots+e_d)}\prod_{j=0}^{\ell-1}\zeta^\star(\mathbb{F}_q[T];s_{d-j}+e_{d-j}+\cdots+s_d+e_d-j).
\end{aligned}$$

□

Corollary 16. *We have*

$$\begin{aligned}
\zeta_d^\star(K;s_1,\dots,s_d) =&R_d(K;s_1,\dots,s_d)+\sum_{\ell=1}^{d}\left(\frac{q^gh_K}{q-1}\right)^\ell R_{d-\ell}(K;s_1,\dots,s_{d-\ell})q^{-(2g-1)(s_{d+1-\ell}+\cdots+s_d)}\\
&\times\sum_{e_{d+1-\ell},\dots,e_d\in\{0,1\}}(-q^g)^{-(e_{d+1-\ell}+\cdots+e_d)}\zeta_\ell^\star(\mathbb{F}_q[T];s_{d+1-\ell}+e_{d+1-\ell},\dots,s_d+e_d).
\end{aligned}\tag{10}$$

Proof. This is a consequence of equation (2). □

For the case $g=1$, we have the following.

Theorem 17. *Let $g=1$, then*

$$\zeta_d^\star(K;s_1,\dots,s_d)=1+\sum_{\ell=1}^{d}h_K^\ell q^{-(s_{d+1-\ell}+\cdots+s_d)}\zeta_\ell^\star(\mathbb{F}_q(T);s_{d+1-\ell},\dots,s_d).$$

Proof. Recall from equation (3) that $b_0=1$. We then conclude that

$$R_m(K;s_1,\dots,s_m)=1\qquad\text{for all } m.$$

Now apply equation (10) together with (8).

□

In the case where we have the function field of an elliptic curve E, we obtain that $h_K=|E(\mathbb{F}_q)|$.

Corollary 18. *Let $g=1$, then*

$$\zeta_d^\star(K;s_1,\dots,s_d)=\zeta_{d-1}^\star(K;s_2,\dots,s_d)+h_K^dq^{-(s_1+\cdots+s_d)}\zeta_d^\star(\mathbb{F}_q(T);s_1,\dots,s_d).$$

Proof. Consider the case $d=1$ in Theorem 17.

$$\zeta^\star(K;s)=1+h_Kq^{-s}\zeta^\star(\mathbb{F}_q(T);s).$$

Similarly, with $d=2$,

$$\begin{aligned}
\zeta_2^\star(K;s_1,s_2)=&1+h_Kq^{-s_2}\zeta^\star(\mathbb{F}_q(T);s_2)+h_K^2q^{-(s_1+s_2)}\zeta_2^\star(\mathbb{F}_q(T);s_1,s_2)\\
=&\zeta^\star(K;s_2)+h_K^2q^{-(s_1+s_2)}\zeta_2^\star(\mathbb{F}_q(T);s_1,s_2).
\end{aligned}$$

The rest of the proof proceeds by induction. □

4. A generalization to polylogarithms

In this section we extend the previous construction of multiple zeta functions to multiple polylogarithms. In this section, all subindexes k denote positive integers.

We start by considering the depth-one case,

$$\mathrm{Li}^\star_k(\mathbb{F}_q[T];z)=\sum_{\substack{f\in\mathbb{F}_q[T]\\ f\text{ monic}\\ 0\le\deg(f)}}\frac{z^{\deg f}}{|f|^k},$$

which is absolutely converging and analytic for $|z|<q^{k-1}$ (here we take the convention $0^0=1$ so that $\mathrm{Li}^\star_k(\mathbb{F}_q[T];0)=1$).

Then

$$\begin{aligned}\mathrm{Li}^\star_k(\mathbb{F}_q[T];z)&=\sum_{n=0}^\infty\frac{z^n\#\{F\text{ monic}|\deg F=n\}}{q^{nk}}\\&=\sum_{n=0}^\infty\frac{z^n}{q^{n(k-1)}}\\&=\left(1-\frac{z}{q^{k-1}}\right)^{-1}\end{aligned}$$

provides an analytic continuation to the complex plane with $z\neq q^{k-1}$.

We define multiple polylogarithms in a similar way.

$$\mathrm{Li}^\star_{k_1,\dots,k_d}(\mathbb{F}_q[T];z_1,\dots,z_d)=\sum_{\substack{(f_1,\dots,f_d)\in(\mathbb{F}_q[T])^d\\ f_1,\dots,f_d\text{ monic}\\ 0\le\deg(f_1)\le\dots\le\deg(f_d)}}\frac{z_1^{\deg(f_1)}\cdots z_d^{\deg(f_d)}}{|f_1|^{k_1}\cdots|f_d|^{k_d}},$$

which is absolutely convergent and analytic in the complex region

$$|z_j\cdots z_d|<q^{k_j+\dots+k_d-d+j-1},\qquad j=1,\dots,d. \tag{11}$$

As before, we can write

$$\begin{aligned}\mathrm{Li}^\star_{k_1,\dots,k_d}(\mathbb{F}_q[T];z_1,\dots,z_d)&=\sum_{0\le n_1\le\dots\le n_d}\frac{z_1^{n_1}\cdots z_d^{n_d}\#\{f_i\text{ monic}|\deg f_i=n_i\}}{q^{n_1k_1}\cdots q^{n_dk_d}}\\&=\sum_{0\le n_1\le\dots\le n_d}\frac{z_1^{n_1}\cdots z_d^{n_d}}{q^{n_1(k_1-1)}\cdots q^{n_d(k_d-1)}}.\end{aligned}$$

Writing $\ell_1=n_1$ and $\ell_i=n_i-n_{i-1}$, we have,

$$\begin{aligned}\mathrm{Li}^\star_{k_1,\dots,k_d}(\mathbb{F}_q[T];z_1,\dots,z_d)&=\sum_{0\le\ell_1,\dots,\ell_d}\frac{z_1^{\ell_1}z_2^{\ell_1+\ell_2}\cdots z_n^{\ell_1+\dots+\ell_n}}{q^{\ell_1(k_1-1)}q^{(\ell_1+\ell_2)(k_2-1)}\cdots q^{(\ell_1+\ell_2+\dots+\ell_d)(k_d-1)}}\\&=\left(1-\frac{z_1\cdots z_d}{q^{k_1+\dots+k_d-d}}\right)^{-1}\left(1-\frac{z_2\cdots z_d}{q^{k_2+\dots+k_d-(d-1)}}\right)^{-1}\cdots\left(1-\frac{z_d}{q^{k_d-1}}\right)^{-1}, \qquad (12)\end{aligned}$$

and this provides an analytic continuation to the complex region

$$z_j\cdots z_d\neq q^{k_j+\dots+k_d-d+j-1},\qquad j=1,\dots,d. \tag{13}$$

We can prove a result that is analogous to equation (2) in the case of polylogarithms.

Theorem 19. *We have*

$$\mathrm{Li}^\star_{k_1,\dots,k_d}(\mathbb{F}_q[T];z_1,\dots,z_d)=\prod_{j=1}^d\mathrm{Li}^\star_{k_j+\dots+k_d-(d-j)}\left(\mathbb{F}_q[T];\prod_{h=j}^d z_h\right). \tag{14}$$

Proof. The proof follows directly from equation (12). □

For a global function field K, consider

$$\mathrm{Li}^{\star}_{k_1,\dots,k_d}(K;z_1,\dots,z_d)=\sum_{\substack{(D_1,\dots,D_d)\in(\mathcal{D}_K^+)^d\\ 0\le\deg(D_1)\le\cdots\le\deg(D_d)}}\frac{z_1^{\deg(D_1)}\cdots z_d^{\deg(D_d)}}{|D_1|^{k_1}\cdots|D_d|^{k_d}},$$

where $\mathcal{D}_K^+$ is the semigroup of effective divisors.

Remark 20. *Observe that*

$$\zeta_d(K,k_1,\dots,k_d)=\mathrm{Li}^{\star}_{k_1,\dots,k_d}(K;1,\dots,1)$$

and we will use the notation $\overline{k_i}$ *to indicate* $z_i=-1$*. For example,*

$$\zeta_d(K,\overline{k_1},\dots,\overline{k_d})=\mathrm{Li}^{\star}_{k_1,\dots,k_d}(K;-1,\dots,-1).$$

The same notation will apply for $\mathbb{F}_q[T]$ *in place of* K*.*

It can be seen that

Theorem 21. $\mathrm{Li}^{\star}_{k_1,\dots,k_d}(K;z_1,\dots,z_d)$ *is absolutely convergent and analytic in the complex region determined by* (11) *and has an analytic continuation to the complex region determined by*

$$z_j\cdots z_d\neq q^{k_j+\cdots+k_d},\dots,q^{k_j+\cdots+k_d-d+j-1},\qquad j=1,\dots,d. \tag{15}$$

This is Theorem 4 from the introduction.

Proof. Define the non-negative integers

$$a_{n_1,\dots,n_d}=\#\{(D_1,\dots,D_d)\in D_K^+\times\cdots\times D_K^+\,|\,\deg(D_j)=n_j, j=1,\dots,d\}$$

and recall that

$$b_{n_j}=\#\{D_j\in D_K^+\,|\,\deg(D_j)=n_j\}.$$

Then we have

$$a_{n_1,\dots,n_d}=\prod_{j=1}^d b_{n_j}.$$

Hence, we can write

$$\begin{aligned}\mathrm{Li}^{\star}_{k_1,\dots,k_d}(K;z_1,\dots,z_d)&=\sum_{0\le n_1\le\cdots\le n_d}a_{n_1,\dots,n_d}\frac{z_1^{n_1}\cdots z_d^{n_d}}{q^{n_1k_1}\cdots q^{n_dk_d}}\\&=\sum_{0\le n_1\le\cdots\le n_d}\prod_{j=1}^d b_{n_j}\frac{z_j^{n_j}}{q^{n_jk_j}}\\&=\sum_{\ell_1=0}^\infty\cdots\sum_{\ell_d=0}^\infty\prod_{j=1}^d b_{\ell_1+\cdots+\ell_j}\frac{z_j^{\ell_1+\cdots+\ell_j}}{q^{k_j(\ell_1+\cdots+\ell_j)}}.\end{aligned}$$

By Lemma 10,

$$|b_{\ell_1+\cdots+\ell_j}|\le C_jq^{\ell_1+\cdots+\ell_j}$$

for some constants $C_j>0$, $j=1,\dots,d$. This yields the estimate

$$\begin{aligned}\prod_{j=1}^d\left|b_{\ell_1+\cdots+\ell_j}\frac{z_j^{\ell_1+\cdots+\ell_j}}{q^{k_j(\ell_1+\cdots+\ell_j)}}\right|&\le\prod_{j=1}^d C_jq^{\ell_1+\cdots+\ell_j}(z_jq^{-k_j})^{\ell_1+\cdots+\ell_j}\\&=\prod_{j=1}^d C_j(z_j\cdots z_d)^{\ell_j}(q^{d-j+1-(k_j+\cdots+k_d)})^{\ell_j}.\end{aligned}$$

Thus, by using the geometric summation, we have in the complex region determined by (11),

$$\sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_d=0}^{\infty}\prod_{j=1}^{d}\left|b_{\ell_1+\cdots+\ell_j}\frac{z_j^{\ell_1+\cdots+\ell_j}}{q^{k_j(\ell_1+\cdots+\ell_j)}}\right|\leq\prod_{j=1}^{d}C_j\sum_{\ell_j=0}^{\infty}(z_j\cdots z_d)^{\ell_j}(q^{d-j+1-(k_j+\cdots+k_d)})^{\ell_j}$$
$$=\prod_{j=1}^{d}C_j\left(1-\frac{z_j\cdots z_d}{q^{(k_j+\cdots+k_d)-d+j-1}}\right)^{-1}.$$

This proves that for any global function field, $\mathrm{Li}^\star_{k_1,\dots,k_d}(K;z_1,\dots,z_d)$ is absolutely convergent and analytic in the complex region determined by (11).

The region for analytic continuation will be a consequence of Corollary 26.

□

5. Some reduction results for multiple polylogarithms

In this section we collect analogous results to those of Section 3 for the case of multiple polylogarithms. The proofs are very similar, and therefore we omit some of them, while repeating others for the sake of clarity.

We start by considering the case $K=\mathbb{F}_q(T)$.

Theorem 22. *We have*

$$\mathrm{Li}^\star_{k_1,\dots,k_d}(\mathbb{F}_q(T);z_1,\dots,z_d)=\frac{q^d}{(q-1)^d}\sum_{e_1,\dots,e_d\in\{0,1\}}(-1)^{e_1+\cdots+e_d}q^{-(e_1+\cdots+e_d)}$$
$$\times\prod_{j=0}^{d-1}\mathrm{Li}^\star_{k_{d-j}+e_{d-j}+\cdots+k_d+e_d-j}\left(\mathbb{F}_q[T];\prod_{h=d-j}^{d}z_h\right)\tag{16}$$
$$=\frac{q^{k_1+\cdots+k_d}}{(q-1)^d\prod_{\ell=1}^d z_\ell}\sum_{e_1,\dots,e_d\in\{0,1\}}(-1)^{e_1+\cdots+e_d}$$
$$\times\prod_{j=0}^{d-1}\mathrm{Li}^\star_{k_{d-j}+e_{d-j}+\cdots+k_d+e_d-j}\left(\mathbb{F}_q[T];\prod_{h=d-j}^{d}z_h\right).\tag{17}$$

Proof. The proof of this result is analogous to that of Theorem 12. □

By combining the above result with equation (14), we obtain the following.

Corollary 23. *We have*

(18)

$$\mathrm{Li}^\star_{k_1,\dots,k_d}(\mathbb{F}_q(T);z_1,\dots,z_d)=\frac{q^d}{(q-1)^d}\sum_{e_1,\dots,e_d\in\{0,1\}}(-1)^{e_1+\cdots+e_d}q^{-(e_1+\cdots+e_d)}\mathrm{Li}^\star_{k_1+e_1,\dots,k_d+e_d}(\mathbb{F}_q[T];z_1,\dots,z_d)$$
$$=\frac{q^{k_1+\cdots+k_d}}{(q-1)^d\prod_{\ell=1}^d z_\ell}\sum_{e_1,\dots,e_d\in\{0,1\}}(-1)^{e_1+\cdots+e_d}\mathrm{Li}^\star_{k_1+e_1,\dots,k_d+e_d}(\mathbb{F}_q[T];z_1,\dots,z_d).$$

We continue with the general case of $g\geq 1$.

Lemma 24. *Assume that* $g\geq 1$ *and let*

$$\mathcal{S}_{k_1,\dots,k_m}(z_1,\dots,z_m)=\sum_{2g-2<n_1\leq\cdots\leq n_m}\frac{(q^{n_1-g+1}-1)\cdots(q^{n_m-g+1}-1)\prod_{\ell=1}^m z_\ell^{n_\ell}}{q^{n_1k_1+\cdots+n_mk_m}}.$$

Then we have

$$\mathcal{S}_{k_1,\dots,k_m}(z_1,\dots,z_m) = q^{gm-(2g-1)(k_1+\cdots+k_m)}\left(\prod_{\ell=1}^m z_\ell\right)^{2g-1}\sum_{e_1,\dots,e_m\in\{0,1\}}(-q^g)^{-(e_1+\cdots+e_m)}$$
$$\times\prod_{j=0}^{m-1}\mathrm{Li}^\star_{k_{m-j}+e_{m-j}+\cdots+k_m+e_m-j}\left(\mathbb{F}_q[T];\prod_{h=m-j}^m z_h\right). \tag{19}$$

Proof. The proof of this result follows very closely the proof of Lemma 14. First notice that the case $m=1$ is given by

$$\mathcal{S}_k(z)=\sum_{2g-2<n}\frac{(q^{n-g+1}-1)z^n}{q^{nk}}=q^{-(2g-1)k}z^{2g-1}\left(q^g\mathrm{Li}^\star_k(\mathbb{F}_q[T];z)-\mathrm{Li}^\star_{k+1}(\mathbb{F}_q[T];z)\right),$$

which satisfies the claim.

By applying the geometric summation on the last term in the sum, we obtain,

$$\begin{aligned}
\mathcal{S}_{k_1,\dots,k_m}(z_1,\dots,z_m) =& \sum_{2g-2<n_1\le\cdots\le n_{m-1}}\frac{(q^{n_1-g+1}-1)\cdots(q^{n_{m-1}-g+1}-1)\prod_{\ell=1}^{m-1}z_\ell^{n_\ell}}{q^{n_1k_1+\cdots+n_{m-1}k_{m-1}}}\sum_{n_m\ge n_{m-1}}\frac{(q^{n_m-g+1}-1)z_m^{n_m}}{q^{n_mk_m}}\\
=&\sum_{2g-2<n_1\le\cdots\le n_{m-1}}\frac{(q^{n_1-g+1}-1)\cdots(q^{n_{m-1}-g+1}-1)\prod_{\ell=1}^{m-1}z_\ell^{n_\ell}}{q^{n_1k_1+\cdots+n_{m-1}k_{m-1}}}\\
&\times\left(\frac{q^{n_{m-1}(1-k_m)-g+1}z_m^{n_{m-1}}}{1-q^{1-k_m}z_m}-\frac{q^{-n_{m-1}k_m}z_m^{n_{m-1}}}{1-q^{-k_m}z_m}\right)\\
=&\frac{q^{-g+1}}{1-q^{1-k_m}z_m}\sum_{2g-2<n_1\le\cdots\le n_{m-1}}\frac{(q^{n_1-g+1}-1)\cdots(q^{n_{m-1}-g+1}-1)\prod_{\ell=1}^{m-2}z_\ell^{n_\ell}(z_{m-1}z_m)^{n_{m-1}}}{q^{n_1k_1+\cdots+n_{m-1}(k_{m-1}+k_m-1)}}\\
&-\frac{1}{1-q^{-k_m}z_m}\sum_{2g-2<n_1\le\cdots\le n_{m-1}}\frac{(q^{n_1-g+1}-1)\cdots(q^{n_{m-1}-g+1}-1)\prod_{\ell=1}^{m-2}z_\ell^{n_\ell}(z_{m-1}z_m)^{n_{m-1}}}{q^{n_1k_1+\cdots+n_{m-1}(k_{m-1}+k_m)}}\\
=&q^{-g+1}\mathrm{Li}_{k_m}(\mathbb{F}_q[T];z_m)\mathcal{S}_{k_1,\dots,k_{m-2},k_{m-1}+k_m-1}(z_1,\dots,z_{d-2},z_{m-1}z_m)\\
&-\mathrm{Li}_{k_m+1}(\mathbb{F}_q[T];z_m)\mathcal{S}_{k_1,\dots,k_{m-2},k_{m-1}+k_m}(z_1,\dots,z_{d-2},z_{m-1}z_m),
\end{aligned}$$

which gives a recurrence relation that reduces the depth.

Proceeding by induction, we replace the formula for $m-1$ and make the change $h=j+1$ in order to obtain

$$\begin{aligned}
\mathcal{S}_{k_1,\dots,k_m}(z_1,\dots,z_m) =& q^{-g+1}\mathrm{Li}^\star_{k_m}(\mathbb{F}_q[T];z_m)q^{g(m-1)-(2g-1)(k_1+\cdots+k_m-1)}\left(\prod_{\ell=1}^m z_\ell\right)^{2g-1}\sum_{e_1,\dots,e_{m-1}\in\{0,1\}}(-q^g)^{-(e_1+\cdots+e_{m-1})}\\
&\times\prod_{h=1}^{m-1}\mathrm{Li}^\star_{k_{m-h}+e_{m-h}+\cdots+k_m-h}\left(\mathbb{F}_q[T];\prod_{\ell=m-h}^m z_\ell\right)\\
&-\mathrm{Li}^\star_{k_m+1}(\mathbb{F}_q[T];z_m)q^{g(m-1)-(2g-1)(k_1+\cdots+k_m)}\left(\prod_{\ell=1}^m z_\ell\right)^{2g-1}\sum_{e_1,\dots,e_{m-1}\in\{0,1\}}(-q^g)^{-(e_1+\cdots+e_{m-1})}\\
&\times\prod_{h=1}^{m-1}\mathrm{Li}^\star_{k_{m-h}+e_{m-h}+\cdots+k_m+1-h}\left(\mathbb{F}_q[T];\prod_{\ell=m-h}^m z_\ell\right),
\end{aligned}$$

and this simplifies to (19). $\square$

Theorem 25. *Assume that $g \geq 1$ and let*

$$\mathcal{R}_{k_1,\ldots,k_m}(K; z_1, \ldots, z_m) = \sum_{0 \leq n_1 \leq \cdots \leq n_m \leq 2g-2} \frac{\prod_{i=1}^m (b_{n_i} z_i^{n_i})}{q^{n_1 k_1 + \cdots + n_m k_m}}. \tag{20}$$

Also set $\mathcal{R}_0 = 1$.

We have

$$\begin{aligned}\mathrm{Li}^\star_{k_1,\ldots,k_d}(K; z_1, \ldots, z_d) =& \mathcal{R}_{k_1,\ldots,k_d}(K; z_1, \ldots, z_d) + \sum_{\ell=1}^d \left(\frac{q^g h_K}{q-1}\right)^\ell \mathcal{R}_{k_1,\ldots,k_{d-\ell}}(K; z_1, \ldots, z_{d-\ell}) \\ & \times q^{-(2g-1)(k_{d+1-\ell}+\cdots+k_d)} \left(\prod_{i=d+1-\ell}^d z_i\right)^{2g-1} \\ & \times \sum_{e_{d+1-\ell},\ldots,e_d \in \{0,1\}} (-q^g)^{-(e_{d+1-\ell}+\cdots+e_d)} \prod_{j=0}^{\ell-1} \mathrm{Li}^\star_{k_{d-j}+e_{d-j}+\cdots+k_d+e_d-j}\left(\mathbb{F}_q[T]; \prod_{i=d-j}^d z_i\right).\end{aligned} \tag{21}$$

This is Theorem 5 from the introduction.

Proof. By direct application of the definition and by replacing the value of $\mathcal{S}_{k_1,\ldots,k_m}(z_1, \ldots, z_m)$ given in Lemma 24, we have

$$\begin{aligned}\mathrm{Li}_{k_1,\ldots,k_d}(K; z_1, \ldots, z_d) =& \sum_{0 \leq n_1 \leq \cdots \leq n_d} \frac{\prod_{i=1}^d (b_{n_i} z_i^{n_i})}{q^{n_1 k_1 + \cdots + n_d k_d}} \\ =& \mathcal{R}_{k_1,\ldots,k_d}(K; z_1, \ldots, z_d) + \sum_{\ell=1}^d \frac{h_K^\ell}{(q-1)^\ell} \mathcal{R}_{k_1,\ldots,k_{d-\ell}}(K; z_1, \ldots, z_{d-\ell}) \mathcal{S}_{k_{d+1-\ell},\ldots,k_d}(z_{d+1-\ell}, \ldots, z_d) \\ =& \mathcal{R}_{k_1,\ldots,k_d}(K; z_1, \ldots, z_d) + \sum_{\ell=1}^d \left(\frac{q^g h_K}{q-1}\right)^\ell \mathcal{R}_{k_1,\ldots,k_{d-\ell}}(K; z_1, \ldots, z_{d-\ell}) \\ & \times q^{-(2g-1)(k_{d+1-\ell}+\cdots+k_d)} \left(\prod_{i=d+1-\ell}^d z_i\right)^{2g-1} \\ & \times \sum_{e_{d+1-\ell},\ldots,e_d \in \{0,1\}} (-q^g)^{-(e_{d+1-\ell}+\cdots+e_d)} \prod_{j=0}^{\ell-1} \mathrm{Li}_{k_{d-j}+e_{d-j}+\cdots+k_d+e_d-j}\left(\mathbb{F}_q[T]; \prod_{h=d-j}^d z_h\right).\end{aligned}$$

□

Corollary 26. *We have*

$$\begin{aligned}\mathrm{Li}^\star_{k_1,\ldots,k_d}(K; z_1, \ldots, z_d) =& \mathcal{R}_{k_1,\ldots,k_d}(K; z_1, \ldots, z_d) + \sum_{\ell=1}^d \left(\frac{q^g h_K}{q-1}\right)^\ell \mathcal{R}_{k_1,\ldots,k_{d-\ell}}(K; z_1, \ldots, z_{d-\ell}) \\ & \times q^{-(2g-1)(k_{d+1-\ell}+\cdots+k_d)} \left(\prod_{i=d+1-\ell}^d z_i\right)^{2g-1} \\ & \times \sum_{e_{d+1-\ell},\ldots,e_d \in \{0,1\}} (-q^g)^{-(e_{d+1-\ell}+\cdots+e_d)} \mathrm{Li}^\star_{k_{d+1-\ell}+e_{d+1-\ell},\ldots,k_d+e_d}(\mathbb{F}_q[T]; z_{d+1-\ell}, \ldots, z_d).\end{aligned} \tag{22}$$

Proof. This is a consequence of equations (14) and (21). □

Corollary 26 shows that $\mathrm{Li}^\star_{k_1,\ldots,k_d}(K; z_1, \ldots, z_d)$ is a rational function with analytic continuation in the region (15) given by

$$z_j \cdots z_d \neq q^{k_j+\cdots+k_d}, \ldots, q^{k_j+\cdots+k_d-d+j-1}, \qquad j = 1, \ldots, d.$$

This concludes the proof of Theorem 21.

For the case $g=1$, we have the following.

Theorem 27. *Let $g=1$, then*

$$\mathrm{Li}^{\star}_{k_1,\dots,k_d}(K;z_1,\dots,z_d) = 1+\sum_{\ell=1}^{d} h_K^{\ell}\left(\prod_{i=d+1-\ell}^{d} z_i\right) q^{-(k_{d+1-\ell}+\cdots+k_d)}\mathrm{Li}^{\star}_{k_{d+1-\ell},\dots,k_d}(\mathbb{F}_q(T);z_{d+1-\ell},\dots,z_d).$$

Proof. Since $b_0=1$ we conclude that

$$\mathcal{R}_{k_1,\dots,k_d}(K;z_1,\dots,z_d)=1 \qquad \text{for all } d.$$

To conclude we combine equations (22) and (18).

□

As before, we obtain a nice recurrence for the case of $g=1$.

Corollary 28. *Let $g=1$, then*

$$\mathrm{Li}^{\star}_{k_1,\dots,k_d}(K;z_1,\dots,z_d) = \mathrm{Li}^{\star}_{k_2,\dots,k_d}(K;z_2,\dots,z_d)+h_K^{d}\left(\prod_{\ell=1}^{d} z_\ell\right) q^{-(k_1+\cdots+k_d)}\mathrm{Li}^{\star}_{k_1,\dots,k_d}(\mathbb{F}_q(T);z_1,\dots,z_d).$$

Proof. The proof proceeds in a similar fashion to that of Corollary 18. □

6. Identities for special values

In this session we explore multiple zeta values and compare them to analogous results in the classical case. All the multiple zeta functions over function fields are rational functions on q^{-s} so these formulas are expected a priori and are not as surprising as the results in the number field case. Nevertheless, they yield interesting comparisons.

6.1. **The $\mathbb{F}_q[T]$ case.** Special values of $\zeta_d^{\star}(\mathbb{F}_q[T];s_1,\dots,s_d)$ are easy to find, thanks to equation (2), which we recall here:

$$\zeta_d^{\star}(\mathbb{F}_q[T];s_1,\dots,s_d)=\prod_{k=1}^{d}\zeta^{\star}(\mathbb{F}_q[T];s_k+\cdots+s_d-(d-k)).$$

We start with one of the most elegant identities due to Hoffman [Hof92] for the classical case and to Aoki, Kombu, and Ohno [AKO08] for the star case. For $d>0$, we have that

$$\zeta(\underbrace{2,\dots,2}_{d})=\frac{\pi^{2d}}{(2d+1)!},\qquad \zeta^{\star}(\underbrace{2,\dots,2}_{d})=2(1-2^{1-2d})\zeta(2d)=(-1)^{d-1}2(2^{2d-1}-1)\frac{B_{2d}\pi^{2d}}{(2d)!}.$$

Remark that the right hand side of each of the above equations can be interpreted as a rational multiple of $\zeta(2)^d$.

We start by considering $\zeta_d^{\star}(\mathbb{F}_q[T];2,\dots,2)$. Following Zagier [Zag12], we set

$$H_d:=\zeta_d^{\star}(\mathbb{F}_q[T];2,\dots,2).$$

In particular

$$H_1=\zeta^{\star}(\mathbb{F}_q[T];2).$$

Proposition 29. *We have for $n\geq 2$,*

$$\zeta^{\star}(\mathbb{F}_q[T];n)=\frac{H_1^{n-1}}{H_1^{n-1}-(H_1-1)^{n-1}}. \tag{23}$$

Proof. It suffices to observe that

$$\zeta^{\star}(\mathbb{F}_q[T];n)=\frac{1}{1-q^{1-n}}=\frac{1}{1-(1-(1-q^{-1}))^{n-1}}=\frac{1}{1-\left(1-\frac{1}{H_1}\right)^{n-1}}=\frac{H_1^{n-1}}{H_1^{n-1}-(H_1-1)^{n-1}}.$$

□

Corollary 30.

$$\zeta_d^\star(\mathbb{F}_q[T];2,\dots,2)=H_d=\frac{H_1^{d(d+1)/2}}{\prod_{n=1}^d(H_1^n-(H_1-1)^n)}. \tag{24}$$

Proof. Recall from (2) that we have

$$\zeta_d^\star(\mathbb{F}_q[T];2,\dots,2)=\prod_{n=2}^{d+1}\zeta^\star(\mathbb{F}_q[T];n).$$

Then apply equation (23). □

The right hand side of equation (24) has total degree d, and we can interpret $\zeta_d^\star(\mathbb{F}_q[T];2,\dots,2)$ as the d power of $\zeta^\star(\mathbb{F}_q[T];2)$ (with some correction factor). This result is consistent with the number field case, even if the final formula is not as simple.

The above result can be easily generalized.

Corollary 31. *We have for $m\geq 2$,*

$$\zeta_d^\star(\mathbb{F}_q[T];m,\dots,m)=\frac{H_1^{(m-1)d(d+1)/2}}{\prod_{n=1}^d(H_1^{(m-1)n}-(H_1-1)^{(m-1)n})} \tag{25}$$

Proof. By (2) we have

$$\zeta_d^\star(\mathbb{F}_q[T];m,\dots,m)=\prod_{n=1}^{d}\zeta^\star(\mathbb{F}_q[T];(m-1)n+1).$$

Then apply equation (23). □

Muneta [Mun08] proves

$$\zeta^\star(\underbrace{2\ell,\dots,2\ell}_{d})=C_{d,\ell}\pi^{2\ell d},$$

where $C_{d,\ell}$ is certain given rational number. By setting $m=2\ell$ in (25) we see that the total degree on the right-hand side is d, which would correspond to π^{2d}, and therefore our formula is different from the one in the classical case.

Ohno and Zudilin [OZ08] consider values of the form $\zeta^\star(1,\underbrace{2,\dots,2}_{b_1},1,\underbrace{2,\dots,2}_{b_2},1,\cdots,\underbrace{2,\dots,2}_{b_{\ell-1}},1,\underbrace{2,\dots,2}_{b_n})$, where $b_1,\dots,b_\ell\geq 0$ and formulate the Two-one Formula to reduce them in terms of classical values of smaller depth. They prove several special cases, while Zhao [Zha16] proves the most general statement.

We consider

$$G(a_1,\dots,a_\ell):=\zeta^\star_{a_1+\cdots+a_\ell+\ell-1}(\mathbb{F}_q[T];\underbrace{2,\dots,2}_{a_1},1,\underbrace{2,\dots,2}_{a_2},1,\cdots,\underbrace{2,\dots,2}_{a_{\ell-1}},1,\underbrace{2,\dots,2}_{a_\ell}).$$

(The case of Ohno and Zudilin is included in the above when $a_1=0$.)

Theorem 32. *We have*

$$G(a,b)=\zeta^\star(\mathbb{F}_q[T];b+1)\prod_{n=2}^{a+b+1}\zeta^\star(\mathbb{F}_q[T];n)=\frac{H_1^{b+(a+b)(a+b+1)/2}}{(H_1^b-(H_1-1)^b)\prod_{n=1}^{a+b}(H_1^n-(H_1-1)^n)},$$

and more generally,

$$G(a_1,\dots,a_\ell)=\prod_{m=2}^{\ell}\zeta^\star(\mathbb{F}_q[T];a_m+\cdots+a_\ell+1)\prod_{n=2}^{a_1+\cdots+a_\ell+1}\zeta^\star(\mathbb{F}_q[T];n).$$

Proof. These formulas are a direct consequence of equation (2). □

In particular, we obtain,

$$\zeta^\star_{a+1}(\mathbb{F}_q[T];1,\underbrace{2,\dots,2}_{a}) = G(0,a) = \zeta^\star(\mathbb{F}_q[T];a+1)\prod_{n=2}^{a+1}\zeta^\star(\mathbb{F}_q[T];n),$$

$$\zeta^\star(\mathbb{F}_q[T];1,\underbrace{2,\dots,2}_{a},1,\underbrace{2,\dots,2}_{b}) = G(0,a,b) = \zeta^\star(\mathbb{F}_q[T];a+b+1)\zeta^\star(\mathbb{F}_q[T];b+1)\prod_{n=2}^{a+b+1}\zeta^\star(\mathbb{F}_q[T];n),$$

and

$$\zeta^\star_{b+1}(\mathbb{F}_q[T];\underbrace{1,\dots,1}_{b},2) = G(\underbrace{0,\dots,0}_{b},1) = \zeta^\star(\mathbb{F}_q[T];2)^{b+1}.$$

From the cyclic sum formula due to Ohno and Wakabayashi [OW06] one can deduce ([OZ08] formula (3a)):

$$\zeta^\star(1,\underbrace{2,\dots,2}_{a}) = 2\zeta(2a+1).$$

Ohno and Zudilin prove ([OZ08], Theorems 1 and 2):

$$\zeta^\star(1,\underbrace{2,\dots,2}_{a},1,\underbrace{2,\dots,2}_{b}) = 4\zeta^\star(2a+1,2b+1) - 2\zeta(2(a+b)+2),$$

$$\zeta^\star(\underbrace{1,\dots,1}_{b},2) = \sum_{j=0}^{b-1}2^{b-j}\sum_{\substack{e_1+\cdots+e_{b-j}=j\\ e_j\geq 0}}\zeta(1+e_{b-j},\dots,1+e_2,3+e_1).$$

The comparison between the above formulas for $\zeta^\star$ and those for $\zeta^\star(\mathbb{F}_q[T])$ is not so clear, due to the relative simplicity of $\zeta^\star(\mathbb{F}_q[T])$.

Zagier [Zag12] considers the multiple zeta values $\zeta(2,\dots,2,3,2,\dots,2)$ and relates them to a convolution of $\zeta(2,\dots,2)$ and $\zeta(2n-1)$. We now explore analogous results for $\zeta^\star(\mathbb{F}_q[T];2,\dots,2,3,2,\dots,2)$.

Again following Zagier we define for $a,b\in\mathbb{Z}_{\geq 0}$,

$$H(a,b) := \zeta^\star_{a+b+1}(\mathbb{F}_q[T];\underbrace{2,\dots,2}_{a},3,\underbrace{2,\dots,2}_{b}),$$

and more generally

$$H(a_1,\dots,a_\ell) := \zeta^\star_{a_1+\cdots+a_\ell+\ell-1}(\mathbb{F}_q[T];\underbrace{2,\dots,2}_{a_1},3,\underbrace{2,\dots,2}_{a_2},3,\cdots,\underbrace{2,\dots,2}_{a_{\ell-1}},3\,\underbrace{2,\dots,2}_{a_\ell}),$$

where we take $a_1,\dots,a_\ell\geq 0$.

Theorem 33. *We have*

$$H(a,b) = \frac{1}{\zeta^\star(\mathbb{F}_q[T];b+2)}\prod_{n=2}^{a+b+3}\zeta^\star(\mathbb{F}_q[T];n) = \frac{H_1^{b+1}-(H_1-1)^{b+1}}{H_1^{b+1}}\frac{H_1^{(a+b+1)(a+b+2)/2}}{\prod_{n=1}^{a+b+1}(H_1^n-(H_1-1)^n)},$$

and more generally,

$$H(a_1,\dots,a_\ell) = \frac{\prod_{n=2}^{a_1+\cdots+a_\ell+2\ell-1}\zeta^\star(\mathbb{F}_q[T];n)}{\prod_{n=2}^{\ell}\zeta^\star(\mathbb{F}_q[T];a_n+\cdots+a_\ell+2(\ell+1-n))}.$$

Proof. These formulas are a direct consequence of equation (2). □

As before, the comparison to Zagier's formulas is not so clear, as our formulas are much simpler.

Theorem 34. *We also have*

$$\zeta^\star_{2d}(\mathbb{F}_q[T];1,3,1,3,\dots,1,3) = \prod_{n=1}^{d}\zeta^\star(\mathbb{F}_q[T];2n+1)^2.$$

Muneta [Mun08] proves

$$\zeta^\star(\underbrace{1,3,1,3,\ldots,1,3}_{2d}) = C_d\pi^{4d},$$

where C_d is certain rational number. Our formula has degree $2d$, corresponding to π^{4d}, and therefore, the degrees coincide.

We close this section by considering some special values of polylogarithms. Recall the notation from Remark 20.

Theorem 35. *We have*

$$\zeta^\star_{d+1}(\mathbb{F}_q[T];1,\ldots,1,\overline{1}) = \frac{1}{2^{d+1}} \tag{26}$$

and

$$\zeta^\star_{d+2}(\mathbb{F}_q[T];\overline{1},1,\ldots,1,2) = \frac{1}{1+q^{-1}}\zeta^\star\left(\mathbb{F}_q[T];2\right)^{d+1}.$$

Proof. By (14), we have

$$\zeta^\star_{d+1}(\mathbb{F}_q[T];1,\ldots,1,\overline{1}) = \mathrm{Li}^\star_{1,\ldots,1}(\mathbb{F}_q[T];\underbrace{1,\ldots,1}_{d},-1) = \mathrm{Li}^\star_1\left(\mathbb{F}_q[T];-1\right)^{d+1} = \frac{1}{2^{d+1}},$$

and

$$\begin{aligned}\zeta^\star_{d+2}(\mathbb{F}_q[T];\overline{1},1,\ldots,1,2) =& \mathrm{Li}^\star_{1,\ldots,1,2}(\mathbb{F}_q[T];-1,\underbrace{1,\ldots,1}_{d},1) = \mathrm{Li}^\star_2(\mathbb{F}_q[T];-1)\mathrm{Li}^\star_2\left(\mathbb{F}_q[T];1\right)^{d+1}\\ =& \mathrm{Li}^\star_2(\mathbb{F}_q[T];-1)\zeta^\star\left(\mathbb{F}_q[T];2\right)^{d+1}.\end{aligned}$$

□

Xu ([Xu18], equation (2.9)) proves

$$\zeta^\star(\underbrace{1,\ldots,1}_{d},\overline{1}) = -\mathrm{Li}_{d+1}\left(\frac{1}{2}\right).$$

It is interesting to compare Xu's formula and (26).

Xu also proves a result ([Xu18], Theorem 2.6) that gives a recursive formula for $\zeta^\star(\overline{1},1,\ldots,1,2)$ involving powers of $\log(2)$ and terms of the form $\mathrm{Li}_j\left(\frac{1}{2}\right)$.

6.2. **The $\mathbb{F}_q(T)$ case.** In this section we consider special values of $\zeta^\star_d(\mathbb{F}_q(T);s_1,\ldots,s_d)$. It is harder to find elegant formulas in this case because, unlike the case of $\zeta^\star_d(\mathbb{F}_q[T];s_1,\ldots,s_d)$, we do not count with the reduction (2).

We will concentrate in the cases in which $s_1=\cdots=s_d$ which are much easier to handle than other cases. Set

$$\mathcal{H}_d(m) := \zeta^\star_d(\mathbb{F}_q(T);m,\ldots,m).$$

Thus,

$$\mathcal{H}_1(m) = \zeta^\star(\mathbb{F}_q(T);m).$$

We also set $\mathcal{H}_0(m):=1$.

Theorem 36. *We have the following reduction formula*

$$\mathcal{H}_d(m) = \frac{1}{d}\sum_{j=0}^{d-1}\frac{\mathcal{H}_{d-1-j}(m)}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);(j+1)(m-1)+\ell+1)q^{j-\ell}.$$

This is Theorem 6 from the introduction.

Proof. We consider the following stuffle[1] product.

$$\begin{aligned}
&\zeta_d^\star(\mathbb{F}_q(T); m,\ldots,m)\zeta^\star(\mathbb{F}_q(T);k)\\
&=\sum_{\substack{(D_1,\ldots,D_d)\in(\mathcal{D}^+_{\mathbb{F}_q(T)})^d\\ 0\le\deg(D_1)\le\cdots\le\deg(D_d)}}\frac{1}{|D_1|^m\cdots|D_d|^m}\sum_{\substack{E\in\mathcal{D}^+_{\mathbb{F}_q(T)}\\ 0\le\deg(E)}}\frac{1}{|E|^k}\\
&=\sum_{j=1}^{d+1}\sum_{\substack{(E,D_1,\ldots,D_d)\in(\mathcal{D}^+_{\mathbb{F}_q(T)})^{d+1}\\ 0\le\deg(D_1)\le\cdots\le\deg(D_{j-1})\le\deg(E)\le\deg(D_j)\le\cdots\le\deg(D_d)}}\frac{1}{|D_1|^m\cdots|D_{j-1}|^m|E|^k|D_j|^m\cdots|D_d|^m}\\
&\quad-\sum_{j=1}^{d}\sum_{\substack{(E,D_1,\ldots,D_d)\in(\mathcal{D}^+_{\mathbb{F}_q(T)})^{d+1}\\ 0\le\deg(D_1)\le\cdots\le\deg(D_{j-1})\le\deg(E)=\deg(D_j)\le\cdots\le\deg(D_d)}}\frac{1}{|D_1|^m\cdots|D_{j-1}|^m|D_j|^{k+m}\cdots|D_d|^m}\\
&=\sum_{j=1}^{d+1}\sum_{\substack{(E,D_1,\ldots,D_d)\in(\mathcal{D}^+_{\mathbb{F}_q(T)})^{d+1}\\ 0\le\deg(D_1)\le\cdots\le\deg(D_{j-1})\le\deg(E)\le\deg(D_j)\le\cdots\le\deg(D_d)}}\frac{1}{|D_1|^m\cdots|D_{j-1}|^m|E|^k|D_j|^m\cdots|D_d|^m}\\
&\quad-\frac{1}{q-1}\sum_{j=1}^{d}\sum_{\substack{(E,D_1,\ldots,D_d)\in(\mathcal{D}^+_{\mathbb{F}_q(T)})^{d+1}\\ 0\le\deg(D_1)\le\cdots\le\deg(D_{j-1})\le\deg(D_j)\le\cdots\le\deg(D_d)}}\frac{q|D_j|-1}{|D_1|^m\cdots|D_{j-1}|^m|D_j|^{k+m}\cdots|D_d|^m},
\end{aligned}$$

where we have used that $b_{\deg(D_j)}=\frac{q|D_j|-1}{q-1}$ to count the number of possible E's with that degree.

Thus,

(27)
$$\zeta_d^\star(\mathbb{F}_q(T);m,\ldots,m)\zeta^\star(\mathbb{F}_q(T);k)=\sum_{j=1}^{d+1}\zeta_{d+1}^\star(\mathbb{F}_q(T);m,\ldots,k,\ldots,m)-\frac{q}{q-1}\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);m,\ldots,k+m-1,\ldots,m)$$

(28)
$$+\frac{1}{q-1}\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);m,\ldots,k+m,\ldots,m).$$

Let

$$N(d,m,k)=\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);m,\ldots,k,\ldots,m)$$

in the above notation. Then equation (27) implies

$$N(d,m,k)=\mathcal{H}_{d-1}(m)\zeta^\star(\mathbb{F}_q(T);k)+\frac{q}{q-1}N(d-1,m,k+m-1)-\frac{1}{q-1}N(d-1,m,k+m).$$

Notice also that

(29)
$$N(d,m,m)=\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);m,\ldots,m,\ldots,m)=d\mathcal{H}_d(m).$$

We claim that

(30)
$$N(d,m,k)=\sum_{j=0}^{d-1}\frac{\mathcal{H}_{d-1-j}(m)}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);j(m-1)+\ell+k)q^{j-\ell}.$$

We proceed by induction on d. When $d=1$, we get

$$N(1,m,k)=\zeta^\star(\mathbb{F}_q(T);k),$$

[1] See ([LQ18], Corollary 1.5) for the stuffle product of the zeta star values in the classical case.

and formula (30) is true in this case. Now assume that the formula is true for $d-1$. Then

$$\begin{aligned}
N(d,m,k) =&\mathcal{H}_{d-1}(m)\zeta^\star(\mathbb{F}_q(T);k) + \frac{q}{q-1}N(d-1,m,k+m-1) - \frac{1}{q-1}N(d-1,m,k+m)\\
=&\mathcal{H}_{d-1}(m)\zeta^\star(\mathbb{F}_q(T);k) + \frac{q}{q-1}\sum_{j=0}^{d-2}\frac{\mathcal{H}_{d-2-j}(m)}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);(j+1)(m-1)+\ell+k)q^{j-\ell}\\
&-\frac{1}{q-1}\sum_{j=0}^{d-2}\frac{\mathcal{H}_{d-2-j}(m)}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);(j+1)(m-1)+\ell+k+1)q^{j-\ell}\\
=&\mathcal{H}_{d-1}(m)\zeta^\star(\mathbb{F}_q(T);k) + \sum_{j=0}^{d-2}\frac{\mathcal{H}_{d-2-j}(m)}{(q-1)^{j+1}}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);(j+1)(m-1)+\ell+k)q^{j+1-\ell}\\
&-\sum_{j=0}^{d-2}\frac{\mathcal{H}_{d-2-j}(m)}{(q-1)^{j+1}}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);(j+1)(m-1)+\ell+k+1)q^{j-\ell}
\end{aligned}$$

Sending $j+1\to j$ above and $\ell+1\to\ell$ in the last line, we have,

$$\begin{aligned}
N(d,m,k) =&\mathcal{H}_{d-1}(m)\zeta^\star(\mathbb{F}_q(T);k) + \sum_{j=1}^{d-1}\frac{\mathcal{H}_{d-1-j}(m)}{(q-1)^j}\sum_{\ell=0}^{j-1}(-1)^\ell\binom{j-1}{\ell}\zeta^\star(\mathbb{F}_q(T);j(m-1)+\ell+k)q^{j-\ell}\\
&-\sum_{j=1}^{d-1}\frac{\mathcal{H}_{d-1-j}(m)}{(q-1)^j}\sum_{\ell=1}^{j}(-1)^{\ell-1}\binom{j-1}{\ell-1}\zeta^\star(\mathbb{F}_q(T);j(m-1)+\ell+k)q^{j-\ell},
\end{aligned}$$

and claim (30) follows from Pascal's formula.

The statement of Theorem 36 then follows from setting $k=m$ in (30) and applying (29). □

We will now proceed to find a formula for $\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{m})$. We set

$$\mathcal{H}_d(\overline{m}) := \zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{m}) = \mathrm{Li}_{m,\dots,m}(\underbrace{-1,\dots,-1}_{d}).$$

Thus,

$$\mathcal{H}_1(\overline{m}) = \zeta^\star(\mathbb{F}_q(T);\overline{m}).$$

We also set $\mathcal{H}_0(\overline{m}) := 1$.

Theorem 37. *We have the following reduction formula*

$$\mathcal{H}_d(\overline{m}) = \frac{1}{d}\sum_{j=0}^{d-1}\frac{\mathcal{H}_{d-1-j}(\overline{m})}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);\overline{(j+1)(m-1)+\ell+1})q^{j-\ell}.$$

Proof. As in the case of the proof of Theorem 36, we have

$$\begin{aligned}
\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{m})\zeta^\star(\mathbb{F}_q(T);k) =&\sum_{j=1}^{d+1}\zeta_{d+1}^\star(\mathbb{F}_q(T);\overline{m},\dots,k,\dots,\overline{m}) - \frac{q}{q-1}\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{k+m-1},\dots,\overline{m})\\
&+\frac{1}{q-1}\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{k+m},\dots,\overline{m}),
\end{aligned}$$

as well as

$$\begin{aligned}
\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{m})\zeta^\star(\mathbb{F}_q(T);\overline{k}) =&\sum_{j=1}^{d+1}\zeta_{d+1}^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{k},\dots,\overline{m}) - \frac{q}{q-1}\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,k+m-1,\dots,\overline{m})\\
&+\frac{1}{q-1}\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,k+m,\dots,\overline{m}).
\end{aligned}$$

Let

$$N(d,\overline{m},\overline{k})=\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{k},\dots,\overline{m})$$

and

$$N(d,\overline{m},k)=\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,k,\dots,\overline{m}).$$

Then we have that

$$N(d,\overline{m},k)=\mathcal{H}_{d-1}(\overline{m})\zeta^\star(\mathbb{F}_q(T);k)+\frac{q}{q-1}N(d-1,\overline{m},\overline{k+m-1})-\frac{1}{q-1}N(d-1,\overline{m},\overline{k+m}),$$

$$N(d,\overline{m},\overline{k})=\mathcal{H}_{d-1}(\overline{m})\zeta^\star(\mathbb{F}_q(T);\overline{k})+\frac{q}{q-1}N(d-1,\overline{m},k+m-1)-\frac{1}{q-1}N(d-1,\overline{m},k+m).$$

Notice also that

$$N(d,\overline{m},\overline{m})=\sum_{j=1}^{d}\zeta_d^\star(\mathbb{F}_q(T);\overline{m},\dots,\overline{m},\dots,\overline{m})=d\mathcal{H}_d(\overline{m}). \tag{31}$$

We claim that

$$N(d,\overline{m},k)=\sum_{j=0}^{d-1}\frac{\mathcal{H}_{d-1-j}(\overline{m})}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);j(m-1)+\ell+k)q^{j-\ell}. \tag{32}$$

$$N(d,\overline{m},\overline{k})=\sum_{j=0}^{d-1}\frac{\mathcal{H}_{d-1-j}(\overline{m})}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);\overline{j(m-1)+\ell+k})q^{j-\ell}. \tag{33}$$

We prove both equations (32) and (33) together by induction on d. When $d=1$, we get

$$N(1,\overline{m},k)=\zeta^\star(\mathbb{F}_q(T);k),\qquad N(1,\overline{m},\overline{k})=\zeta^\star(\mathbb{F}_q(T);\overline{k}),$$

and formulas (32) and (33) are true in this case. Now assume the formulas are true for $d-1$. Then for (32) we have

$$\begin{aligned}N(d,\overline{m},k)=&\mathcal{H}_{d-1}(\overline{m})\zeta^\star(\mathbb{F}_q(T);\overline{k})+\frac{q}{q-1}N(d-1,\overline{m},\overline{k+m-1})-\frac{1}{q-1}N(d-1,\overline{m},\overline{k+m})\\=&\mathcal{H}_{d-1}(\overline{m})\zeta^\star(\mathbb{F}_q(T);\overline{k})+\frac{q}{q-1}\sum_{j=0}^{d-2}\frac{\mathcal{H}_{d-2-j}(\overline{m})}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);\overline{(j+1)(m-1)+\ell+k})q^{j-\ell}\\&-\frac{1}{q-1}\sum_{j=0}^{d-2}\frac{\mathcal{H}_{d-2-j}(\overline{m})}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);\overline{(j+1)(m-1)+\ell+k+1})q^{j-\ell}.\end{aligned}$$

We obtain (32) by proceeding similarly to the proof of (30).

For (33) we have

$$\begin{aligned}N(d,\overline{m},\overline{k})=&\mathcal{H}_{d-1}(\overline{m})\zeta^\star(\mathbb{F}_q(T);k)+\frac{q}{q-1}N(d-1,\overline{m},k+m-1)-\frac{1}{q-1}N(d-1,\overline{m},k+m)\\=&\mathcal{H}_{d-1}(\overline{m})\zeta^\star(\mathbb{F}_q(T);k)+\frac{q}{q-1}\sum_{j=0}^{d-2}\frac{\mathcal{H}_{d-2-j}(\overline{m})}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);(j+1)(m-1)+\ell+k)q^{j-\ell}\\&-\frac{1}{q-1}\sum_{j=0}^{d-2}\frac{\mathcal{H}_{d-2-j}(\overline{m})}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);(j+1)(m-1)+\ell+k+1)q^{j-\ell}.\end{aligned}$$

As before, we obtain (33) by proceeding similarly to the proof of (30).

After setting $k=m$ in (33) and applying (31), we obtain the desired result. □

We now show how the recurrence relations of Theorems 36 and 37 lead to closed formulas for $\mathcal{H}_d(m)$ and $\mathcal{H}_d(\overline{m})$.

Theorem 38. *Let $\mathcal{H}_d$ be defined recursively by $\mathcal{H}_0 = 1$ and*

$$\mathcal{H}_d = \frac{1}{d}\sum_{j=0}^{d-1} \mathcal{H}_{d-1-j}a_{j+1}.$$

Then we have

$$\mathcal{H}_d = \sum_{\ell_1+2\ell_2+\cdots+d\ell_d=d} \frac{a_1^{\ell_1}\cdots a_d^{\ell_d}}{\ell_1!\cdots\ell_d!1^{\ell_1}\cdots d^{\ell_d}},$$

where the sum is taken over all the possible non-negative integers ℓ_i such that $\ell_1+2\ell_2+\cdots+d\ell_d = d$.

Proof. We proceed by induction. If $d=1$, we have

$$\mathcal{H}_1 = \mathcal{H}_0 a_1 = a_1,$$

and the statement is true. Assume it is true for up to $d-1$. Then

$$\begin{aligned}
\mathcal{H}_d =& \frac{1}{d}\sum_{j=0}^{d-1}\mathcal{H}_{d-1-j}a_{j+1}\\
=&\frac{a_d}{d} + \frac{1}{d}\sum_{j=1}^{d-1}\left(\sum_{\ell_1+2\ell_2+\cdots+d\ell_d=d-j}\frac{a_1^{\ell_1}\cdots a_d^{\ell_d}}{\ell_1!\cdots\ell_d!1^{\ell_1}\cdots d^{\ell_d}}\right)a_j\\
=&\frac{a_d}{d} + \sum_{j=1}^{d-1}\frac{j}{d}\sum_{\ell_1+2\ell_2+\cdots+j(\ell_j+1)+\cdots+d\ell_d=d}\frac{(\ell_j+1)a_1^{\ell_1}\cdots a_j^{\ell_j+1}\cdots a_d^{\ell_d}}{\ell_1!\cdots(\ell_j+1)!\cdots\ell_d!1^{\ell_1}\cdots j^{\ell_j+1}\cdots d^{\ell_d}}.
\end{aligned}$$

Now we do the change $\ell_j+1\to\ell_j$ and the above becomes

$$\begin{aligned}
&\frac{a_d}{d} + \sum_{j=1}^{d-1}\frac{j}{d}\sum_{\substack{\ell_1+2\ell_2+\cdots+d\ell_d=d\\ \ell_j\geq 1}}\frac{\ell_j a_1^{\ell_1}\cdots a_d^{\ell_d}}{\ell_1!\cdots\ell_d!1^{\ell_1}\cdots d^{\ell_d}}\\
=&\frac{a_d}{d} + \sum_{j=1}^{d-1}\frac{j}{d}\sum_{\ell_1+2\ell_2+\cdots+d\ell_d=d}\frac{\ell_j a_1^{\ell_1}\cdots a_d^{\ell_d}}{\ell_1!\cdots\ell_d!1^{\ell_1}\cdots d^{\ell_d}}\\
=&\frac{a_d}{d} + \sum_{\ell_1+2\ell_2+\cdots+d\ell_d=d}\frac{a_1^{\ell_1}\cdots a_d^{\ell_d}}{\ell_1!\cdots\ell_d!1^{\ell_1}\cdots d^{\ell_d}}\sum_{j=1}^{d-1}\frac{j\ell_j}{d}.
\end{aligned}$$

Notice that $\ell_d = 1$ only when all the other ℓ_j are equal to 0 and in that case the quotient inside the first sum is just equal to a_d/d. Thus we have

$$\begin{aligned}
\mathcal{H}_d =& \sum_{\ell_1+2\ell_2+\cdots+d\ell_d=d}\frac{a_1^{\ell_1}\cdots a_d^{\ell_d}}{\ell_1!\cdots\ell_d!1^{\ell_1}\cdots d^{\ell_d}}\sum_{j=1}^{d}\frac{j\ell_j}{d}\\
=& \sum_{\ell_1+2\ell_2+\cdots+d\ell_d=d}\frac{a_1^{\ell_1}\cdots a_d^{\ell_d}}{\ell_1!\cdots\ell_d!1^{\ell_1}\cdots d^{\ell_d}}
\end{aligned}$$

and the result follows. $\square$

Theorem 38 can be combined with Theorems 36 and 37 by taking

$$a_{j+1} = \frac{1}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);(j+1)(m-1)+\ell+1)q^{j-\ell}$$

and

$$a_{j+1} = \frac{1}{(q-1)^j}\sum_{\ell=0}^{j}(-1)^\ell\binom{j}{\ell}\zeta^\star(\mathbb{F}_q(T);\overline{(j+1)(m-1)+\ell+1})q^{j-\ell}$$

respectively.

Remark 39. *By using* (5) *and* (16) *one can see that the above expressions for a_{j+1} are equivalent to*

$$a_{j+1} = \frac{1}{(q-1)^{j+1}} \sum_{\ell=0}^{j+1} (-1)^{\ell} \binom{j+1}{\ell} \zeta^{\star}(\mathbb{F}_q[T]; (j+1)(m-1)+\ell+1) q^{j+1-\ell}$$

and

$$a_{j+1} = \frac{1}{(q-1)^{j+1}} \sum_{\ell=0}^{j+1} (-1)^{\ell} \binom{j+1}{\ell} \zeta^{\star}(\mathbb{F}_q[T]; \overline{(j+1)(m-1)+\ell+1}) q^{j+1-\ell}$$

respectively

6.3. **The $g=1$ case.** We close our discussion by briefly considering the case of K with $g=1$. By Theorems 17 and 27, we have

$$\zeta_d^{\star}(K; m, \ldots, m) = 1 + \sum_{\ell=1}^{d} h_K^{\ell} q^{-\ell m} \zeta_{\ell}^{\star}(\mathbb{F}_q(T); m, \ldots, m),$$

and

$$\zeta_d^{\star}(K; \overline{m}, \ldots, \overline{m}) = 1 + \sum_{\ell=1}^{d} h_K^{\ell} (-1)^{\ell} q^{-\ell m} \zeta_{\ell}^{\star}(\mathbb{F}_q(T); \overline{m}, \ldots, \overline{m}).$$

These formulas can be combined with Theorems 36, 37, and 38 to give closed formulas for $\zeta_d^{\star}(K; m, \ldots, m)$ and $\zeta_d^{\star}(K; \overline{m}, \ldots, \overline{m})$.

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