THE MAHLER MEASURE OF A GENUS 3 FAMILY

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Abstract. We use the elliptic regulator to prove an identity between the Mahler measures of a genus 3 polynomial family and of a genus 1 polynomial family that was initially conjectured by Liu and Qin.

1. Introduction

The (logarithmic) Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)$ is defined by

$$\mathrm{m}(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where the integration is taken over the unit torus $\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\}$ with respect to the Haar measure.

This object was first considered for one-variable polynomials by Lehmer [Leh33], and in this case, the formula depends on the roots of the polynomial. The interest in the construction in several variables lies in connections to special values of functions such as the Riemann zeta-function (discovered by Smyth [Smy81] and also investigated by Boyd [Boy81]) and L-functions associated to arithmetic-geometric objects such as elliptic curves. Such connections were predicted by Deninger [Den97], who related the Mahler measure to a regulator value. He discovered that for certain polynomials, this regulator was expected to yield a special value of an L-function by means of Beĭlinson's conjectures. This was further investigated in detail by Boyd [Boy98], who conducted a systematic study of certain families of two variable polynomials and found many numerical examples. For example, Boyd conjectured that for k integral,

$$\mathrm{m}(R_k(x,y)) \stackrel{?}{=} r_k L'(E_k, 0) \quad \text{for} \quad R_k(x,y) = (x+1)y^2 + (x^2+(2-k)x+1)y + x^2 + x,$$

where E_k is the elliptic curve corresponding to the zero loci of the polynomial $R_k(x,y) = 0$ (which has genus 1 for $k \neq -1, 0, 8$) and r_k is a rational number. This conjecture was supported by numerical evidence, as the question mark indicates an identity verified to at least 20 decimal places, for $|k| \leq 40$.

The first cases of this conjecture were settled by Rodriguez-Villegas [RV99, Boy98] for $r_{-4} = 2$ and $r_2 = 1/2$. Other cases were settled by Mellit [Mel] (in 2009) for $r_{-8} = 10, r_1 = 1, r_7 = 6$, and Rogers and Zudilin [RZ12] for $r_{-2} = 3$ and $r_4 = 2$. The time difference between the first results and the rest is explained by

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the fact that Rodriguez-Villegas' cases correspond to complex multiplication and the L-function is better understood.

While most of Boyd's families correspond to genus 1 curves, he also studied a few cases involving families of genus 2 curves. For example, he considered

$$S_k(x,y) = y^2 + (x^4 + kx^3 + 2kx^2 + kx + 1)y + x^4.$$

The zero loci is a genus 2 curve for $k \neq -1, 0, 4, 8$. The Jacobian associated to this curve splits as a product of two elliptic curves, one of which is the same E_k given by $R_k(x,y) = 0$. Boyd found numerical relations of the type

$$\mathrm{m}(S_k) \stackrel{?}{=} s_k L'(E_k, 0). \tag{1}$$

Equating the numerical formulas for $\mathrm{m}(R_k)$ and $\mathrm{m}(S_k)$ led Boyd to conjecture for k real

$$\mathrm{m}(S_k) \stackrel{?}{=} \begin{cases} 2\mathrm{m}(R_k) & 0 \leq k \leq 4, \\ \\ \mathrm{m}(R_k) & k \leq -1. \end{cases} \tag{2}$$

The Mahler measure $\mathrm{m}(S_k)$ was first studied by Bosman in his thesis [Bos04]. He considered the relationship with the regulator and proved exact formulas of the type (1) in the cases $k = 8$, corresponding to genus 0, and $k = -1$ corresponding to genus 1. He also proved the case $k = 2$, where the relevant Jacobian factor has complex multiplication. The case $k = 4$, which also corresponds to a genus 1 curve, can be settled by techniques of modular unit parametrizations [Zud14, BZ16]. The case $k = 0$ results in a product of two genus 0 components.

Identity (2) was settled by Bertin and Zudilin [BZ16] by differentiating both Mahler measures as functions on the parameter k and by using hypergeometric identities. Another such identity, also conjectured by Boyd, involving the Mahler measures of two genus 2 families, was also settled by Bertin and Zudilin [BZ17] with similar methods. Alternative proofs of both of these results were given by the authors of the current manuscript in [LW19] by establishing identities between regulators, essentially completing the ideas initiated by Bosman [Bos04].

Most of the families studied by Boyd may be described as reciprocal polynomials of the form

$$P_k(x,y) = A(x)y^2 + B_k(x)y + C(x). \tag{3}$$

Here we say that the polynomial $P(x_1, \ldots, x_n)$ is reciprocal if $P(x_1, \ldots, x_n)$ equals $P(1/x_1, \ldots, 1/x_n)$ multiplied by a monomial. In addition, the polynomials considered by Boyd are tempered, i.e., the zeros of the face polynomials are roots of unity.

Recently Liu and Qin [LQ] extended Boyd's ideas (particularly allowing more general expressions for $B_k(x)$) to obtain many more conjectural families generically corresponding to genus 2 and genus 3 curves. The polynomials (3) are again reciprocal, tempered, and the quotient of the zero loci by the automorphism $\sigma : (x,y) \to (1/x, 1/y)$ corresponds to a genus 1 curve. Their work is full of intriguing conjectures that also include shifted Mahler measures, involving not only families of the form $P_k(x,y)$ but also $P_k(x-1,y)$ and $P_k(x+1,y)$ for some favorably cases.

Our goal is to prove the following result conjectured by Liu and Qin.

Theorem 1. *Let*

$$P_k(x,y) = y^2 + (x^6 + kx^5 - x^4 + (2-2k)x^3 - x^2 + kx + 1)y + x^6$$

and

$$Q_k(x,y) = xy^2 + (kx-1)y - x^2 + x.$$

Then, for $k \geq 2$,

$$\mathrm{m}(Q_k) = \mathrm{m}(P_k).$$

The curve defined by $P_k(x,y) = 0$ has genus 3, except for $k = \pm 2$, when it has genus 2, while the curve defined by $Q_k(x,y) = 0$ has genus 1 for all k.

More precisely, Liu and Qin conjectured that the common value of $\mathrm{m}(Q_k)$ and $\mathrm{m}(P_k)$ is given by $s_k L'(E_k, 0)$, where

$$E_k : Y^2 = X^3 + (k^2-4)X^2 - 8kX + 16.$$

The first few values for $1/s_k$ are given in Table 1. It is common to consider the reciprocal of s_k because it has the tendency to be an integer.

k	2	3	4	5	6	7	8	9	10
$1/s_k$	$-1/2$	-1	-2	-4	6	14	-18	36	52
N_k	37	79	197	469	997	1907	3349	5497	8549

Table 1. Numerical values of $1/s_k$ for the conjectural formulas $\mathrm{m}(Q_k) = \mathrm{m}(P_k) = s_k L'(E_k, 0)$ found by Liu and Qin [LQ]. N_k indicates the conductor of E_k.

To our knowledge, Theorem 1 is the first result shedding light on the Mahler measure of a genus 3 curve. Our method of proof is similar to the one employed in [LW19], establishing identities between the regulators, but the regulator of the genus 3 curve is quite difficult to evaluate and we employ a few strategies to simplify it before comparing it to the regulator of the genus 1 curve. The major new idea for evaluating the regulator of the genus 3 curve is to use equation (9) as opposed to (10) to simplify the evaluation of the diamond operator on $(x_1) \diamond (y_1)$. This simple idea has potential for other cases. Another interesting feature of this example is that the regulators are supported in powers of a point of infinite order in the elliptic curve. The majority of the examples that have been proven so far have the regulators supported in torsion points. An exception is the very first identity proven with this technique by Rodriguez-Villegas [RV02], which involve the elliptic curve $37a1$. Coincidentally, the case $k = 2$ in Theorem 1 also corresponds to this curve.

This article is organized as follows. In Section 2 we review the theory of K_2 and the regulator, and its relationship to Mahler measure. We evaluate and compare the regulators of both curves in Section 3. Finally, in Section 4 we prove that the homology classes of the integration cycles are the same and we conclude the proof of the theorem.

2. K_2 and the regulator

We recall the definition of K_2 of an elliptic curve E and the regulator given by Bloch and Beĭlinson and explain how it can be computed in terms of the elliptic dilogarithm. We then explain the role of the Mahler measure in this setting.

Let F be a field. By Matsumoto's theorem, the second K-group of F can be described as

$$K_2(F) \cong F^* \otimes_{\mathbb{Z}} F^*/\langle a \otimes (1-a) : a \in F, a \neq 0, 1\rangle.$$

The class $\{a, b\}$ of $a \otimes b$ in $K_2(F)$ is called the Steinberg symbol.

Let $C/\mathbb{Q}$ be a smooth projective geometrically irreducible curve. Then $K_2(C) \otimes \mathbb{Q}$ can be thought of as a subset of $K_2(\mathbb{Q}(C)) \otimes \mathbb{Q}$ determined by certain extra conditions, including the triviality of tame symbols. If C is given by an equation $P(x, y) = 0$, a necessary condition is that P be tempered, namely, that the face polynomials have Mahler measure zero, or in other words, that they be products of cyclotomic polynomials and monomials. A detailed discussion of this can be found in [RV99] and [LQ].

Let $x, y \in \mathbb{Q}(C)$. We will consider the differential form

$$\eta(x, y) := \log|x| d \arg y - \log|y| d \arg x, \tag{4}$$

where $d \arg x$ is defined by $\operatorname{Im}(dx/x)$. η is defined outside the zeros and poles of x and y.

The Bloch–Wigner dilogarithm is given by

$$D(x) = \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1-x)\log|x|, \tag{5}$$

where

$$\operatorname{Li}_2(x) = -\int_0^x \frac{\log(1-z)}{z} dz$$

is an analytic extension of the classical dilogarithm.

The form $\eta(x, y)$ is closed in its domain of definition, multiplicative, antisymmetric, and satisfies

$$\eta(x, 1-x) = dD(x).$$

The fact that η is exact for $x \otimes (1-x)$ allows us to consider its cohomology class in $K_2(\mathbb{Q}(C))$. The regulator map given by Bloch [Blo00] and Beĭlinson [Bl80] is

$$\begin{array}{rcl} r_C : K_2(C) \otimes \mathbb{Q} & \to & H^1(C, \mathbb{R}) \\ \{x, y\} & \to & \left\{ [\gamma] \to \int_\gamma \eta(x, y) \right\}. \end{array}$$

In the above definition, $[\gamma] \in H_1(C, \mathbb{Z})$ and we interpret $H^1(C, \mathbb{R})$ as the dual of $H_1(C, \mathbb{Z})$. The regulator map η is trivial for the classes that remain invariant by complex conjugation, denoted by $H_1(C, \mathbb{Z})^+$. It therefore suffices to consider the regulator as a function on $H_1(C, \mathbb{Z})^-$, a g-dimensional space where g is a genus of C.

Let σ be an automorphism of order 2 of C and let $f : C \to C/\langle\sigma\rangle$. Let $M \in K_2(C)$. Then Bosman [Bos04] observed that

$$\int_\gamma \eta(f^* f_*(M)) = \int_{f(\gamma)} \eta(f_*(M)),$$

where f_* is the transfer homomorphism and f^* is the restriction homomorphism. See [LQ] for more details. In particular, if $C/\langle\sigma\rangle$ has genus 1, this construction permits to relate the regulator on C to the regulator of an elliptic curve.

Now consider the case of $E/\mathbb{Q}$ an elliptic curve. Then we have

$$\begin{array}{ccccc} E(\mathbb{C}) & \stackrel{\sim}{\to} & \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) & \stackrel{\sim}{\to} & \mathbb{C}^\times/q^{\mathbb{Z}} \\ P=(\wp(u),\wp'(u)) & \to & u \bmod \Lambda & \to & z=e^{2\pi i u}, \end{array} \tag{6}$$

where $\wp$ is the Weierstrass function, Λ is the lattice $\mathbb{Z}+\tau\mathbb{Z}$, $\tau\in\mathbb{H}$, and $q=e^{2\pi i\tau}$.

Bloch [Blo00] defines the elliptic dilogarithm as a function on $E(\mathbb{C})$ given for $P\in E(\mathbb{C})$ corresponding to $z\in\mathbb{C}^\times/q^{\mathbb{Z}}$ by

$$D^E(P):=\sum_{n\in\mathbb{Z}} D(q^n z), \tag{7}$$

where D is the Bloch–Wigner dilogarithm defined by (5).

Let $\mathbb{Z}[E(\mathbb{C})]$ be the group of divisors on E and let

$$\mathbb{Z}[E(\mathbb{C})]^-\cong\mathbb{Z}[E(\mathbb{C})]/\langle(P)+(-P):P\in E(\mathbb{C})\rangle.$$

Let $x,y\in\mathbb{C}(E)^\times$. We define a diamond operation by

$$\begin{array}{rcl} \diamond:\Lambda^2\mathbb{C}(E)^\times & \to & \mathbb{Z}[E(\mathbb{C})]^- \\ (x)\diamond(y) & = & \displaystyle\sum_{i,j} m_i n_j (S_i-T_j), \end{array}$$

where

$$(x)=\sum_i m_i(S_i) \text{ and } (y)=\sum_j n_j(T_j).$$

Theorem 2. *(Bloch* [Blo00]*) The elliptic dilogarithm D^E extends by linearity to a map from $\mathbb{Z}[E(\mathbb{Q})]^-$ to $\mathbb{C}$. Let $x,y\in\mathbb{Q}(E)$ and $\{x,y\}\in K_2(E)$. Then*

$$r_E(\{x,y\})[\gamma]=D^E((x)\diamond(y)),$$

where $[\gamma]$ is a generator of $H_1(E,\mathbb{Z})^-$.

Let $P(x,y)\in\mathbb{C}[x,y]$ be a polynomial of degree 2 on y. We may then write

$$P(x,y)=P^*(x)(y-y_+(x))(y-y_-(x)),$$

where $y_+(x),y_-(x)$ are algebraic functions and $P^*(x)$ is a one-variable polynomial (the sign notation reflects a choice of sign while solving the quadratic equation).

Jensen's formula implies for $\alpha\in\mathbb{C}$ that

$$\frac{1}{2\pi i}\int_{\mathbb{T}^1}\log|z-\alpha|\frac{dz}{z}=\begin{cases}\log|\alpha| & |\alpha|\geq 1,\\ 0 & |\alpha|\leq 1.\end{cases}$$

By applying Jensen's formula in the Mahler measure formula of $P(x,y)$ with respect to the variable y, we obtain

$$\begin{aligned}\mathrm{m}(P)-\mathrm{m}(P^*)=&\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}\log|P(x,y)|\frac{dx}{x}\frac{dy}{y}-\mathrm{m}(P^*)\\ =&\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2}(\log|y-y_+(x)|+\log|y-y_-(x)|)\frac{dx}{x}\frac{dy}{y}\\ =&\frac{1}{2\pi i}\int_{|x|=1,|y_+(x)|\geq1}\log|y_+(x)|\frac{dx}{x}+\frac{1}{2\pi i}\int_{|x|=1,|y_-(x)|\geq1}\log|y_-(x)|\frac{dx}{x}.\end{aligned}$$

Recalling formula (4) for $\eta(x,y)$, we have,

$$\mathrm{m}(P)-\mathrm{m}(P^*)=-\frac{1}{2\pi}\int_{|x|=1,|y_+(x)|\geq 1}\eta(x,y_+)-\frac{1}{2\pi}\int_{|x|=1,|y_-(x)|\geq 1}\eta(x,y_-).$$

If P is reciprocal, then one of the roots has always absolute value greater than or equal to 1 as $|x|=1$ and the other root has always absolute value smaller than or equal to 1 as $|x|=1$. Then we can write the right-hand side as a single term, an integral over the closed path $\{|x|=1,|y_\pm(x)|\geq 1\}$, which is seen as a cycle in $H_1(C,\mathbb{Z})$. This leads to a formula of the form

$$\mathrm{m}(P)=\frac{1}{2\pi}r(\{x,y\})[\gamma].$$

Deninger [Den97] related the Mahler measure to the regulator. This was also explored by Rodriguez-Villegas [RV99, RV02] who was the first to prove an identity between the Mahler measures of two genus 1 curves (originally conjectured by Boyd [Boy98]) This was the first result of the type of result that we consider in this article.

3. The regulator relationship

3.1. **The genus 3 curve.** We start by considering the regulator in

$$C_k : P_k(x_1,y_1)=0,$$

where

$$P_k(x_1,y_1)=y_1^2+(x_1^6+kx_1^5-x_1^4+(2-2k)x_1^3-x_1^2+kx_1+1)y_1+x_1^6.$$

A standard procedure to obtain a hyperelliptic function from a polynomial of type (3) is to complete squares and write $(2A(x)y+B_k(x))^2=B_k(x)^2-4A(x)C(x)$ and set $X=\frac{x+1}{x-1}$, $Y=\frac{2A(x)y+B_k(x)}{\delta(x,y)}$ for a conveniently chosen polynomial $\delta(x,y)$. In our case, the following birational transformation

$$\begin{aligned}X(x_1,y_1)=&\frac{x_1+1}{x_1-1},\\ Y(x_1,y_1)=&\frac{8(2y_1+(x_1^6+kx_1^5-x_1^4+(2-2k)x_1^3-x_1^2+kx_1+1))}{(x_1^2-1)(x_1-1)^4},\end{aligned}$$

leads to

$$Y^2=(k+2)X^8+4(k^2+3k+3)X^6-2(4k^2-3k-16)X^4+4(k^2-5k+5)X^2+k-2.$$

If we further set

$$\begin{aligned}Z=&\frac{4(X^2-1)}{(k+2)X^2+(2-k)},\\ W=&\frac{8Y}{((k+2)X^2+(2-k))^2}\end{aligned}$$

We obtain the family of elliptic curves

$$E_k : W^2=Z^3+(k^2-4)Z^2-8kZ+16. \tag{8}$$

In sum, we have,

$$f: C_k\to C_k/\langle\sigma\rangle\cong E_k,$$

given by

$$\begin{aligned} Z(x_1,y_1) =& \frac{4x_1}{x_1^2+kx_1+1}, \\ W(x_1,y_1) =& \frac{4(2y_1+(x_1^6+kx_1^5-x_1^4+(2-2k)x_1^3-x_1^2+kx_1+1))}{(x_1^2-1)(x_1^2+kx_1+1)^2} \\ =& \frac{4(y_1^2-x_1^6)}{y_1(x_1^2-1)(x_1^2+kx_1+1)^2}, \\ y_1(X,W) =& \frac{((2+k)X^2+2-k)^2XW-2(X^6+(8k+13)X^4-(8k-19)X^2-1)}{2(X-1)^6}. \end{aligned}$$

We can then write

$$\int_{\gamma_{p,k}} \eta_{C_k}(x_1,y_1) = \int_{f(\gamma_{p,k})} \eta_{E_k}(f_*(\{x_1,y_1\}),$$

where

$$\gamma_{p,k} = \{(x_1,y_1) \,:\, |x_1|=1, |y_1|\geq 1\},$$

and our goal in this section is to evaluate $\eta_{E_k}(f_*(\{x_1,y_1\})$. For simplicity of notation, we will refer to $(x_1)\diamond(y_1)$ but we will think of x_1, y_1 as functions on Z,W.

However, x_1, y_1 are rational functions on X,W and not on Z,W. In order to find their divisors in E_k, we follow a version of an idea of Bosman. We search for rational functions such that

$$a(X^2,W)x_1(X,W)+b(X^2,W)\frac{y_1(X,W)}{x_1(X,W)^3}=1. \tag{9}$$

While Bosman [Bos04] and [LW19] work with the equation

$$a(X^2,W)x_1(X,W)+b(X^2,W)y_1(X,W)=1, \tag{10}$$

we have modified the y_1 component to $\frac{y_1}{x_1^3}$ in order to eliminate as much as possible the number of monomials with odd degree in X. Indeed, y_1 has the factor $(X-1)^6$ in the denominator, but this becomes $(X^2-1)^3$ in the denominator of $\frac{y_1}{x_1^3}$.

Lemma 3. *Let $a,b\in\mathbb{Q}(E)$ satisfying* (9). *We have*

$$r_{C_k}(\{x_1(X,W),y_1(X,W)\})([\gamma_{p,k}]) = -r_{E_k}(\{a(Z,W),b(Z,W)\})([f(\gamma_{p,k})]).$$

Proof. From Bosman [Bos04] and Lemma 7 in [LW19] we have that

$$(a(Z,W))\diamond(b(Z,W)) \sim -(x_1(X,W))\diamond\left(\frac{y_1(X,W)}{x_1(X,W)^3}\right).$$

The result follows because

$$(x_1)\diamond\left(\frac{y_1}{x_1^3}\right) \sim (x_1)\diamond(y_1).$$

□

In our computations it will be convenient to introduce another variable to name the even powers of X:

$$Z_1 := X^2 = \frac{(k-2)Z-4}{(k+2)Z-4}.$$

For simplicity of notation, we also let

$$\begin{aligned}A(Z_1) =&((2+k)Z_1+2-k)^2,\\ B(Z_1) =&Z_1^3+(8k+13)Z_1^2-(8k-19)Z_1-1.\end{aligned}$$

Then we write

$$y_1(X,W)=\frac{AXW-2B}{2(X-1)^6}.$$

We have to solve

$$a\frac{(X+1)^2}{Z_1-1}+b\frac{AXW-2B}{2(Z_1-1)^3}=1,$$

which leads to

$$\begin{cases} a(Z_1+1)(Z_1-1)^2-bB &= (Z_1-1)^3,\\ 4a(Z_1-1)^2+bAW &= 0,\end{cases}$$

and then

$$\begin{aligned}a(Z,W)=&\frac{(Z_1-1)AW}{(Z_1+1)AW+4B}=\frac{4ZW}{2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32},\\ b(Z,W)=&-\frac{4(Z_1-1)^3}{(Z_1+1)AW+4B}=-\frac{Z^3}{4(2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32)}.\end{aligned}$$

After ignoring constants, and grouping together terms, the diamond operation gives

$$\begin{aligned}(a)\diamond(b)=&2(Z)\diamond(2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32),\\ &-(W)\diamond(2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32)+3(W)\diamond(Z).\end{aligned}\tag{11}$$

If we proceeded as usual (see, for example, [LW19]), we would compute the divisors $(Z),(W),(2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32)$. With the exception of (Z), these divisors are supported in non-rational points and it is difficult to find relationships among them. Instead of directly computing the divisors, we consider some further manipulations. Notice from (8) that

$$2(4-kZ)W+Z^3+2(k^2-4)Z^2-16kZ+32=2(4-kZ+W)W-Z^3.$$

We consider the following trivial symbol (trivial because it is of the form $(g)\diamond(1-g)$):

$$\begin{aligned}0\sim&\left(\frac{2(4-kZ+W)W}{Z^3}\right)\diamond\left(\frac{Z^3-2(4-kZ+W)W}{Z^3}\right)\\ \sim&(4-kZ+W)\diamond(2(4-kZ+W)W-Z^3)-3(4-kZ+W)\diamond(Z)\\ &+(W)\diamond(2(4-kZ+W)W-Z^3)-3(W)\diamond(Z)\\ &-3(Z)\diamond(2(4-kZ+W)W-Z^3).\end{aligned}$$

Combining with (11), we have,

$$\begin{aligned}(a)\diamond(b)=&2(Z)\diamond(2(4-kZ+W)W-Z^3)-(W)\diamond(2(4-kZ+W)W-Z^3)+3(W)\diamond(Z)\\ \sim&2(Z)\diamond(2(4-kZ+W)W-Z^3)+(4-kZ+W)\diamond(2(4-kZ+W)W-Z^3)\\ &-3(4-kZ+W)\diamond(Z)-3(Z)\diamond(2(4-kZ+W)W-Z^3)\\ \sim&\left(\frac{4-kZ+W}{Z}\right)\diamond(2(4-kZ+W)W-Z^3)-3(4-kZ+W)\diamond(Z).\end{aligned}\tag{12}$$

The advantage of working with (12) as opposed to (11) is that we do not have to consider the divisor (W) anymore. Instead, we will compute the divisor $(4 - kZ + W)$, supported on rational points.

The family E_k given by (8) has a point $P = (0, 4)$ of infinite order that satisfies $2P = (4, 4(k-1))$, $3P = (4(1-k), 4(k^2 - 3k + 1))$.

$$(Z) = (P) + (-P) - 2O, \tag{13}$$

$$(4 - kZ + W) = 2(-P) + (2P) - 3O, \tag{14}$$

$$(2(4 - kZ + W)W - Z^3) = 2(-P) + (U) + (P - U) + (V) + (P - V) - 6O, \tag{15}$$

for certain points U, V. We remark that

$$\left(\frac{4 - kZ + W}{Z}\right) = (-P) + (2P) - (P) - O,$$

and

$$\left(\frac{4 - kZ + W}{Z}\right) \diamond (2(4 - kZ + W)W - Z^3) = 10(P) - 8(2P) + 2(3P),$$

since the terms involving U cancel themselves, and the same applies for the terms involving V.

From (12), we obtain

$$\begin{aligned}(a) \diamond (b) \sim& 10(P) - 8(2P) + 2(3P) - 3(5(P) - 4(2P) + (3P))\\ =& -(5(P) - 4(2P) + (3P)).\end{aligned}$$

Finally,

$$(x_1) \diamond (y_1) \sim 5(P) - 4(2P) + (3P).$$

In sum, we have

$$\int_{\gamma_{p,k}} \eta_{C_k}(x_1, y_1) = c_k D^{E_k}(5(P) - 4(2P) + (3P)), \tag{16}$$

where c_k is a constant defined by

$$[f(\gamma_{p,k})] = c_k[\gamma_k],$$

and $[\gamma_k]$ is a generator of $H_1(E_k, Z)^-$.

3.2. **The genus 1 curve.** We now consider the genus 1 family given by

$$Q_k(x_2, y_2) = x_2 y_2^2 + (kx_2 - 1)y_2 - x_2^2 + x_2.$$

The following birational transformation

$$\begin{aligned}Z(x_2, y_2) =& 4x_2, & x_2(Z, W) =& \frac{Z}{4},\\ W(x_2, y_2) =& 4(2x_2y_2 + kx_2 - 1), & y_2(Z, W) =& \frac{W - kZ + 4}{2Z},\end{aligned}$$

leads directly to the Weierstrass form (8)

$$W^2 = Z^3 + (k^2 - 4)Z^2 - 8kZ + 16.$$

We can compute the relevant divisors by using equations (13) and (14),

$$\begin{aligned}(x_2) =& (P) + (-P) - 2O,\\ (y_2) =& (-P) + (2P) - (P) - O,\end{aligned}$$

Then

$$(x_2) \diamond (y_2) = -5(P) + 4(2P) - (3P).$$

In sum, we have

$$\int_{\gamma_{q,k}} \eta_{E_k}(x_2, y_2) = -d_k D^{E_k}(5(P) - 4(2P) + (3P)), \tag{17}$$

where d_k is a constant defined by

$$[\gamma_{q,k}] = d_k[\gamma_k],$$

and $[\gamma_k]$ is a generator of $H_1(E_k, Z)^-$.

4. The cycles of integration

In this section we consider the relationship between the cycles of integration. From (16) and (17), we understand the relationship between the regulators. It remains to compare the cycles of integration, namely, to find the relationship between c_k and d_k. It suffices to compare the integral of the holomorphic differential in E_k respect to each cycle. The strategy is to evaluate $\omega = \frac{dZ}{W}$ both in terms of x_1, y_1 and x_2, y_2, integrate over $f(\gamma_{p,k})$ and $\gamma_{q,k}$ respectively, and compare both integrals.

In our calculations we ignore the sign in front, since the Mahler measure is always non-negative.

4.1. **The genus 3 curve.**

Since $P_k(x, y)$ is reciprocal, the path $f(\gamma_{p,k})$ to be considered corresponds to a fixed choice of a root y_+ or y_-. We do not need to specify the choice, since working with the wrong root will only lead to the opposite sign in the integral of the holomorphic differential. We have,

$$\begin{aligned} \frac{dZ}{W} &= -\frac{y_1(1-x_1^2)^2 dx_1}{y_1^2 - x_1^6} \\ &= \pm \frac{(1-x_1^2)^2 dx_1}{\sqrt{(x_1^6 + kx_1^5 - x_1^4 + (2-2k)x_1^3 - x_1^2 + kx_1 + 1)^2 - 4x_1^6}} \\ &= \pm \frac{(1-x_1^2) dx_1}{\sqrt{(x_1^2 + kx_1 + 1)(x_1^6 + kx_1^5 - x_1^4 + 2(2-k)x_1^3 - x_1^2 + kx_1 + 1)}}. \end{aligned}$$

We see from the change of variables $x_1 \to \frac{1}{x_1}$ that the integral over $|x_1| = 1$ is purely imaginary.

By writing $x_1 = e^{i\theta}$, we have

$$\frac{dZ}{W} = \pm \frac{\sin\theta d\theta}{\sqrt{(2\cos\theta + k)(2\cos^3\theta + k\cos^2\theta - 2\cos\theta + 1 - k)}}.$$

Setting $t = \cos\theta$,

$$\frac{dZ}{W} = \pm \frac{dt}{\sqrt{(2t + k)(2t^3 + kt^2 - 2t + 1 - k)}}.$$

Now set $s = \frac{1}{2t+k}$,

$$\frac{dZ}{W} = \pm \frac{ds}{\sqrt{4s^3 + (k^2 - 4)s^2 - 2ks + 1}}.$$

For $k > 1$, the polynomial $p(s)$ inside the square root has one negative root θ_0 and two roots θ_1, θ_2 between 0 and 1 (where $\theta_2 = 1$ for $k = 1$). More precisely, assume that $k > 2$, then $p(-1) = (k+1)^2 - 8 > 0$, $p\left(\frac{1}{k+2}\right) = \frac{4}{(k+2)^3} > 0$, $p\left(\frac{1}{k}\right) = \frac{4(1-k)}{k^3} < 0$, $p\left(\frac{1}{k-2}\right) = \frac{4}{(k-2)^3} > 0$. In addition notice that $p(1) = (k-1)^2 > 0$. In conclusion, for $k > 2$, $p(s)$ has three real roots satisfying

$$\theta_0 < -1 < 0 < \frac{1}{k+2} < \theta_1 < \frac{1}{k} < \theta_2 < \min\left\{1, \frac{1}{k-2}\right\}.$$

And the above is also true for $k = 2$ by taking the limit.

Then we must integrate

$$\begin{aligned}\int_{\gamma_{p,k}} \omega(Z(x_1,y_1), W(x_1,y_1)) &= \pm i2\,\mathrm{Im}\left(\int_{\frac{1}{k+2}}^{\frac{1}{k-2}} \frac{ds}{\sqrt{4s^3+(k^2-4)s^2-2ks+1}}\right)\\ &= \pm 2\int_{\theta_1}^{\theta_2} \frac{ds}{\sqrt{4s^3+(k^2-4)s^2-2ks+1}},\end{aligned} \tag{18}$$

where we have multiplied by 2 because the change of variable $t = \cos\theta$ implies that there are two values of x_1 yielding the same value of s.

4.2. **The genus 1 curve.** Since $Q_k(x_2, y_2)$ is not reciprocal, we must first verify that the integration path $\gamma_{q,k}$ is closed. First we prove that $Q_k(x_2, y_2) = 0$ does not intersect the unit torus $\{|x_2| = |y_2| = 1\}$ for $k \geq 2$. This means that $|y_{2,+}|$ and $|y_{2,-}|$ stay always > 1 or < 1 while $|x| = 1$. The case $k = 2$ will then follow by continuity. We start by making the change $x_3 = x_2/y_2$ and by writing the equation as

$$x_3 - (y_2 + k + y_2^{-1}) + x_3^{-1}y_2^{-1} = 0.$$

We look for a solution with $|x_3| = |y_2| = 1$. Assuming such solution exists, it must also verify that

$$x_3^{-1} - (y_2 + k + y_2^{-1}) + x_3 y_2 = 0.$$

By combining both equations, we obtain

$$x_3(1 - y_2) = x_3^{-1}(1 - y_2^{-1}),$$

and this implies that either $y_2 = 1$ or $x_3^2 y_2 = -1$. In the first case, we get $x_3 + x_3^{-1} = 2 + k$, which has no solution in $|x_3| = 1$ for $k > 0$. In the second case we get $x_3^{-2} - k + x_3^2 = 0$, which has no solution on $|x_3| = 1$ for $k > 2$ (for $k = 2$ the only solutions are $x_3 = \pm 1$).

This proves that the paths $\{|x_2| = 1, |y_{2+}| \geq 1\}$ and $\{|x_2| = 1, |y_{2-}| \geq 1\}$ are either closed or empty. Notice that $|y_{2+}y_{2-}| = |1 - x_2|$. Since $|1 - x_2| > 1$ for $x_2 = -1$, we conclude that at least one of these cycles is not empty, since we must have $|y_{2\pm}| > 1$ for a certain choice of the sign. On the other hand $|1 - x_2| < 1$ for $x_2 = 1$ and we conclude that at least one of these cycles is empty since we must have $|y_{2\pm}| < 1$ for a certain choice of the sign. Thus, we obtain exactly one nonempty cycle $\gamma_{q,k}$.

We have,

$$\frac{dZ}{W} = \frac{dx_2}{2x_2y_2 + kx_2 - 1} = \pm\frac{dx_2}{\sqrt{4x_2^3 + (k^2-4)x_2^2 - 2kx_2 + 1}}.$$

Then we must integrate

$$\int_{\gamma_{q,k}} \omega(Z(x_2,y_2), W(x_2,y_2)) = \pm\int_{|x_2|=1} \frac{dx_2}{\sqrt{4x_2^3 + (k^2-4)x_2^2 - 2kx_2 + 1}}, \tag{19}$$

4.3. **The end of the proof.** To prove that $c_k = \pm d_k$, we must combine equations (18) and (19) and prove

$$\pm 2 \int_{\theta_1}^{\theta_2} \frac{ds}{\sqrt{4s^3 + (k^2-4)s^2 - 2ks + 1}} = \int_{|x_2|=1} \frac{dx_2}{\sqrt{4x_2^3 + (k^2-4)x_2^2 - 2kx_2 + 1}},$$

but this is true because $\theta_0 < -1 < 0 < \theta_1 < \theta_2 < 1$ and therefore, exactly θ_1 and θ_2 are in the interior of the cycle $|x_2| = 1$. Thus, integrating over $|x_2| = 1$ gives the complex period of E_k, which is twice the semi-period obtained by integrating between θ_1 and θ_2.

By combining equations (16) and (17), we conclude the proof of Theorem 1.

5. Conclusion

It would be very interesting to see if this method can be extended to prove other similar results, particularly some of those conjectured by Liu and Qin [LQ]. For example, one could consider identities between different genus 3 families or between a genus 3 family and a genus 2 family. It would also be interesting to find an example relating a high genus curve with a genus 1 curve, for a case where the Mahler measure of the genus 1 curve is actually proven. Unfortunately, in our case, the Mahler measure of $Q_k(x, y)$ has not been proven for any value of $k \geq 2$.

Another possible direction would be to prove some of the identities between shifted Mahler measures that were conjectured by Liu and Qin.

We remark that several attempts have been made to prove other identities, with no success. The most delicate part of the method is the proof of the relationship between the regulators. In most cases, the divisors to be considered are very complicated, or are supported in points that are difficult to understand.

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