THE DISTRIBUTION OF PRIME INDEPENDENT MULTIPLICATIVE FUNCTIONS OVER FUNCTION FIELDS

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ABSTRACT. We consider the family of multiplicative functions of $\mathbb{F}_q[T]$ with the property that the value at a power of an irreducible polynomial depends only on the exponent, but does not depend on the polynomial or its degree. We study variances of such functions in various regimes, relating them to variances of the divisor function $d_k(f)$. We examine different settings that can be related to distributions over the ensembles of unitary matrices, symplectic matrices, and orthogonal matrices as in the works of [KRRGR18, KL22a, KL22b]. While most questions give very similar answers as the distributions of the divisor function, some of the symplectic problems, dealing with quadratic characters, are different and vary according to the values of the function at the square of the primes.

1. Introduction

The goal of this work is to explore the connection between the distribution of certain multiplicative functions and integrals over the ensembles of unitary, symplectic, and orthogonal matrices. Let $\mathbb{F}_q$ be the field of q elements, where q is an odd prime power with $q \equiv 1 \pmod 4$. Let $\mathfrak{g} : \mathbb{F}_q[T] \to \mathbb{R}$ be a multiplicative function such that $\mathfrak{g}(a) = 1$ for any $a \in \mathbb{F}_q^\times$, and for $P \in \mathbb{F}_q[T]$ irreducible, $\mathfrak{g}(P^k) = \mathfrak{d}_k$, where $\{\mathfrak{d}_k\}_{k=1}^\infty$ is an arbitrary sequence of reals. Yudelevich [Yud20] considers the problem of determining

$$\mathcal{T}_{\mathfrak{g}}(N) := \sum_{f \in \mathcal{M}_N} \mathfrak{g}(f),$$

where we recall that $\mathcal{M}$ denotes the set of monic polynomials in $\mathbb{F}_q[T]$ and $\mathcal{M}_N$ denotes the elements of $\mathcal{M}$ that have degree N. Similarly, $\mathcal{P}$ and $\mathcal{P}_N$ and $\mathcal{H}$ and $\mathcal{H}_N$ denote analogue sets of monic irreducible polynomials and monic square-free polynomials respectively.

Let

$$f(t) := 1 + \mathfrak{d}_1 t + \mathfrak{d}_2 t^2 + \cdots \tag{1}$$

be the generating series for $\{\mathfrak{d}_k\}_{k=1}^\infty$ and define the sequence $\{h_k\}_{k=1}^\infty$ recursively by $h_1 = \mathfrak{d}_1$ and

$$f(t)(1-t)^{h_1}(1-t^2)^{h_2}\cdots(1-t^n)^{h_n} = 1 + h_{n+1}t^{n+1} + \cdots . \tag{2}$$

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In [Yud20, Theorem 1], Yudelevich proves that

$$\mathcal{T}_{\mathfrak{g}}(N) = \sum_{\ell=0}^{N-1} A_\ell(N) q^{N-\ell},$$

where

$$A_\ell(N) = \sum_{\substack{k_1+2k_2+\cdots+(\ell+1)k_{\ell+1}=N\\ k_2+2k_3+\cdots+\ell k_{\ell+1}=\ell}} \binom{-h_1}{k_1}\binom{-h_2}{k_2}\cdots\binom{-h_{\ell+1}}{k_{\ell+1}}(-1)^{k_1+k_2+\cdots+k_{\ell+1}}.$$

Yudelevich then uses this to derive several asymptotics in the cases q, q and N or q^N approach infinity. These results are applied to various functions $\mathfrak{g}$, including the reciprocal of the divisor function.

In [KRRGR18], Keating, Rodgers, Roditty-Gershon, and Rudnick study variances of the divisor function

$$d_k(f) := \#\{(f_1,\ldots,f_k) : f = f_1\cdots f_k,\, f_j \in \mathcal{M}\}.$$

The function $d_k(f)$ appears naturally as the coefficient of kth power of the Riemann zeta function $\zeta_q(s)^k$ (see (5) for the definition in the function field setting).

For $Q \in \mathcal{H}$ and $N \le k(\deg(Q)-1)$, they prove in [KRRGR18, Theorem 3.1] that the variance of

$$\mathcal{S}^U_{d_k;N;Q}(A) := \sum_{\substack{f\in\mathcal{M}_N\\ f\equiv A\,(\mathrm{mod}\,Q)}} d_k(f)$$

defined by

$$\mathrm{Var}(\mathcal{S}^U_{d_k;N;Q}) := \frac{1}{\Phi(Q)} \sum_{\substack{A\,(\mathrm{mod}\,Q)\\ (A,Q)=1}} \left|\mathcal{S}^U_{d_k;N;Q}(A) - \langle \mathcal{S}^U_{d_k;N;Q}\rangle\right|^2,$$

is given, as $q\to\infty$, by

$$\mathrm{Var}(\mathcal{S}^U_{d_k;N;Q}) \sim \frac{q^N}{q^{\deg(Q)}} \int_{\mathrm{U}(\deg(Q)-1)} \left| \sum_{\substack{j_1+\cdots+j_k=N\\ 0\le j_1,\ldots,j_k\le \deg(Q)-1}} \mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_k}(U)\right|^2 \mathrm{d}U, \tag{3}$$

where the integral ranges over complex unitary matrices of dimension $\deg(Q)-1$, and the $\mathrm{Sc}_j(U)$ are the secular coefficients, defined for a $\Delta\times\Delta$ matrix U via

$$\det(I+Ux) = \sum_{j=0}^{\Delta} \mathrm{Sc}_j(U)x^j.$$

This result relies on Katz's equidistribution theorem [Kat13b] for primitive Dirichlet characters with square-free conductors. It was later extended by Roditty-Gershon, Hall, and Keating [HKRG20] to the coefficients of kth powers of more general L-functions, such as those associated with elliptic curves.

Keating et al obtain a similar statement in [KRRGR18, Theorem 1.2] by considering the variance of the divisor function over short intervals.

In [KL22a], Kuperberg and Lalín consider the distribution of the divisor function $d_k(f)$ when restricted to quadratic residues modulo an irreducible polynomial P of odd degree

$2g+1$. More precisely, they prove in [KL22a, Theorem 1.1] that for $P \in \mathcal{P}_{2g+1}$, $N \leq 2gk$ and

$$\mathcal{S}^S_{d_k,N}(P) := \sum_{\substack{f \in \mathcal{M}_N \\ f \equiv \square \,(\mathrm{mod}\, P) \\ P \nmid f}} d_k(f),$$

as $q \to \infty$,

$$\mathcal{S}^S_{d_k,N}(P) \sim \frac{1}{2} \sum_{\substack{f \in \mathcal{M}_N \\ P \nmid f}} d_k(f) \sim \frac{q^N}{2} \binom{k+N-1}{k-1},$$

and

$$\mathrm{Var}(\mathcal{S}^S_{d_k,N}) := \frac{1}{\#\mathcal{P}_{2g+1}} \sum_{P \in \mathcal{P}_{2g+1}} \left(\mathcal{S}^S_{d_k,N}(P) - \frac{1}{2} \sum_{\substack{f \in \mathcal{M}_N \\ P \nmid f}} d_k(f) \right)^2$$

$$\sim \frac{q^N}{4} \int_{\mathrm{Sp}(2g)} \left(\sum_{\substack{j_1+\cdots+j_k=N \\ 0 \leq j_1,\ldots,j_k \leq 2g}} \mathrm{Sc}_{j_1}(U) \cdots \mathrm{Sc}_{j_k}(U) \right)^2 dU. \tag{4}$$

The integral is very similar to the one appearing in (3), but it ranges over the set of symplectic unitary matrices of dimension $2g$. The restriction to quadratic residues modulo an irreducible polynomial P can be detected by twisting by a quadratic character, which ultimately yields the connection to the symplectic matrix integral via some monodromy arguments due to Katz (see [KL22a, Section 3]). A similar problem over integers, involving the sum of the divisor function weighted by the quadratic character modulo d, is also considered in [KL22b, Conjecture 1.10].

In addition, Kuperberg and Lalín consider the distribution of the divisor function over monic polynomials of fixed degree with a condition that can be interpreted as the function field analogue of having the argument of a complex number lying in a specific sector of a unit circle. This setting follows a model of Gaussian integers in the function fields that was proposed by Bary-Soroker, Smilansky, and Wolf [BSSW16] and later developed by Rudnick and Waxman [RW19]. In this context, the result on the variance proven in [KL22a] also leads to an integral over the symplectic matrices similar to the one from (4). A variation of this question leads to distributions over the ensemble of orthogonal questions in [KL22b, Theorem 1.12].

In this note, we are concerned with generalizing these results for d_k to the class of functions $\mathfrak{g}$. We remark that the divisor function is a particular case of the class of functions $\mathfrak{g}$. Moreover, since the generating function for the divisor function is the kth power of the zeta function, which over $\mathbb{F}_q[T]$ takes the shape $(1-qu)^{-1}$, we can reinterpret equation (2) as a sequence of approximations of $\mathfrak{g}$ by divisor functions.

More precisely, we extend the results of [KRRGR18] by proving the following.

Theorem 1.1. *Let $\mathfrak{g}$ be defined as before. Let $Q \in \mathcal{H}$, $A \in \mathbb{F}_q[T]$ coprime to Q, and*

$$\mathcal{S}^U_{\mathfrak{g};N,Q}(A) := \sum_{\substack{f \in \mathcal{M}_N \\ f \equiv A \,(\mathrm{mod}\, Q)}} \mathfrak{g}(f).$$

Then, as $q \to \infty$,

$$\mathcal{S}^U_{\mathfrak{g};N,Q}(A) \sim \frac{1}{\Phi(Q)} \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1 \in \mathcal{M}_{m_1} \\ \cdots \\ f_N \in \mathcal{M}_{m_N} \\ (f_1 f_2 \cdots f_N, Q)=1}} d_{h_1}(f_1) d_{h_2}(f_2) \cdots d_{h_N}(f_N)$$

$$\sim \frac{q^N}{\Phi(Q)} \binom{\mathfrak{d}_1 + N - 1}{N}.$$

If $\mathfrak{d}_1$ is a positive integer, $\deg(Q) \geq 2$, and $N \leq \mathfrak{d}_1(\deg(Q)-1)$ then, as $q \to \infty$,

$$\mathrm{Var}(\mathcal{S}^U_{\mathfrak{g};N,Q})$$

$$:= \frac{1}{\Phi(Q)} \sum_{\substack{A \,(\mathrm{mod}\, Q) \\ (A,Q)=1}} \left| \mathcal{S}^U_{\mathfrak{g};N,Q}(A) - \frac{1}{\Phi(Q)} \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1 \in \mathcal{M}_{m_1} \\ \cdots \\ f_N \in \mathcal{M}_{m_N} \\ (f_1 f_2 \cdots f_N, Q)=1}} d_{h_1}(f_1) d_{h_2}(f_2) \cdots d_{h_N}(f_N) \right|^2$$

$$\sim q^{N-\deg(Q)} \int_{\mathrm{U}(\deg(Q)-1)} \left| \sum_{\substack{j_1+\cdots+j_{h_1}=N \\ 0 \leq j_1,\ldots,j_{h_1} \leq \deg(Q)-1}} \mathrm{Sc}_{j_1}(U) \cdots \mathrm{Sc}_{j_{h_1}}(U) \right|^2 dU.$$

For any nonzero $f \in \mathbb{F}_q[T]$, the norm is given by $|f| = q^{\deg(f)}$. For $A \in \mathbb{F}_q[T]$, The set

$$I_{\mathfrak{h}}(A) := \{f \in \mathbb{F}_q[T] \,:\, |f - A| \leq q^{\mathfrak{h}}\}$$

is the interval of radius $\mathfrak{h}$ centered in A.

Theorem 1.2. *Let $\mathfrak{g}$ be defined as before. Let $A \in \mathcal{M}_N$ and $0 \leq \mathfrak{h} \leq N-2$, and consider*

$$\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}(A) := \sum_{f \in I_{\mathfrak{h}}(A)} \mathfrak{g}(f).$$

The mean value is given by

$$\frac{1}{q^N} \sum_{A \in \mathcal{M}_N} \mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}(A) = \frac{q^{\mathfrak{h}+1}}{q^N} \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1 \in \mathcal{M}_{m_1} \\ \cdots \\ f_N \in \mathcal{M}_{m_N}}} d_{h_1}(f_1) d_{h_2}(f_2) \cdots d_{h_N}(f_N)$$

$$\sim q^{\mathfrak{h}+1} \binom{\mathfrak{d}_1 + N - 1}{N}.$$

Suppose that $\mathfrak{d}_1$ is a positive integer and $0 \leq \mathfrak{h} \leq \min\{N-5, (1-\frac{1}{\mathfrak{d}_1})N - 2\}$. Suppose also that the $\mathfrak{d}_k$ are essentially uniformly bounded in the sense that $\mathfrak{d}_k = o(q^{\varepsilon})$ for any $\varepsilon > 0$.

Then, as $q \to \infty$,

$$\begin{aligned}
&\mathrm{Var}(\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}})\\
&:= \frac{1}{q^N}\sum_{A\in\mathcal{M}_N}\left|\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}(A) - \frac{q^{\mathfrak{h}+1}}{q^N}\sum_{m_1+2m_2+\cdots+Nm_N=N}\ \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\right|^2\\
&\sim q^{\mathfrak{h}+1}\int_{\mathrm{U}(N-\mathfrak{h}-2)}\left|\sum_{\substack{j_1+\cdots+j_{h_1}=N\\ 0\le j_1,\ldots,j_{h_1}\le N-\mathfrak{h}-2}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_{h_1}}(U)\right|^2 dU.
\end{aligned}$$

The short interval problem was vastly generalized by Rodgers [Rod18] to fixed factorization functions, that is, functions that depend solely on the extended factorization type of the element, which involves not only the exponents of the irreducible factors, but also their degrees. The functions $\mathfrak{g}$ that we consider are a particular case of this, since they have the same value for powers of irreducible polynomials, independently of the degree of the underlying polynomial. Gorodetsky and Rodgers [GR21] further extended the short interval problem to the case of d_z for z real.

We also extend the results of [KL22a, KL22b] as follows.

Theorem 1.3. *Let $\mathfrak{g}$ be defined as before and let $P \in \mathcal{P}_{2g+1}$. Consider*

$$\mathcal{S}^S_{\mathfrak{g},N}(P) := \sum_{\substack{f \text{ monic},\deg(f)=N\\ f\equiv\square \pmod P\\ P\nmid f}} \mathfrak{g}(f).$$

Then, as $q \to \infty$,

$$\mathcal{S}^S_{\mathfrak{g},N}(P) \sim \frac{1}{2}\sum_{m_1+2m_2+\cdots+Nm_N=N}\ \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ P\nmid f_1f_2\cdots f_N}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N) \sim \frac{q^N}{2}\binom{\mathfrak{d}_1+N-1}{N}.$$

Assume that $\mathfrak{d}_1$ is positive integer and $N \le 2g\mathfrak{d}_1$. As $q \to \infty$,

$$\begin{aligned}
\mathrm{Var}(\mathcal{S}^S_{\mathfrak{g},N}) &:= \frac{1}{\#\mathcal{P}_{2g+1}}\sum_{P\in\mathcal{P}_{2g+1}}\left(\mathcal{S}^S_{\mathfrak{g};N}(P) - \frac{1}{2}\sum_{m_1+2m_2+\cdots+Nm_N=N}\ \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ P\nmid f_1f_2\cdots f_N}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\right)^2\\
&\sim \frac{q^N}{4}\int_{\mathrm{Sp}(2g)}\left(\sum_{\ell=0}^{\lfloor\frac{N}{2}\rfloor}\binom{-h_2}{\ell}(-1)^\ell\sum_{\substack{j_1+\cdots+j_{h_1}=N-2\ell\\ 0\le j_1,\ldots,j_{h_1}\le 2g}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_{h_1}}(U)\right)^2 dU.
\end{aligned}$$

Theorem 1.4. *Let $\mathfrak{g}$ be defined as before and let $R \in \mathcal{H}_{2g+1}$. Consider*

$$\mathcal{T}^S_{\mathfrak{g};N}(R) := \sum_{\substack{f\in\mathcal{M}_N\\ (f,R)=1}} \mathfrak{g}(f)\chi_R(f).$$

Then, as $q \to \infty$, $\mathcal{T}^S_{\mathfrak{g};N}(R) = O\left(q^{\frac{N}{2}}\right)$. Suppose that $\mathfrak{d}_1$ is positive integer and $N \le 2g\mathfrak{d}_1$. As $q \to \infty$,

$$\begin{aligned}\mathrm{Var}(\mathcal{T}^S_{\mathfrak{g};N}) :=& \frac{1}{\#\mathcal{H}_{2g+1}} \sum_{R \in \mathcal{H}_{2g+1}} \left(\mathcal{T}^S_{\mathfrak{g};N}(R)\right)^2 \\ \sim& q^N \int_{\mathrm{Sp}(2g)} \left(\sum_{\ell=0}^{\lfloor \frac{N}{2} \rfloor} \binom{-h_2}{\ell} (-1)^\ell \sum_{\substack{j_1+\cdots+j_{h_1}=N-2\ell \\ 0\le j_1,\dots,j_{h_1}\le 2g}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_{h_1}}(U) \right)^2 dU.\end{aligned}$$

In [KL22a, KL22b], the authors worked with a model of the Gaussian integers in the function field setting which was initially developed by Bary-Soroker, Smilansky, and Wolf in [BSSW16] and considered by Rudnick and Waxman in [RW19]. The authors of [KL22a, KL22b] studied the variances of

$$\mathcal{N}^S_{d_\ell,k,n}(v) = \sum_{\substack{f\in\mathcal{M}_n \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} d_\ell(f) \qquad \text{and} \qquad \mathcal{N}^O_{d_\ell,k,n}(v) = \sum_{\substack{f\in\mathcal{M}_n \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} d_\ell(f)\left(\frac{1+\chi_2(f)}{2}\right),$$

where the sums are taken over monic polynomials of fixed degree under a condition (see (18)) corresponding to the function field analogue of having the argument of a complex number lying in certain specific sector of the unit circle, and where χ_2 is the quadratic character over $\mathbb{F}_q[T]$ defined by $\chi_2(f) := \chi_2(f(0))$, where for $x \in \mathbb{F}_q$, $\chi_2(x) := 1$ if x is a non-zero square, -1 if x is a non square, and 0 if $x = 0$.

We obtain the following generalizations.

Theorem 1.5. *Let $\mathfrak{g}$ be defined as before and consider*

$$\mathcal{N}^S_{\mathfrak{g},k,N}(v) = \sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} \mathfrak{g}(f) \qquad \text{and} \qquad \mathcal{N}^O_{\mathfrak{g},k,N}(v) = \sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} \mathfrak{g}(f)\left(\frac{1+\chi_2(f)}{2}\right).$$

Let $\kappa = \left\lfloor \frac{k}{2} \right\rfloor$. As $q \to \infty$, the mean values are given by

$$\begin{aligned}\frac{1}{q^\kappa}\sum_{v\in\mathbb{S}^1_k} \mathcal{N}^S_{\mathfrak{g},k,N}(v) =& \frac{1}{q^\kappa} \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1\in\mathcal{M}_{m_1} \\ \cdots \\ f_N\in\mathcal{M}_{m_N} \\ f_1f_2\cdots f_N(0)\neq 0}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N) \\ \sim& q^{N-\kappa}\binom{\mathfrak{d}_1+N-1}{N},\end{aligned}$$

and

$$\frac{1}{q^\kappa}\sum_{v\in\mathbb{S}_k^1}\mathcal{N}^O_{\mathfrak{g},k,N}(v) \sim \frac{1}{2q^\kappa}\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ f_1f_2\cdots f_N(0)\neq 0}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)$$

$$\sim \frac{q^{N-\kappa}}{2}\binom{\mathfrak{d}_1+N-1}{N}.$$

Assume that $\mathfrak{d}_1$ is a positive integer and $N\leq \mathfrak{d}_1(2\kappa-2)$. As $q\to\infty$,

$$\mathrm{Var}(\mathcal{N}^S_{\mathfrak{g},k,N}) := \frac{1}{q^\kappa}\sum_{v\in\mathbb{S}_k^1}\left|\mathcal{N}^S_{\mathfrak{g},k,N}(v)-\frac{1}{2q^\kappa}\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}}\mathfrak{g}(f)\right|^2$$

$$\sim \frac{q^N}{q^\kappa}\int_{\mathrm{Sp}(2\kappa-2)}\left(\sum_{\substack{j_1+\cdots+j_\ell=N\\ 0\leq j_1,\ldots,j_\ell\leq 2\kappa-2}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_\ell}(U)\right)^2 dU.$$

Assume that $\mathfrak{d}_1$ is a positive integers and $N\leq \mathfrak{d}_1(2\kappa-1)$ with $\kappa=\left\lfloor\frac{k}{2}\right\rfloor\geq 3$. As $q\to\infty$,

$$\mathrm{Var}_S(\mathcal{N}^O_{\mathfrak{g},k,N}) := \frac{1}{q^\kappa}\sum_{v\in\mathbb{S}_k^1}\left|\mathcal{N}^O_{\mathfrak{g},k,N}(v)-\frac{1}{2q^\kappa}\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0\\ U(f)\in\mathrm{Sect}(v,k)}}\mathfrak{g}(f)\right|^2$$

$$\sim \frac{q^N}{4q^\kappa}\int_{\mathrm{O}(2\kappa-1)}\left(\sum_{\substack{j_1+\cdots+j_\ell=N\\ 0\leq j_1,\ldots,j_\ell\leq 2\kappa-1}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_\ell}(U)\right)^2 dU.$$

The integrals appearing as expressions for the variances in all the above results have been studied in [KRRGR18, KL22a, KL22b]. For example, Keating et al prove that

$$\int_{\mathrm{U}(M)}\Big|\sum_{\substack{j_1+\cdots+j_k=m\\ 0\leq j_1,\ldots,j_k\leq M}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_k}(U)\Big|^2\mathrm{d}U=\gamma_k(m/M)M^{k^2-1}+O_k(M^{k^2-2}),$$

where $\gamma_k(c)$ is seen to be a piecewise polynomial function of degree k^2-1. Similar results are proven by Kuperberg and Lalín for the symplectic and orthogonal cases (with the degrees changing in these cases). Basor, Ge, and Rubinstein [BGR18] further analyze $\gamma_k(c)$ in the unitary case. See also Medjedovic's MSc thesis [Med21] for more on the properties of $\gamma_k(c)$ in the symplectic and orthogonal cases.

We see that, while Theorems 1.1, 1.2, and 1.5 give similar results to (3) and (4) in the sense that the variance is entirely determined by $\mathfrak{d}_1$, Theorems 1.3 and 1.4 give a different result from those, as the variance depends not only on $\mathfrak{d}_1$, but also on $h_2=\mathfrak{d}_2-\frac{\mathfrak{d}_1(\mathfrak{d}_1+1)}{2}$.

This article is organized as follows. Section 2 exposes some general background on the arithmetics of function fields, while Section 3 treats some auxiliary results on evaluations of sums of $\mathfrak{g}$ twisted by characters that are needed for the results. Sections 4, 5, 6, and 7 contain the proofs of Theorems 1.1, 1.2, 1.3, and 1.4 respectively. A discussion on the evaluation of the integrals appearing in Theorems 1.3 and 1.4 is included in Section 8. Finally, Section 9 contains the proof of Theorem 1.5 together with some necessary background.

2. Some general background

In this section we give some general background about the arithmetics of $\mathbb{F}_q[T]$. For $f \in \mathcal{M}$, let $d_k(f)$ denote the kth divisor function given by

$$d_k(f) := \#\{(f_1, \dots, f_k) \, : \, f = f_1 \cdots f_k, \, f_j \in \mathcal{M}, \}.$$

The divisor function can be extended to non-zero elements of $\mathbb{F}_q[T]$ by setting $d_k(cf) := d_k(f)$ for $c \in \mathbb{F}_q^\times$. Notice that d_k is a multiplicative function, defined on powers of $P \in \mathcal{P}$ by $d_k(P^j) = \binom{j+k-1}{j}$.

The zeta function of $\mathbb{F}_q[T]$ is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_q(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s}, \tag{5}$$

where for any non-zero f, the norm is given by $|f| = q^{\deg(f)}$. By counting monic polynomials of a fixed degree, one obtains that

$$\zeta_q(s) = \frac{1}{1 - q^{1-s}},$$

which provides a meromorphic continuation of $\zeta_q(s)$, with simple poles when $q^s = q$. Making the change of variables $u = q^{-s}$, the zeta function becomes

$$\mathcal{Z}_q(u) = \sum_{f \in \mathcal{M}} u^{\deg(f)} = \prod_{P \in \mathcal{P}} \left(1 - u^{\deg(P)}\right)^{-1},$$

for $|u| < 1/q$. As before, we have the expression

$$\mathcal{Z}_q(u) = \frac{1}{1 - qu}.$$

We remark that the kth power of the zeta function is the generating function of d_k, namely,

$$\mathcal{Z}_q(u)^k = \sum_{f \in \mathcal{M}} d_k(f) u^{\deg(f)}.$$

For $\kappa \in \mathbb{R}$ and $|u| < 1/q$, we can define $(1 - qu)^{-\kappa}$ by taking the principal branch of the logarithm. Writing the generating function gives

$$\mathcal{Z}_q(u)^\kappa = \sum_{f \in \mathcal{M}} d_\kappa(f) u^{\deg(f)},$$

which leads to an extension of $d_\kappa(f)$ to the case where κ is a real parameter.

For a general arithmetic function $\mathfrak{g} : \mathbb{F}_q[T] \to \mathbb{C}$, we define

$$\mathcal{M}(N; \mathfrak{g}) := \sum_{f \in \mathcal{M}_N} \mathfrak{g}(f).$$

In particular, we have that

$$\mathcal{Z}_q(u)^\kappa = \sum_{N=0}^{\infty} \mathcal{M}(N; d_\kappa) u^N.$$

For a polynomial $Q(T) \in \mathbb{F}_q[T]$ of positive degree, the order of $(\mathbb{F}_q[T]/(Q))^\times$ is given by the Euler function $\Phi(Q)$. A Dirichlet character modulo Q is a homomorphism

$$\chi : (\mathbb{F}_q[T]/(Q))^\times \to \mathbb{C}^\times,$$

which is extended to $\chi(R) = 0$ for $R \in \mathbb{F}_q[T]$ such that $(R, Q) \neq 1$. The orthogonality relations give, for $A(T) \in \mathbb{F}_q[T]$,

$$\frac{1}{\Phi(Q)} \sum_{\chi \,(\mathrm{mod}\, Q)} \overline{\chi}(A)\chi(R) = \begin{cases} 1 & R \equiv A \,(\mathrm{mod}\, Q), \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Given a Dirichlet character χ, the corresponding Dirichlet L-series is given by

$$L_q(s, \chi) = \sum_{f \in \mathcal{M}} \frac{\chi(f)}{|f|^s}$$

for $\mathrm{Re}(s) > 1$. As in the zeta function case, we can make the change of variables $u = q^{-s}$ and write

$$\mathcal{L}_q(u, \chi) = \sum_{f \in \mathcal{M}} \chi(f) u^{\deg(f)} = \prod_{P \in \mathcal{P}} \left(1 - \chi(P) u^{\deg(P)}\right)^{-1}.$$

Notice that

$$\mathcal{L}_q(u, \chi)^k = \sum_{f \in \mathcal{M}} \chi(f) d_k(f) u^{\deg(f)} = \sum_{N=0}^{\infty} \mathcal{M}(N; \chi d_k) u^N.$$

A Dirichlet character χ is called even if $\chi(c) = 1$ for any $c \in \mathbb{F}^\times$ and is called odd otherwise.

By orthogonality, when χ is a nontrivial character, $\mathcal{L}_q(u, \chi)$ is a polynomial of degree $\Delta \leq \deg(Q) - 1$ ([Ros02, Proposition 4.3]). If we consider the reciprocals of the roots, i.e.,

$$\mathcal{L}_q(u, \chi) = \prod_{j=1}^{\Delta} (1 - \alpha_j u).$$

Then, for χ odd, the Riemann hypothesis implies that $|\alpha_j| = \sqrt{q}$. If χ is even, one root equals 1 and the others satisfy the Riemann hypothesis. This implies that

$$\mathcal{L}_q(u, \chi) = (1 - u)^\lambda \det(1 - u q^{1/2} \Theta_\chi),$$

where $\lambda = 0$ if χ is odd and 1 if χ is even, and Θ_χ is a conjugacy class in the unitary matrix ensemble of dimension $N = \Delta - \lambda$.

For an $N \times N$ matrix U, the secular coefficients $\mathrm{Sc}_j(U)$ are defined by

$$\det(I + xU) = \sum_{j=0}^{N} \mathrm{Sc}_j(U) x^j.$$

Thus, the coefficients of $\mathcal{L}_q(u, \chi)$ can be expressed in terms of the secular coefficients of Θ_χ.

For χ_0 the trivial character modulo Q, we have $\mathcal{L}_q(u, \chi_0) = \mathcal{Z}_q(u) \prod_{P|Q} (1 - u^{\deg(P)})$.

Let $P(T) \in \mathcal{P}$ and $f \in \mathbb{F}_q[T]$. The quadratic residue symbol is defined by

$$\left(\frac{f}{P}\right) \equiv f^{\frac{|P|-1}{2}} \,(\mathrm{mod}\, P)$$

if $P \nmid f$, and $\left(\frac{f}{P}\right) = 0$ otherwise. If $Q = P_1^{e_1} \cdots P_r^{e_r}$ with each P_j irreducible, then the Jacobi symbol is given by

$$\left(\frac{f}{Q}\right) = \prod_{j=1}^{r} \left(\frac{f}{P_j}\right)^{e_j}.$$

From now on we will assume that $q \equiv 1 \pmod 4$; in this case, quadratic reciprocity implies $\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right)$ for any $A, B \in \mathcal{M}$ non-zero such that $(A, B) = 1$.

For $D \in \mathcal{H}$, we consider the quadratic character

$$\chi_D(f) = \left(\frac{D}{f}\right).$$

We have that χ_D is an odd character if $\deg(D)$ is odd and even otherwise. In both cases $\mathcal{L}_q(u, \chi_D)$ is a polynomial of degree $\deg(D) - 1$.

3. Evaluation of twisted averages of $\mathfrak{g}$

In this section we will establish a general setting for evaluating $\mathcal{M}(N; \chi\mathfrak{g})$, where χ is a Dirichlet character, and $\mathfrak{g} : \mathbb{F}_q[T] \to \mathbb{R}$ is multiplicative function such that $\mathfrak{g}(a) = 1$ for any $a \in \mathbb{F}_q^\times$, and for $P \in \mathbb{F}_q[T]$ irreducible, $\mathfrak{g}(P^k) = \mathfrak{d}_k$, where $\{\mathfrak{d}_k\}_{k=1}^\infty$ is an arbitrary sequence of reals. Many of the arguments of this section follow ideas in the proof of [Yud20, Theorem 1].

We start by considering the generating series

$$\mathcal{G}(u; \chi\mathfrak{g}) := \sum_{f \in \mathcal{M}} \chi(f)\mathfrak{g}(f)u^{\deg(f)} = \sum_{N=0}^{\infty} \mathcal{M}(N; \chi\mathfrak{g})u^N.$$

Since $\mathfrak{g}$ is multiplicative, we have

$$\begin{aligned}
\mathcal{G}(u; \chi\mathfrak{g}) &= \prod_{\pi \in \mathcal{P}} \left(1 + \chi(\pi)\mathfrak{g}(\pi)u^{\deg(\pi)} + \chi(\pi^2)\mathfrak{g}(\pi^2)u^{2\deg(\pi)} + \cdots\right) \\
&= \prod_{\ell=1}^{\infty} \prod_{\pi \in \mathcal{P}_\ell} \left(1 + \chi(\pi)\mathfrak{d}_1 u^\ell + \chi(\pi)^2 \mathfrak{d}_2 u^{2\ell} + \cdots\right) \\
&= \prod_{\ell=1}^{\infty} \prod_{\pi \in \mathcal{P}_\ell} f(\chi(\pi)u^\ell) \\
&= \prod_{\ell=1}^{\infty} \prod_{\pi \in \mathcal{P}_\ell} (1 - \chi(\pi)u^\ell)^{-h_1}(1 - \chi(\pi)^2 u^{2\ell})^{-h_2} \cdots (1 - \chi(\pi)^n u^{n\ell})^{-h_n} f_{n+1}(\chi(\pi)u^\ell),
\end{aligned}$$

where we have applied (1) and (2) and we have set

$$f_{n+1}(t) := f(t)(1-t)^{h_1}(1-t^2)^{h_2} \cdots (1-t^n)^{h_n}.$$

Gathering the first n factors gives

$$\mathcal{G}(u; \chi\mathfrak{g}) = \mathcal{L}_q(u, \chi)^{h_1} \mathcal{L}_q(u^2, \chi^2)^{h_2} \cdots \mathcal{L}_q(u^n, \chi^n)^{h_n} \prod_{\ell=1}^{\infty} \prod_{\pi \in \mathcal{P}_\ell} f_{n+1}(\chi(\pi)u^\ell). \tag{7}$$

Taking $n = N$ and comparing coefficients in (7), we get

$$\mathcal{M}(N;\chi\mathfrak{g}) = \sum_{m_1+2m_2+\cdots+Nm_N=N} \mathcal{M}(m_1;\chi d_{h_1})\mathcal{M}(m_2;\chi^2 d_{h_2})\cdots\mathcal{M}(m_N;\chi^N d_{h_N}). \tag{8}$$

The above expression allows us to relate $\mathcal{M}(N;\chi\mathfrak{g})$ to a product of $\mathcal{M}(m;\chi d_h)$. The advantage of this process is that several tools are available to us regarding how to evaluate or estimate these sums.

We recall the following results.

Lemma 3.1. [KL22a, Lemma 2.1] *Let χ be odd and k be a positive integer. For $N \leq k\Delta$ we have*

$$\mathcal{M}(N;\chi d_k) = (-1)^N q^{\frac{N}{2}} \sum_{\substack{j_1+\cdots+j_k=N\\ 0\leq j_1,\ldots j_k\leq\Delta}} \mathrm{Sc}_{j_1}(\Theta_\chi)\cdots\mathrm{Sc}_{j_k}(\Theta_\chi)$$

and $\mathcal{M}(N;d_k\chi) = 0$ otherwise.

Let χ be even. For $N \leq k\Delta$, as $q\to\infty$ we have

$$\mathcal{M}(N;\chi d_k) = (-1)^N q^{\frac{N}{2}} \sum_{\substack{j_1+\cdots+j_k=N\\ 0\leq j_1,\ldots j_k\leq\Delta}} \mathrm{Sc}_{j_1}(\Theta_\chi)\cdots\mathrm{Sc}_{j_k}(\Theta_\chi) + O_{k,\Delta,N}\left(q^{\frac{N-1}{2}}\right).$$

For $k\Delta < N \leq k(\Delta+1)$, as $q\to\infty$,

$$|\mathcal{M}(N;\chi d_k)| \ll_{k,\Delta} q^{\frac{N-1}{2}}.$$

Finally, $\mathcal{M}(N;\chi d_k) = 0$ for $N > k(\Delta+1)$.

Lemma 3.2. *Let κ be real and let χ_0 be the trivial character modulo $Q\in\mathbb{F}_q[T]$. As $q\to\infty$, we have that*

$$\mathcal{M}(N;\chi_0 d_\kappa) = \sum_{\substack{f\in\mathcal{M}_N\\ (f,Q)=1}} d_\kappa(f) = q^N\binom{\kappa+N-1}{N} + O_{\deg(P_j),\kappa,N}(q^{N-1}),$$

where the implied constant depends on the degrees of the prime factors P_j of Q, but not on Q itself.

Proof. This is essentially [KL22a, Lemma 5.2]. Without loss of generality we can assume that Q is monic. Let $Q = P_1^{e_1}\cdots P_r^{e_r}$ be the prime decomposition of Q. We can estimate the sum by considering its generating function.

$$\sum_{\substack{f\in\mathcal{M}\\ (f,Q)=1}} d_\kappa(f)u^{\deg(f)} = \left(\mathcal{Z}_q(u)\prod_{j=1}^r(1-u^{\deg(P_j)})\right)^\kappa = \left(\frac{\prod_{j=1}^r(1-u^{\deg(P_j)})}{1-qu}\right)^\kappa$$

$$= \sum_{N=0}^\infty\sum_{m_1=0}^\infty\cdots\sum_{m_r=0}^\infty\left(\prod_{j=1}^r\binom{\kappa}{m_j}\right)\binom{-\kappa}{N-\sum_{j=1}^r m_j\deg(P_j)} q^{N-\sum_{j=1}^r m_j\deg(P_j)}(-1)^N u^N.$$

By focusing on the coefficient of u^N, we obtain

$$\sum_{\substack{f\in\mathcal{M}_N\\(f,Q)=1}} d_\kappa(f) =(-1)^N \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\left(\prod_{j=1}^{r}\binom{\kappa}{m_j}\right)\binom{-\kappa}{N-\sum_{j=1}^r m_j\deg(P_j)} q^{N-\sum_{j=1}^r m_j\deg(P_j)}$$

$$=q^N\binom{\kappa+N-1}{N} + O_{\deg(P_j),\kappa,N}(q^{N-1}).$$

□

Proposition 3.3. *Let χ be a nontrivial Dirichlet character of conductor $\mathbb{Q}$. Then we have*

$$|\mathcal{M}(N;\chi\mathfrak{g})|^2 = \sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\n_1+2n_2+\cdots+Nn_N=N}}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi^j d_{h_j})\overline{\mathcal{M}(n_j;\chi^j d_{h_j})}$$

$$=\sum_{\substack{m_1+2m_2=N\\n_1+2n_2=N}}\mathcal{M}(m_1;\chi d_{h_1})\overline{\mathcal{M}(n_1;\chi d_{h_1})}\mathcal{M}(m_2;\chi^2 d_{h_2})\overline{\mathcal{M}(n_2;\chi^2 d_{h_2})} + O_{h_j,N,\deg(Q)}\left(q^{N-\frac{1}{6}}\right).$$

Proof. By equation (8) and Lemmas 3.1 and 3.2, we have that

$$|\mathcal{M}(N;\chi\mathfrak{g})|^2 = \sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\n_1+2n_2+\cdots+Nn_N=N}}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi^j d_{h_j})\overline{\mathcal{M}(n_j;\chi^j d_{h_j})}$$

$$\ll_{h_j,N,\deg(Q)}\sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\n_1+2n_2+\cdots+Nn_N=N}} q^{\frac{m_1}{2}+m_2+\cdots+m_N+\frac{n_1}{2}+n_2+\cdots+n_N}.$$

Using that $\mathcal{M}(0;\chi d_h)=1$, we have, as $q\to\infty$,

$$\sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\n_1+2n_2+\cdots+Nn_N=N}}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi^j d_{h_j})\overline{\mathcal{M}(n_j;\chi^j d_{h_j})}$$

$$=\sum_{\substack{m_1+2m_2=N\\n_1+2n_2=N}}\mathcal{M}(m_1;\chi d_{h_1})\overline{\mathcal{M}(n_1;\chi d_{h_1})}\mathcal{M}(m_2;\chi^2 d_{h_2})\overline{\mathcal{M}(n_2;\chi^2 d_{h_2})}$$

$$+\sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\n_1+2n_2+\cdots+Nn_N=N\\m_1+2m_2<N\text{ or }n_1+2n_2<N}}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi^j d_{h_j})\overline{\mathcal{M}(n_j;\chi^j d_{h_j})}$$

$$=\sum_{\substack{m_1+2m_2=N\\n_1+2n_2=N}}\mathcal{M}(m_1;\chi d_{h_1})\overline{\mathcal{M}(n_1;\chi d_{h_1})}\mathcal{M}(m_2;\chi^2 d_{h_2})\overline{\mathcal{M}(n_2;\chi^2 d_{h_2})}$$

$$+O\left(\sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\n_1+2n_2+\cdots+Nn_N=N\\m_1+2m_2<N\text{ or }n_1+2n_2<N}} q^{\frac{m_1}{2}+m_2+\cdots+m_N+\frac{n_1}{2}+n_2+\cdots+n_N}\right). \tag{9}$$

Note that, for $m_1+2m_2+\cdots+Nm_N=N$, we have

$$\begin{aligned}\frac{m_1}{2}+m_2+\cdots+m_N=&\frac{m_1+2m_2}{2}+\frac{1}{3}\left(3m_3+3m_4+\cdots+3m_N\right)\\ \leq&\frac{m_1+2m_2}{2}+\frac{1}{3}\left(3m_3+4m_4+\cdots+Nm_N\right)\\ =&\frac{m_1+2m_2}{2}+\frac{1}{3}\left(N-(m_1+2m_2)\right)\\ =&\frac{N}{3}+\frac{m_1+2m_2}{6}.\end{aligned}$$

Thus, when $m_1+2m_2<N$, the above is at most $\frac{N}{2}-\frac{1}{6}$. Incorporating this in (9),

$$\begin{aligned}&\sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\ n_1+2n_2+\cdots+Nn_N=N}}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi^j d_{h_j})\overline{\mathcal{M}(n_j;\chi^j d_{h_j})}\\ =&\sum_{\substack{m_1+2m_2=N\\ n_1+2n_2=N}}\mathcal{M}(m_1;\chi d_{h_1})\overline{\mathcal{M}(n_1;\chi d_{h_1})}\mathcal{M}(m_2;\chi^2 d_{h_2})\overline{\mathcal{M}(n_2;\chi^2 d_{h_2})}+O\left(q^{N-\frac{1}{6}}\right).\end{aligned}$$

□

4. Sums over arithmetic progressions

In this section we prove Theorem 1.1. Let $Q\in\mathcal{H}$ and let $A\in\mathbb{F}_q[T]$ be coprime to Q. We consider the following sum

$$\mathcal{S}^U_{\mathfrak{g};N,Q}(A):=\sum_{\substack{f\in\mathcal{M}_N\\ f\equiv A\,(\mathrm{mod}\,Q)}}\mathfrak{g}(f).$$

4.1. **The average.** We use the orthogonality relation of Dirichlet characters (6) to isolate the arithmetic progression as in Section 4.1 of [KR16]. This gives

$$\begin{aligned}\mathcal{S}^U_{\mathfrak{g};N,Q}(A)=&\sum_{f\in\mathcal{M}_N}\mathfrak{g}(f)\left(\frac{1}{\Phi(Q)}\sum_{\chi\,(\mathrm{mod}\,Q)}\overline{\chi}(A)\chi(f)\right)\\ =&\frac{1}{\Phi(Q)}\sum_{\chi\,(\mathrm{mod}\,Q)}\overline{\chi}(A)\mathcal{M}(N;\chi\mathfrak{g})\\ =&\frac{1}{\Phi(Q)}\sum_{\substack{f\in\mathcal{M}_N\\ (f,Q)=1}}\mathfrak{g}(f)+\frac{1}{\Phi(Q)}\sum_{\substack{\chi\,(\mathrm{mod}\,Q)\\ \chi\neq\chi_0}}\overline{\chi}(A)\mathcal{M}(N;\chi\mathfrak{g}),\end{aligned}$$

where χ_0 denotes the trivial character modulo Q.

We have the following result.

Lemma 4.1.

$$\mathcal{S}^U_{\mathfrak{g};N,Q}(A)=\sum_{m_1+2m_2+\cdots+Nm_N=N}\ \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ f_1f_2^2\cdots f_N^N\equiv A\,(\mathrm{mod}\,Q)}}d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N).$$

Proof. By equation (8) and the previous expression, we have

$$\begin{aligned}
\mathcal{S}^{U}_{\mathfrak{g};N,Q}(A) =&\frac{1}{\Phi(Q)}\sum_{\chi \,(\mathrm{mod}\, Q)} \overline{\chi}(A)\mathcal{M}(N;\chi\mathfrak{g})\\
=&\frac{1}{\Phi(Q)}\sum_{\chi \,(\mathrm{mod}\, Q)} \overline{\chi}(A)\sum_{m_1+2m_2+\cdots+Nm_N=N}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi^j d_{h_j})\\
=&\frac{1}{\Phi(Q)}\sum_{\chi \,(\mathrm{mod}\, Q)} \overline{\chi}(A)\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}}}\prod_{j=1}^{N}\chi^j(f_j)d_{h_j}(f_j)\\
=&\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}}}\prod_{j=1}^{N}d_{h_j}(f_j)\frac{1}{\Phi(Q)}\sum_{\chi \,(\mathrm{mod}\, Q)}\overline{\chi}(A)\chi(f_1f_2^2\cdots f_N^N)\\
=&\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ f_1f_2^2\cdots f_N^N\equiv A\,(\mathrm{mod}\, Q)}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N),
\end{aligned}$$

where the last identity follows from orthogonality of Dirichlet characters (6). □

We also have the following average result.

Lemma 4.2. *As $q\to\infty$,*

$$\begin{aligned}
\mathcal{S}^{U}_{\mathfrak{g};N,Q}(A) =&\frac{1}{\Phi(Q)}\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ (f_1f_2\cdots f_N,Q)=1}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)+O\left(q^{\frac{N}{2}}\right)\\
=&\frac{q^N}{\Phi(Q)}\binom{\mathfrak{d}_1+N-1}{N}\left(1+O\left(q^{-1}\right)\right).
\end{aligned}$$

Proof. By equation (8) and Lemma 3.2, we have that, as $q \to \infty$,

$$\begin{aligned}
\sum_{\substack{f \in \mathcal{M}_N \\ (f,Q)=1}} \mathfrak{g}(f) =& \mathcal{M}(N; \chi_0 \mathfrak{g}) \\
=& \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1 \in \mathcal{M}_{m_1} \\ \cdots \\ f_N \in \mathcal{M}_{m_N} \\ (f_1 f_2 \cdots f_N, Q)=1}} d_{h_1}(f_1) d_{h_2}(f_2) \cdots d_{h_N}(f_N) \\
=& \sum_{m_1+2m_2+\cdots+Nm_N=N} \prod_{j=1}^{N} \mathcal{M}(m_j; \chi_0 d_{h_j}) \\
=& \sum_{m_1+2m_2+\cdots+Nm_N=N} q^{m_1+\cdots+m_N} \prod_{j=1}^{N} \binom{h_j+m_j-1}{m_j} \left(1+O\left(q^{-1}\right)\right) \\
=& q^N \binom{h_1+m_1-1}{m_1} \left(1+O\left(q^{-1}\right)\right), \qquad (10)
\end{aligned}$$

since the only way to maximize the contributions of the powers of q is to take $m_1 = N$ and $m_2 = \cdots = m_N = 0$.

Now equation (8) and Lemmas 3.1 and 3.2 imply that, for χ non-trivial,

$$\begin{aligned}
\mathcal{M}(N; \chi \mathfrak{g}) =& \sum_{m_1+2m_2+\cdots+Nm_N=N} \prod_{j=1}^{N} \mathcal{M}(m_j; \chi^j d_{h_j}) \\
=& O\left(\sum_{m_1+2m_2+\cdots+Nm_N=N} q^{\frac{m_1}{2}+m_2+\cdots+m_N} \right).
\end{aligned}$$

As in the proof of Proposition 3.3, we have that

$$\frac{m_1}{2} + m_2 + \cdots + m_N \leq \frac{N}{3} + \frac{m_1+2m_2}{6} \leq \frac{N}{2}.$$

Thus, we obtain

$$\mathcal{S}^U_{\mathfrak{g};N,Q}(A) = \frac{1}{\Phi(Q)} \sum_{\substack{f \in \mathcal{M}_N \\ (f,Q)=1}} \mathfrak{g}(f) + \frac{1}{\Phi(Q)} \sum_{\substack{\chi \,(\mathrm{mod}\, Q) \\ \chi \neq \chi_0}} \overline{\chi}(A) \mathcal{M}(N; \chi \mathfrak{g}) = \frac{1}{\Phi(Q)} \sum_{\substack{f \in \mathcal{M}_N \\ (f,Q)=1}} \mathfrak{g}(f) + O\left(q^{\frac{N}{2}}\right).$$

Combining the above with (10) and noticing that $h_1 = \mathfrak{d}_1$, we conclude. □

4.2. **The variance.** We now consider the variance, which is defined by

$$\mathrm{Var}(\mathcal{S}^U_{\mathfrak{g};N,Q}) := \frac{1}{\Phi(Q)} \sum_{\substack{A \,(\mathrm{mod}\, Q) \\ (A,Q)=1}} \left| \mathcal{S}^U_{\mathfrak{g};N,Q}(A) - \frac{1}{\Phi(Q)} \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1 \in \mathcal{M}_{m_1} \\ \cdots \\ f_N \in \mathcal{M}_{m_N} \\ (f_1 f_2 \cdots f_N, Q)=1}} d_{h_1}(f_1) d_{h_2}(f_2) \cdots d_{h_N}(f_N) \right|^2.$$

We prove the following result.

Theorem 4.3. *Let $Q \in \mathcal{H}_{\geq 2}$. Assume that $\mathfrak{d}_1$ is a positive integer and $N \leq \mathfrak{d}_1(\deg(Q)-1)$. As $q \to \infty$,*

$$\mathrm{Var}(\mathcal{S}^U_{\mathfrak{g};N,Q}) \sim q^{N-\deg(Q)} \int_{\mathrm{U}(\deg(Q)-1)} \left| \sum_{\substack{j_1+\cdots+j_{h_1}=N\\ 0\leq j_1,\dots,j_{h_1}\leq \deg(Q)-1}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_{h_1}}(U)\right|^2 dU.$$

Proof. By the discussion in the proof of Lemma 4.2 and (8), we have that

$$\mathrm{Var}(\mathcal{S}^U_{\mathfrak{g};N,Q}) := \frac{1}{\Phi(Q)} \sum_{\substack{A \,(\mathrm{mod}\, Q)\\ (A,Q)=1}} \left| \frac{1}{\Phi(Q)} \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{\chi \,(\mathrm{mod}\, Q)\\ \chi\neq\chi_0}} \overline{\chi}(A) \prod_{j=1}^{N} \mathcal{M}(m_j;\chi^j d_{h_j})\right|^2$$

$$= \frac{1}{\Phi(Q)} \sum_{\substack{A \,(\mathrm{mod}\, Q)\\ (A,Q)=1}} \frac{1}{\Phi(Q)^2}$$

$$\times \sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\ n_1+2n_2+\cdots+Nn_N=N}} \sum_{\substack{\chi_1,\chi_2 \,(\mathrm{mod}\, Q)\\ \chi_1,\chi_2\neq\chi_0}} \overline{\chi_1}(A)\chi_2(A) \prod_{j=1}^{N} \mathcal{M}(m_j;\chi_1^j d_{h_j})\overline{\mathcal{M}(n_j;\chi_2^j d_{h_j})}.$$

Applying orthogonality, we obtain

$$\frac{1}{\Phi(Q)} \sum_{\substack{A \,(\mathrm{mod}\, Q)\\ (A,Q)=1}} \overline{\chi_1}(A)\chi_2(A) = \begin{cases} 1 & \chi_1=\chi_2,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, this gives

$$\mathrm{Var}(\mathcal{S}^U_{\mathfrak{g};N,Q}) = \frac{1}{\Phi(Q)^2} \sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\ n_1+2n_2+\cdots+Nn_N=N}} \sum_{\substack{\chi \,(\mathrm{mod}\, Q)\\ \chi\neq\chi_0}} \prod_{j=1}^{N} \mathcal{M}(m_j;\chi^j d_{h_j})\overline{\mathcal{M}(n_j;\chi^j d_{h_j})}.$$

Now we apply Proposition 3.3 to get

$$\mathrm{Var}(\mathcal{S}^U_{\mathfrak{g};N,Q}) = \frac{1}{\Phi(Q)^2} \sum_{\substack{m_1+2m_2=N\\ n_1+2n_2=N}} \sum_{\substack{\chi \,(\mathrm{mod}\, Q)\\ \chi\neq\chi_0}} \mathcal{M}(m_1;\chi d_{h_1})\overline{\mathcal{M}(n_1;\chi d_{h_1})}\mathcal{M}(m_2;\chi^2 d_{h_2})\overline{\mathcal{M}(n_2;\chi^2 d_{h_2})}$$
$$+ O\left(q^{N-\frac{1}{6}-\deg(Q)}\right),$$

where we have used that $\Phi(Q) \sim |Q| = q^{\deg(Q)}$ as $q \to \infty$.

There are two cases in which the terms in the main term of the above sum are maximal with respect to powers of q (and lead to a final size of $q^{N-\deg(Q)}$). The first case is when $m_1 = n_1 = N$ and χ is arbitrary. The second case is when χ is a quadratic character, so that

$\chi^2 = \chi_0$. All the other terms will have size at most $O\left(q^{N-\frac{1}{2}-\deg(Q)}\right)$. Therefore, we have

(11)

$$\begin{aligned}\mathrm{Var}(\mathcal{S}^U_{\mathfrak{g};N,Q}) =& \frac{1}{\Phi(Q)^2} \sum_{\substack{\chi \,(\mathrm{mod}\, Q)\\ \chi\neq\chi_0}} |\mathcal{M}(N;\chi d_{h_1})|^2 \\ &+ \frac{1}{\Phi(Q)^2} \sum_{\substack{m_1+2m_2=N\\ n_1+2n_2=N\\ m_1<N \text{ or } n_1<N}} \sum_{\substack{\chi \,(\mathrm{mod}\, Q)\\ \chi^2=\chi_0\\ \chi\neq\chi_0}} \mathcal{M}(m_1;\chi d_{h_1})\mathcal{M}(n_1;\chi d_{h_1})\mathcal{M}(m_2;\chi_0 d_{h_2})\mathcal{M}(n_2;\chi_0 d_{h_2}) \\ &+ O\left(q^{N-\frac{1}{6}-\deg(Q)}\right).\end{aligned}$$

We proceed to bound the second sum above. To do this, we estimate the number of quadratic residues modulo Q, for Q square-free. This is equivalent to the number of conductors, which is $d_2(Q) = 2^{\omega(Q)}$. The expected size is then

$$\frac{2^{\omega(Q)}}{\Phi(Q)^2} q^N \ll q^{N+(\varepsilon-2)\deg(Q)}.$$

The first sum in (11) was estimated by Keating et al (sum (3.12) in [KRRGR18]). First one discards the contribution of even and non-primitive characters, then one applies Lemma 3.1, where the nontrivial condition is $N \leq h_1(\deg(Q)-1)$. This gives

$$\begin{aligned}\mathrm{Var}(\mathcal{S}^U_{\mathfrak{g};N,Q}) =& \frac{1}{\Phi(Q)^2} \sum_{\substack{\chi \,(\mathrm{mod}\, Q)\\ \chi\neq\chi_0 \text{ odd primitive}}} |\mathcal{M}(N;\chi d_{h_1})|^2 + O\left(q^{N-\frac{1}{6}-\deg(Q)}\right) \\ \sim& q^{N-\deg(Q)} \int_{\mathrm{U}(\deg(Q)-1)} \left| \sum_{\substack{j_1+\cdots+j_{h_1}=N\\ 0\leq j_1,\ldots,j_{h_1}\leq \deg(Q)-1}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_{h_1}}(U)\right|^2 dU,\end{aligned}$$

where the last identity follows from applying Katz's equidistribution theorem [Kat13b], which requires Q to be square-free and $2 \leq \deg(Q)$.

□

5. Sums over short intervals

In this section we prove Theorem 1.2. Given $A \in \mathcal{M}_N$ and $0 \leq \mathfrak{h} \leq N-2$, we recall that

$$I_{\mathfrak{h}}(A) := \{f \in \mathbb{F}_q[T] \,:\, |f-A| \leq q^{\mathfrak{h}}\}$$

is the interval of radius $\mathfrak{h}$ centered in A. Consider

$$\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}(A) := \sum_{f\in I_{\mathfrak{h}}(A)} \mathfrak{g}(f).$$

5.1. **The average.** We have the following result.

Lemma 5.1. *As $q \to \infty$,*

$$\begin{aligned}\frac{1}{q^N}\sum_{A\in\mathcal{M}_N}\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}(A) =&\frac{q^{\mathfrak{h}+1}}{q^N}\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\\ =&q^{\mathfrak{h}+1}\binom{\mathfrak{d}_1+N-1}{N}\left(1+O\left(q^{-1}\right)\right).\end{aligned}$$

Proof. Adding over all the possible A's means that we are adding over all the $f \in \mathcal{M}_N$, and that each f is counted $q^{\mathfrak{h}+1}$ times, since there are $q^{\mathfrak{h}+1}$ possible A's to which f is close, as the lowest $\mathfrak{h}+1$ coefficients are completely free. Applying (8) in the trivial case, together with Lemma 3.2, the mean value is given by

$$\begin{aligned}\frac{1}{q^N}\sum_{A\in\mathcal{M}_N}\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}(A) =&\frac{q^{\mathfrak{h}+1}}{q^N}\sum_{f\in\mathcal{M}_N}\mathfrak{g}(f)\\ =&\frac{q^{\mathfrak{h}+1}}{q^N}\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\\ =&q^{\mathfrak{h}+1-N}\sum_{m_1+2m_2+\cdots+Nm_N=N} q^{m_1+\cdots+m_N}\prod_{j=1}^N\binom{h_j+m_j-1}{m_j}\\ =&q^{\mathfrak{h}+1}\binom{h_1+N-1}{N}\left(1+O\left(q^{-1}\right)\right),\end{aligned}$$

since the only way to maximize the contributions of the powers of q is to take $m_1 = N$ and $m_2 = \cdots = m_N = 0$. ☐

5.2. **The variance.** Our goal here is to compute the variance

$$\begin{aligned}&\mathrm{Var}(\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}})\\ &:=\frac{1}{q^N}\sum_{A\in\mathcal{M}_N}\left|\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}(A)-\frac{q^{\mathfrak{h}+1}}{q^N}\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\right|^2.\end{aligned}$$

In order to compute the variance we need some preliminary results. A function $\alpha : \mathbb{F}_q[T] \to \mathbb{C}$ is called even if $\alpha(cf) = \alpha(f)$ for any nonzero $c \in \mathbb{F}_q$, and symmetric if $\alpha(f) = \alpha(t^{\deg(f)}f(1/t))$. Notice that the functions $\mathfrak{g}$ under our consideration satisfy both properties.

We need the following result of Keating and Rudnick.

Lemma 5.2. [KR16, Lemma 5.3] *If $\alpha : \mathbb{F}_q[T] \to \mathbb{C}$ is even, symmetric, and multiplicative, and $0 \le \mathfrak{h} \le n-2$, then for all $B \in \mathcal{M}_{n-\mathfrak{h}-1}$,*

$$\mathcal{N}^U_{\alpha;\mathfrak{h}}(T^{\mathfrak{h}+1}B) =\langle\mathcal{N}^U_{\alpha;\mathfrak{h}}\rangle+\frac{1}{\Phi_{even}(T^{n-\mathfrak{h}})}\sum_{m=0}^n\alpha(T^{n-m})\sum_{\substack{\chi\,(\mathrm{mod}\,T^{n-\mathfrak{h}})\\ \chi\neq\chi_0\ even}}\overline{\chi}(\theta_{n-\mathfrak{h}-1}(B))\mathcal{M}(m;\chi\alpha),$$

where

$$\langle \mathcal{N}^U_{\alpha;\mathfrak{h}} \rangle = \frac{q^{\mathfrak{h}+1}}{q^n} \sum_{f \in \mathcal{M}_n} \alpha(f)$$

denotes the average, and $\theta_n : \mathbb{F}_q[T]_{\leq n} \to \mathbb{F}_q[T]_{\leq n}$, *is given by*

$$\theta_n(f)(T) = T^n f\left(\frac{1}{T}\right).$$

One can apply the above Lemma by observing that if $A(T) \in \mathcal{M}_N$, then

$$\mathcal{N}^U_{\alpha;\mathfrak{h}}(A) = \mathcal{N}^U_{\alpha;\mathfrak{h}}(T^{\mathfrak{h}+1}B), \tag{12}$$

where B is the polynomial of degree $n-\mathfrak{h}-1$ that is made of the highest $n-\mathfrak{h}-1$ coefficients of A. Notice that there are $q^{\mathfrak{h}+1}$ polynomials A corresponding to each B.

We are now ready to prove the following result.

Theorem 5.3. *Suppose that* $\mathfrak{d}_1$ *is a positive integer and* $0 \leq \mathfrak{h} \leq \min\{N-5, (1-\frac{1}{\mathfrak{d}_1})N-2\}$. *Suppose also that* $\mathfrak{d}_k = o(q^\varepsilon)$ *for any* $\varepsilon > 0$. *Then, as* $q \to \infty$,

$$\mathrm{Var}(\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}) \sim q^{\mathfrak{h}+1} \int_{\mathrm{U}(N-\mathfrak{h}-2)} \Bigg| \sum_{\substack{j_1+\cdots+j_{h_1}=N \\ 0\leq j_1,\ldots,j_{h_1}\leq N-\mathfrak{h}-2}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_{h_1}}(U) \Bigg|^2 dU.$$

Proof. We apply Lemma 5.2 with B as in equation (12). This gives

$$\begin{aligned}
&\mathrm{Var}(\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}) \\
&= \frac{q^{\mathfrak{h}+1}}{q^N} \sum_{B\in\mathcal{M}_{N-\mathfrak{h}-1}} \frac{1}{\Phi_{\mathrm{even}}(T^{N-\mathfrak{h}})^2} \Bigg| \sum_{m=0}^{N} \mathfrak{d}_{N-m} \sum_{\substack{\chi \,(\mathrm{mod}\, T^{N-\mathfrak{h}}) \\ \chi\neq\chi_0 \text{ even}}} \overline{\chi}(\theta_{N-\mathfrak{h}-1}(B)) \mathcal{M}(m;\chi\mathfrak{g}) \Bigg|^2 \\
&= \frac{q^{\mathfrak{h}+1}}{q^N} \frac{1}{\Phi_{\mathrm{even}}(T^{N-\mathfrak{h}})^2} \sum_{m,n=0}^{N} \mathfrak{d}_{N-m}\overline{\mathfrak{d}_{N-n}} \sum_{\substack{\chi_1,\chi_2 \,(\mathrm{mod}\, T^{N-\mathfrak{h}}) \\ \chi_1,\chi_2\neq\chi_0 \text{ even}}} \mathcal{M}(m;\chi_1\mathfrak{g})\overline{\mathcal{M}(n;\chi_2\mathfrak{g})} \\
&\times \sum_{B\in\mathcal{M}_{N-\mathfrak{h}-1}} \overline{\chi_1}(\theta_{N-\mathfrak{h}-1}(B))\chi_2(\theta_{N-\mathfrak{h}-1}(B)).
\end{aligned}$$

As in [KR16] (equations (5.34) and (5.35)), one can see that

$$\frac{q^{\mathfrak{h}+1}}{q^N} \sum_{B\in\mathcal{M}_{N-\mathfrak{h}-1}} \overline{\chi_1}(\theta_{N-\mathfrak{h}-1}(B))\chi_2(\theta_{N-\mathfrak{h}-1}(B)) = \begin{cases} 1 & \chi_1=\chi_2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$\begin{aligned}\mathrm{Var}(\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}) =&\frac{1}{\Phi_{\text{even}}(T^{N-\mathfrak{h}})^2}\sum_{m,n=0}^{N}\mathfrak{d}_{N-m}\overline{\mathfrak{d}_{N-n}}\sum_{\substack{\chi \,(\mathrm{mod}\, T^{N-\mathfrak{h}})\\ \chi\neq\chi_0 \text{ even}}}\mathcal{M}(m;\chi\mathfrak{g})\overline{\mathcal{M}(n;\chi\mathfrak{g})}\\ =&\frac{1}{\Phi_{\text{even}}(T^{N-\mathfrak{h}})^2}\sum_{\substack{m_1+2m_2+\cdots+Nm_N\leq N\\ n_1+2n_2+\cdots+Nn_N\leq N}}\mathfrak{d}_{N-\sum jm_j}\mathfrak{d}_{N-\sum jn_j}\\ &\times\sum_{\substack{\chi \,(\mathrm{mod}\, T^{N-\mathfrak{h}})\\ \chi\neq\chi_0 \text{ even}}}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi^j d_{h_j})\overline{\mathcal{M}(n_j;\chi^j d_{h_j})},\end{aligned}$$

where we have applied equation (8).

Now we apply Proposition 3.3 and make use of the fact that $\mathfrak{d}=o(q^{\varepsilon})$. This leads to

$$\begin{aligned}\mathrm{Var}(\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}) =&\frac{1}{\Phi_{\text{even}}(T^{N-\mathfrak{h}})^2}\sum_{\substack{m_1+2m_2=N\\ n_1+2n_2=N}}\sum_{\substack{\chi \,(\mathrm{mod}\, T^{N-\mathfrak{h}})\\ \chi\neq\chi_0 \text{ even}}}\mathcal{M}(m_1;\chi d_{h_1})\overline{\mathcal{M}(n_1;\chi d_{h_1})}\mathcal{M}(m_2;\chi^2 d_{h_2})\overline{\mathcal{M}(n_2;\chi^2 d_{h_2})}\\ &+O\left(q^{\mathfrak{h}+\frac{5}{6}}\right),\end{aligned}$$

where we have applied the fact that that $\Phi_{\text{even}}(T^{N-\mathfrak{h}})=\frac{1}{q-1}\Phi(T^{N-\mathfrak{h}})\sim q^{N-\mathfrak{h}-1}$ as $q\to\infty$.

As in the proof of Theorem 4.3, we can consider the case of $m_1=n_1=N$ separately from the rest. For the non-quadratic characters, the cases $m_1<N$ or $n_1<N$ contribute at most $O\left(q^{\mathfrak{h}+\frac{1}{2}}\right)$. For the quadratic characters, the contribution is bounded by $O\left(q^{2\mathfrak{h}+2+(\varepsilon-1)N}\right)$. Following [KRRGR18], one can discard the non-primitive characters, then apply Lemma 3.1, where the nontrivial condition is $N\leq h_1(N-\mathfrak{h}-2)$ (since for primitive even characters modulo T^n, the degree of the L-function is $n-2$). This leads to

$$\begin{aligned}\mathrm{Var}(\mathcal{N}^U_{\mathfrak{g};\mathfrak{h}}) =&\frac{q^{\mathfrak{h}+1-N}}{\Phi_{\text{even primitive}}(T^{N-\mathfrak{h}})}\sum_{\substack{\chi \,(\mathrm{mod}\, T^{N-\mathfrak{h}})\\ \chi\neq\chi_0 \text{ even primitive}}}|\mathcal{M}(N;\chi d_{h_1})|^2+O\left(q^{\mathfrak{h}+\frac{5}{6}}\right)\\ \sim& q^{\mathfrak{h}+1}\int_{\mathrm{U}(N-\mathfrak{h}-2)}\left|\sum_{\substack{j_1+\cdots+j_{h_1}=N\\ 0\leq j_1,\dots,j_{h_1}\leq N-\mathfrak{h}-2}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_{h_1}}(U)\right|^2 dU,\end{aligned}$$

where the last identity follows from an equidistribution result of Katz [Kat13a], which applies to $5\leq N-\mathfrak{h}$. □

6. Sums over squares modulo P

In this section we prove Theorem 1.3. For simplicity, we fix $P\in\mathcal{P}_{2g+1}$ of odd degree. Thus, the corresponding character χ_P will be odd. Our aim is to study the following sum.

$$\mathcal{S}^S_{\mathfrak{g};N}(P):=\sum_{\substack{f\in\mathcal{M}_N\\ f\equiv\square \,(\mathrm{mod}\, P)\\ P\nmid f}}\mathfrak{g}(f).$$

6.1. **The average.** Recalling that if $P \nmid f$, then f is a square modulo P if and only if $1+\chi_P(f)=2$, we obtain

$$\mathcal{S}^S_{\mathfrak{g};N}(P) := \frac{1}{2}\sum_{\substack{f\in\mathcal{M}_N\\ P\nmid f}} \mathfrak{g}(f) + \frac{1}{2}\mathcal{M}(N;\chi_P\mathfrak{g}).$$

Denoting by χ_0 the trivial character modulo P, we have, by (8),

$$\frac{1}{2}\sum_{\substack{f\in\mathcal{M}_N\\ P\nmid f}} \mathfrak{g}(f) + \frac{1}{2}\mathcal{M}(N;\chi_P\mathfrak{g}) = \frac{1}{2}\mathcal{M}(N;\chi_0\mathfrak{g}) + \frac{1}{2}\mathcal{M}(N;\chi_P\mathfrak{g})$$

$$= \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\frac{1}{2}\Big(\chi_0(f_1f_2^2\cdots f_N^N)+\chi_P(f_1f_2^2\cdots f_N^N)\Big).$$

Thus, we immediately conclude the following result.

Lemma 6.1.

$$\mathcal{S}^S_{\mathfrak{g};N}(P) = \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ f_1f_2^2\cdots f_N^N\equiv\square \,(\mathrm{mod}\, P)\\ P\nmid f_1f_2\cdots f_N}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N).$$

We can also obtain the following average result.

Lemma 6.2. *As $q\to\infty$,*

$$\begin{aligned}\mathcal{S}^S_{\mathfrak{g},N}(P) =& \frac{1}{2}\sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ P\nmid f_1f_2\cdots f_N}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N) + O\left(q^{\frac{N}{2}}\right)\\ =&\frac{q^N}{2}\binom{\mathfrak{d}_1+N-1}{N}\left(1+O\left(q^{-1}\right)\right).\end{aligned}$$

Proof. By equation (8) and Lemma 3.2, we have that, as $q\to\infty$

$$\begin{aligned}\mathcal{M}(N;\chi_0\mathfrak{g}) =& \sum_{m_1+2m_2+\cdots+Nm_N=N} \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ P\nmid f_1\cdots f_N}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\\ =& \sum_{m_1+2m_2+\cdots+Nm_N=N} \prod_{j=1}^{N}\mathcal{M}(m_j;\chi_0 d_{h_j})\\ =& \sum_{m_1+2m_2+\cdots+Nm_N=N} q^{m_1+\cdots+m_N}\prod_{j=1}^{N}\binom{h_j+m_j-1}{m_j}\left(1+O\left(q^{-1}\right)\right)\\ =&q^N\binom{h_1+m_1-1}{m_1}\left(1+O\left(q^{-1}\right)\right),\end{aligned}\tag{13}$$

since the only way to obtain a maximal contribution for powers of q is to take $m_1 = N$ and $m_2 = \cdots = m_N = 0$.

Notice that equation (8) together with Lemmas 3.1 and 3.2 imply that, since χ_P is non-trivial,

$$\begin{aligned}\mathcal{M}(N;\chi_P\mathfrak{g}) =& \sum_{m_1+2m_2+\cdots+Nm_N=N}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi_P^j d_{h_j})\\ =&O\left(\sum_{m_1+2m_2+\cdots+Nm_N=N} q^{\frac{m_1}{2}+m_2+\cdots+m_N}\right).\end{aligned}$$

As in the proof of Proposition 3.3, we have that

$$\frac{m_1}{2}+m_2+\cdots+m_N \leq \frac{N}{3}+\frac{m_1+2m_2}{6}\leq \frac{N}{2}.$$

Thus, we obtain that

$$\mathcal{S}^S_{\mathfrak{g},N}(P) = \frac{1}{2}\mathcal{M}(N;\chi_0\mathfrak{g})+\frac{1}{2}\mathcal{M}(N;\chi_P\mathfrak{g}) = \frac{1}{2}\mathcal{M}(N;\chi_0\mathfrak{g}) + O\left(q^{\frac{N}{2}}\right).$$

We conclude by combining the above with (13) and by noticing that $h_1 = \mathfrak{d}_1$.

□

6.2. **The variance.** We now consider the variance, which is defined by

$$\mathrm{Var}(\mathcal{S}^S_{\mathfrak{g};N}) := \frac{1}{\#\mathcal{P}_{2g+1}}\sum_{P\in\mathcal{P}_{2g+1}}\left(\mathcal{S}^S_{\mathfrak{g};N}(P) - \frac{1}{2}\sum_{m_1+2m_2+\cdots+Nm_N=N}\sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ P\nmid f_1f_2\cdots f_N}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\right)^2.$$

We prove the following result.

Theorem 6.3. *Assume that $\mathfrak{d}_1$ is a positive integer and that $N \leq 2g\mathfrak{d}_1$. As $q \to \infty$,*

$$\mathrm{Var}(\mathcal{S}^S_{\mathfrak{g};N}) \sim \frac{q^N}{4}\int_{\mathrm{Sp}(2g)}\left(\sum_{\ell=0}^{\lfloor\frac{N}{2}\rfloor}\binom{-h_2}{\ell}(-1)^\ell \sum_{\substack{j_1+\cdots+j_{h_1}=N-2\ell\\ 0\leq j_1,\ldots,j_{h_1}\leq 2g}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_{h_1}}(U)\right)^2 dU.$$

Proof. By the discussion from the proof of Lemma 6.2, we have that

$$\begin{aligned}\mathrm{Var}(\mathcal{S}^S_{\mathfrak{g};N}) =& \frac{1}{4\#\mathcal{P}_{2g+1}}\mathcal{M}(N;\chi_P\mathfrak{g})^2\\ =& \frac{1}{4\#\mathcal{P}_{2g+1}}\sum_{P\in\mathcal{P}_{2g+1}}\left(\sum_{m_1+2m_2+\cdots+Nm_N=N}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi_P^j d_{h_j})\right)^2.\end{aligned}$$

We apply Proposition 3.3 to obtain

$$\begin{aligned}\mathrm{Var}(\mathcal{S}^S_{\mathfrak{g};N}) =&\frac{1}{4\#\mathcal{P}_{2g+1}}\sum_{P\in\mathcal{P}_{2g+1}}\sum_{\substack{m_1+2m_2=N\\ n_1+2n_2=N}}\mathcal{M}(m_1;\chi_P d_{h_1})\mathcal{M}(n_1;\chi_P d_{h_1})\mathcal{M}(m_2;\chi_P^2 d_{h_2})\mathcal{M}(n_2;\chi_P^2 d_{h_2})\\ &+O\left(q^{N-\frac{1}{6}}\right).\end{aligned}$$

By Lemmas 3.1 and 3.2, the inner sum above becomes

$$\left(\sum_{m_1+2m_2=N}(-1)^{m_1}q^{\frac{m_1}{2}}\sum_{\substack{j_1+\cdots+j_{h_1}=m_1\\0\le j_1,\ldots,j_{h_1}\le 2g}}\mathrm{Sc}_{j_1}(\Theta_{\chi_P})\cdots\mathrm{Sc}_{j_{h_1}}(\Theta_{\chi_P})\left(q^{m_2}\binom{h_2+m_2-1}{m_2}+O(q^{m_2-1})\right)\right)^2$$

$$+O\left(q^{N-\frac{1}{6}}\right)$$

$$=q^N\left(\sum_{m_1+2m_2=N}(-1)^{m_1}\sum_{\substack{j_1+\cdots+j_{h_1}=m_1\\0\le j_1,\ldots,j_{h_1}\le 2g}}\mathrm{Sc}_{j_1}(\Theta_{\chi_P})\cdots\mathrm{Sc}_{j_{h_1}}(\Theta_{\chi_P})\binom{h_2+m_2-1}{m_2}\right)^2\left(1+O\left(q^{-\frac{1}{6}}\right)\right).$$

Thus, we obtain

$$\mathrm{Var}(\mathcal{S}^S_{\mathfrak{g};N}):=\frac{q^N}{4\#\mathcal{P}_{2g+1}}$$

$$\times\sum_{P\in\mathcal{P}_{2g+1}}\left((-1)^N\sum_{m_1+2m_2=N}\sum_{\substack{j_1+\cdots+j_{h_1}=m_1\\0\le j_1,\ldots,j_{h_1}\le 2g}}\mathrm{Sc}_{j_1}(\Theta_{\chi_P})\cdots\mathrm{Sc}_{j_{h_1}}(\Theta_{\chi_P})\binom{h_2+m_2-1}{m_2}\right)^2$$

$$\times\left(1+O\left(q^{-\frac{1}{6}}\right)\right)$$

$$=\frac{q^N}{4\#\mathcal{P}_{2g+1}}\sum_{P\in\mathcal{P}_{2g+1}}\left(\sum_{\ell=0}^{\lfloor\frac{N}{2}\rfloor}\binom{h_2+\ell-1}{\ell}\sum_{\substack{j_1+\cdots+j_{h_1}=N-2\ell\\0\le j_1,\ldots,j_{h_1}\le 2g}}\mathrm{Sc}_{j_1}(\Theta_{\chi_P})\cdots\mathrm{Sc}_{j_{h_1}}(\Theta_{\chi_P})\right)^2$$

$$\times\left(1+O\left(q^{-\frac{1}{6}}\right)\right).$$

We conclude by invoking the following equidistribution result of Katz. □

Theorem 6.4. [KL22a, Theorem 3.2]*Let F be a continuous $\mathbb{C}$-valued central function on* $\mathrm{Sp}(2g)$ *and let σ be any fixed partition of $2g+1$. Then*

$$\lim_{q\to\infty}\frac{1}{\#\mathcal{P}_\sigma}\sum_{Q\in\mathcal{P}_\sigma}F(Q)=\int_{\mathrm{Sp}(2g)}F(U)dU.$$

7. Sums over fundamental discriminants

Here we prove Theorem 1.4 by considering, for $R\in\mathcal{H}_{2g+1}$,

$$\mathcal{T}^S_{\mathfrak{g};N}(R):=\sum_{\substack{f\in\mathcal{M}_N\\(f,R)=1}}\mathfrak{g}(f)\chi_R(f).$$

7.1. **The average and the variance.** Our first observation is that the average is very small. By equation (8),

$$\mathcal{T}^S_{\mathfrak{g};N}(R)=\sum_{m_1+2m_2+\cdots+Nm_N=N}\prod_{j=1}^{N}\mathcal{M}(m_j;\chi_R^jd_{h_j})=O\left(q^{\frac{N}{2}}\right),\tag{14}$$

as in the estimate for $\mathcal{M}(N, \chi_P \mathfrak{g})$ in Subsection 6.1.

Given the size of the average, we consider

$$\mathrm{Var}(\mathcal{T}^S_{\mathfrak{g};N}) := \frac{1}{\#\mathcal{H}_{2g+1}} \sum_{R \in \mathcal{H}_{2g+1}} \left(\mathcal{T}^S_{\mathfrak{g};N}(R) \right)^2 .$$

As in the case of Theorem 1.3, we obtain a contribution that depends on h_2.

Theorem 7.1. *Assume that $\mathfrak{d}_1$ is a positive integer and that $N \leq 2g\mathfrak{d}_1$. As $q \to \infty$,*

$$\mathrm{Var}(\mathcal{T}^S_{\mathfrak{g};N}) \sim q^N \int_{\mathrm{Sp}(2g)} \left(\sum_{\ell=0}^{\lfloor \frac{N}{2} \rfloor} \binom{-h_2}{\ell} (-1)^\ell \sum_{\substack{j_1+\cdots+j_{h_1}=N-2\ell \\ 0 \leq j_1,\ldots,j_{h_1} \leq 2g}} \mathrm{Sc}_{j_1}(U) \cdots \mathrm{Sc}_{j_{h_1}}(U) \right)^2 dU.$$

Proof. By (14), we have that

$$\mathrm{Var}(\mathcal{T}^S_{\mathfrak{g};N}) = \frac{1}{\#\mathcal{H}_{2g+1}} \sum_{R \in \mathcal{H}_{2g+1}} \left(\sum_{m_1+2m_2+\cdots+Nm_N=N} \prod_{j=1}^{N} \mathcal{M}(m_j; \chi_R^j d_{h_j}) \right)^2 .$$

By Proposition 3.3,

$$\begin{aligned}\mathrm{Var}(\mathcal{T}^S_{\mathfrak{g};N}) = &\frac{1}{\#\mathcal{H}_{2g+1}} \sum_{R \in \mathcal{H}_{2g+1}} \sum_{\substack{m_1+2m_2=N \\ n_1+2n_2=N}} \mathcal{M}(m_1; \chi_R d_{h_1}) \mathcal{M}(n_1; \chi_R d_{h_1}) \mathcal{M}(m_2; \chi_R^2 d_{h_2}) \mathcal{M}(n_2; \chi_R^2 d_{h_2}) \\ &+ O\left(q^{N-\frac{1}{6}}\right).\end{aligned}$$

Applying Lemmas 3.1 and 3.2, the inner sum above becomes

$$\begin{aligned}&\left(\sum_{m_1+2m_2=N} (-1)^{m_1} q^{\frac{m_1}{2}} \sum_{\substack{j_1+\cdots+j_{h_1}=m_1 \\ 0 \leq j_1,\ldots,j_{h_1} \leq 2g}} \mathrm{Sc}_{j_1}(\Theta_{\chi_R}) \cdots \mathrm{Sc}_{j_{h_1}}(\Theta_{\chi_R}) \left(q^{m_2} \binom{h_2+m_2-1}{m_2} + O(q^{m_2-1}) \right) \right)^2 \\ &+ O\left(q^{N-\frac{1}{6}}\right) \\ &= q^N \left(\sum_{m_1+2m_2=N} (-1)^{m_1} \sum_{\substack{j_1+\cdots+j_{h_1}=m_1 \\ 0 \leq j_1,\ldots,j_{h_1} \leq 2g}} \mathrm{Sc}_{j_1}(\Theta_{\chi_R}) \cdots \mathrm{Sc}_{j_{h_1}}(\Theta_{\chi_R}) \binom{h_2+m_2-1}{m_2} \right)^2 \left(1 + O\left(q^{-\frac{1}{6}}\right)\right).\end{aligned}$$

Thus, we have

$$\operatorname{Var}(\mathcal{T}^S_{\mathfrak{g};N}) := \frac{q^N}{\#\mathcal{H}_{2g+1}}$$
$$\times \sum_{R\in\mathcal{H}_{2g+1}} \left((-1)^N \sum_{m_1+2m_2=N} \sum_{\substack{j_1+\cdots+j_{h_1}=m_1\\ 0\le j_1,\dots,j_{h_1}\le 2g}} \mathrm{Sc}_{j_1}(\Theta_{\chi_R})\cdots \mathrm{Sc}_{j_{h_1}}(\Theta_{\chi_R}) \binom{h_2+m_2-1}{m_2} \right)^2$$
$$\times \left(1+O\left(q^{-\frac{1}{6}}\right)\right)$$
$$= \frac{q^N}{\#\mathcal{H}_{2g+1}} \sum_{R\in\mathcal{H}_{2g+1}} \left(\sum_{\ell=0}^{\lfloor \frac{N}{2}\rfloor} \binom{h_2+\ell-1}{\ell} \sum_{\substack{j_1+\cdots+j_{h_1}=N-2\ell\\ 0\le j_1,\dots,j_{h_1}\le 2g}} \mathrm{Sc}_{j_1}(\Theta_{\chi_R})\cdots \mathrm{Sc}_{j_{h_1}}(\Theta_{\chi_R}) \right)^2$$
$$\times \left(1+O\left(q^{-\frac{1}{6}}\right)\right).$$

We conclude by standard equidistribution results for the hyperelliptic ensemble. □

8. Analyzing the integral

Consider the integral

$$\int_{\mathrm{Sp}(2g)} \frac{\det(1+Ux)^{h_1}}{(1-x^2)^{h_2}} \frac{\det(1+Uy)^{h_1}}{(1-y^2)^{h_2}} dU = \sum_{m,n=0}^{\infty} x^m y^n K^S_{d_{h_1},2}(m,n;g). \tag{15}$$

The diagonal coefficients on the right hand side are then given by

$$K^S_{d_{h_1},2}(N,N;g) := \int_{\mathrm{Sp}(2g)} \left(\sum_{\ell=0}^{\lfloor \frac{N}{2}\rfloor} \binom{-h_2}{\ell} (-1)^\ell \sum_{\substack{j_1+\cdots+j_{h_1}=N-2\ell\\ 0\le j_1,\dots,j_{h_1}\le 2g}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_{h_1}}(U) \right)^2 dU,$$

which is the integral appearing in Theorems 6.3 and 7.1.

Our goal is to find a formula for $K^S_{d_{h_1},2}(N,N;g)$. We can give a partial result.

Proposition 8.1. *Let $N \le g + \frac{1+h_1}{2}$. We have*

$$K^S_{d_{h_1},2}(N,N;g) = \sum_{\ell=0}^{\lfloor \frac{N}{2}\rfloor} \binom{\ell+h_2+\binom{h_1+1}{2}-1}{\ell}^2 \binom{N-2\ell+h_1^2-1}{N-2\ell}. \tag{16}$$

We will need the following lemma, which is a particular case of the generalized Vandermonde matrix in [Kal84] (see also [KL22b, Lemma 2.2]).

Lemma 8.2. *For any positive integer k,*

$$\det_{\substack{1\le i_1,i_2,\le k\\ 1\le j\le 2k}} \begin{bmatrix} \frac{(j-1)!}{(j-i_1)!} x^{j-i_1} \\ \\ \frac{(j-1)!}{(j-i_2)!} y^{j-i_2} \end{bmatrix} = G(1+k)^2 (y-x)^{k^2},$$

where $G(1+k) = 0!\cdot 1! \cdots (k-1)!$ is the Barnes G-function.

Proof of Proposition 8.1. Using results of Medjedovic [Med21] and the Désarménien–Stembridge–Proctor formula [BK72, BW16, Pro90, BG06], the authors of [KL22b] obtain that

$$\int_{\mathrm{Sp}(2g)} \det(1+Ux)^h \det(1+Uy)^h dU = \frac{1}{(1-x^2)^{\binom{h+1}{2}}(1-y^2)^{\binom{h+1}{2}}(1-xy)^{h^2}(y-x)^{h^2}} \times \det_{\substack{1\le i_1,i_2\le h\\ 1\le j\le 2h}} \begin{bmatrix} \binom{2g+4h+1-j}{i_1-1}x^{2g+4h+2-j-i_1} - \binom{j-1}{i_1-1}x^{j-i_1} \\ \\ \binom{2g+4h+1-j}{i_2-1}y^{2g+4h+2-j-i_2} - \binom{j-1}{i_2-1}y^{j-i_2} \end{bmatrix}.$$

Combining the above with (15), we immediately see that

$$\sum_{m,n=0}^{2gh_1} x^m y^n K^S_{d_{h_1},2}(m,n;g) = \frac{1}{(1-x^2)^{h_2+\binom{h_1+1}{2}}(1-y^2)^{h_2+\binom{h_1+1}{2}}(1-xy)^{h_1^2}(y-x)^{h_1^2}} \times \det_{\substack{1\le i_1,i_2\le h_1\\ 1\le j\le 2h_1}} \begin{bmatrix} \binom{2g+4h_1+1-j}{i_1-1}x^{2g+4h_1+2-j-i_1} - \binom{j-1}{i_1-1}x^{j-i_1} \\ \\ \binom{2g+4h_1+1-j}{i_2-1}y^{2g+4h_1+2-j-i_2} - \binom{j-1}{i_2-1}y^{j-i_2} \end{bmatrix}.$$

From the right-hand side of the above equation, the contribution to $K^S_{d_{h_1},2}(N,N;g)$, that is, the coefficient of $x^N y^N$, for $N \le g + \frac{1+h_1}{2}$, comes from the determinant that takes exclusively terms of the form $\binom{j-1}{i-1}x^{j-i}$ and similarly with y. In other words, $K^S_{d_{h_1},2}(N,N;g)$ comes exclusively from the diagonal terms of

$$\frac{1}{(1-x^2)^{h_2+\binom{h_1+1}{2}}(1-y^2)^{h_2+\binom{h_1+1}{2}}(1-xy)^{h_1^2}(y-x)^{h_1^2}} \det_{\substack{1\le i_1,i_2\le h_1\\ 1\le j\le 2h_1}} \begin{bmatrix} \binom{j-1}{i_1-1}x^{j-i_1} \\ \\ \binom{j-1}{i_2-1}y^{j-i_2} \end{bmatrix}.$$

Lemma 8.2 implies that we should consider the diagonal terms in

$$\frac{1}{(1-x^2)^{h_2+\binom{h_1+1}{2}}(1-y^2)^{h_2+\binom{h_1+1}{2}}(1-xy)^{h_1^2}}$$
$$= \sum_{\ell_1=0}^{\infty}\binom{\ell_1+h_2+\binom{h_1+1}{2}-1}{\ell_1}x^{2\ell_1}\sum_{\ell_2=0}^{\infty}\binom{\ell_2+h_2+\binom{h_1+1}{2}-1}{\ell_2}y^{2\ell_2}\sum_{m=0}^{\infty}\binom{m+h_1^2-1}{h_1^2-1}(xy)^m.$$

The result follows by fixing the coefficient of $(xy)^N$ with $N = m+2\ell_1 = m+2\ell_2$. □

In the case where h_2 is an integer satisfying $h_2 + \binom{h_1+1}{2} - 1 \ge 0$, equation (16) gives a quasi-polynomial in N, since the upper bound in the sum depends on the parity of N. Each term in the sum has total degree $2\left(h_2+\binom{h_1+1}{2}-1\right)+h_1^2-1 = 2h_2+2h_1^2+h_1-3$ in the variable ℓ. By summing over ℓ, each term leads to a polynomial of degree $2h_2+2h_1^2+h_1-2$ on N, or more precisely, $\left\lfloor \frac{N}{2}\right\rfloor$. Moreover, the coefficients are all positive, which guarantees that there is no cancellation that could lower the final degree.

Now we focus on the case $h_1 = 1$. Equation (16) gives

$$K^S_{d_1,2}(N,N;g) = \sum_{\ell=0}^{\lfloor \frac{N}{2}\rfloor}\binom{\ell+h_2}{\ell}^2 = \sum_{\ell=0}^{\lfloor \frac{N}{2}\rfloor}\binom{-1-h_2}{\ell}^2.$$

Thus, for h_2 a nonnegative integer, we get a polynomial in $\left\lfloor \frac{N}{2}\right\rfloor$ of degree $2h_2+1$.

As an example, consider the multiplicative functions $D_k : \mathbb{F}_q[T] \to \mathbb{Z}$ given by

$$D_k(F) = d_k\left(\frac{F}{\prod_{P|F} P}\right).$$

We have that $\mathfrak{d}_\ell = D_k(P^\ell) = d_k(P^{\ell-1}) = \binom{k+\ell-2}{k-1}$. In particular, $\mathfrak{d}_1 = 1$ and $\mathfrak{d}_2 = k$. Therefore, $h_1 = \mathfrak{d}_1 = 1$ and $h_2 = \mathfrak{d}_2 - \frac{\mathfrak{d}_1(\mathfrak{d}_1+1)}{2} = k-1$. Thus, this leads to a polynomial of degree $2k-1$. For example, if we take $k=2$, then

$$K^S_{d_1,2}(N,N;D_2) = \sum_{\ell=0}^{\lfloor \frac{N}{2} \rfloor} (\ell+1)^2 = \frac{1}{6}\left(\left\lfloor \frac{N}{2}\right\rfloor + 1\right)\left(\left\lfloor \frac{N}{2}\right\rfloor + 2\right)\left(2\left\lfloor \frac{N}{2}\right\rfloor + 3\right).$$

9. Sums over short arcs of the unit circle

We consider here a couple of problems involving the divisor function in function fields that can be described by integrals over the unitary symplectic matrices and over the orthogonal matrices. We follow part of the exposition by Rudnick and Waxman [RW19].

Let a $P(T) \in \mathcal{P}$. Then, there exist $A(T), B(T) \in \mathbb{F}_q[T]$ such that

$$P(T) = A(T)^2 + TB(T)^2 \tag{17}$$

if and only if $P(0)$ is a square in $\mathbb{F}_q$. Accordingly, let χ_2 be the character on $\mathbb{F}_q$ defined by

$$\chi_2(a) = \begin{cases} 0 & a = 0, \\ 1 & a \text{ is a nonzero square in } \mathbb{F}_q, \\ -1 & \text{otherwise.} \end{cases}$$

We can extend χ_2 to $\mathcal{M}$ by defining $\chi_2(f) := \chi_2(f(0))$. Thus, (17) is solvable if and only if $\chi_2(P) \neq -1$.

Set $S := \sqrt{-T}$ and we think of $\mathbb{F}_q[T] \subseteq \mathbb{F}_q[S]$. The goal here is for S to play the role of a square-root of -1 in the function field setting. Equation (17) then becomes

$$P(T) = (A+BS)(A-BS) = \mathfrak{p}\overline{\mathfrak{p}}$$

in $\mathbb{F}_q[S]$. One can define the analogue of complex conjugation by considering the automorphism over the ring of formal power series

$$\sigma : \mathbb{F}_q[[S]] \to \mathbb{F}_q[[S]], \qquad \sigma(S) = -S.$$

Complex conjugation can be used to construct the norm map as

$$\mathrm{Norm} : \mathbb{F}_q[[S]]^\times \to \mathbb{F}_q[[T]]^\times, \qquad \mathrm{Norm}(f) = f\sigma(f) = f(S)f(-S).$$

We define

$$\mathbb{S}^1 := \{g \in \mathbb{F}_q[[S]]^\times \,:\, g(0) = 1, \mathrm{Norm}(g) = 1\},$$

a group that can be seen as an analogue of the unit circle in this setting. For $f \in \mathbb{F}_q[[S]]$ let $\mathrm{ord}(f) = \max\{j \,:\, S^j \mid f\}$ and $|f| := q^{-\mathrm{ord}(f)}$, the absolute value associated with the place at infinity.

A sector on the unit circle is given by

$$\mathrm{Sect}(v;k) := \{w \in \mathbb{S}^1 \,:\, |w-v| \leq q^{-k}\}. \tag{18}$$

Thus, we have that $w \in \mathrm{Sect}(v;k)$ if and only if $w \equiv v \pmod{S^k}$.

The following modular group then parametrizes the sectors of the unit circle:

$$\mathbb{S}^1_k := \{f \in \mathbb{F}_q[S]/(S^k) \,:\, f(0) = 1,\, \mathrm{Norm}(f) \equiv 1 \pmod{S^k}\}.$$

Lemma 9.1. [Kat17, Lemma 2.1], [RW19, Lemma 6.1],

(1) *The cardinality of* $\mathbb{S}^1_k$ *is given by*

$$\#\mathbb{S}^1_k = q^\kappa, \text{ with } \kappa := \left\lfloor \frac{k}{2} \right\rfloor.$$

(2) *There is a direct product decomposition*

$$\left(\mathbb{F}_q[S]/(S^k)\right)^\times = H_k \times \mathbb{S}^1_k,$$

where

$$H_k := \{f \in \left(\mathbb{F}_q[S]/(S^k)\right)^\times \,:\, f(-S) \equiv f(S) \pmod{S^k}\},$$

and

$$\#H_k = (q-1)q^{\left\lfloor \frac{k-1}{2} \right\rfloor}.$$

For $f \in \mathbb{F}_q[S]$ coprime to S, the square-root of $\frac{f}{\sigma(f)} \in \mathbb{S}^1$ is well-defined, since $v \mapsto v^2$ is an automorphism of $\mathbb{S}^1$ by Hensel's Lemma. Thus, we can consider

$$U(f) := \sqrt{\frac{f}{\sigma(f)}}.$$

Thus, $U(f)$ defines an analogue for the complex argument of f. Note that $U(cf) = U(f)$ for scalars $c \in \mathbb{F}_q^\times$.

The modular analogue of U is given by

$$U_k : \left(\mathbb{F}_q[S]/(S^k)\right)^\times \to \mathbb{S}^1_k, \qquad f \mapsto \sqrt{\frac{f}{\sigma(f)}} \pmod{S^k}$$

and is a surjective homomorphism whose kernel is H_k ([RW19, Lemma 6.2]).

A super-even character modulo S^k is a Dirichlet character

$$\Xi : \left(\mathbb{F}_q[S]/(S^k)\right)^\times \to \mathbb{C}^\times$$

which is trivial on H_k (see [RW19] and [Kat17]). Super-even characters modulo S^k can be seen as the characters of $\left(\mathbb{F}_q[S]/(S^k)\right)^\times / H_k \cong \mathbb{S}^1_k$. They are the analogues of the Hecke characters in this setting.

The super-even characters modulo S satisfy the orthogonality relations,

$$\sum_{v \in \mathbb{S}^1_k} \overline{\Xi_1(v)} \Xi_2(v) = \begin{cases} q^\kappa & \Xi_1 = \Xi_2, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

(See the proof of Lemma 6.8 in [RW19, page 192].)

Proposition 9.2. [RW19, Proposition 6.3] *For* $f \in \left(\mathbb{F}_q[S]/(S^k)\right)^\times$ *and* $v \in \mathbb{S}^1_k$, *the following are equivalent:*

(1) $U_k(f) \in \mathrm{Sect}(v;k)$,
(2) $U_k(f) = U_k(v)$,
(3) $fH_k = vH_k$,

(4) $\Xi(f) = \Xi(v)$ *for all super-even characters* $\pmod{S^k}$.

The *Swan conductor* of Ξ is the maximal integer $d = d(\Xi) < k$ for which Ξ is nontrivial on the subgroup

$$\Gamma_d := \left(1 + (S^d)\right)/(S^k) \subset \left(\mathbb{F}_q[S]/(S^k)\right)^\times .$$

In other words, Ξ is a primitive character modulo $S^{d(\Xi)+1}$. The Swan conductor of a super-even character is always odd, since these characters are trivial on Γ_d for d even.

The L-function associated to a non-trivial Ξ is given by

$$\mathcal{L}(u,\Xi) = \sum_{\substack{f \in \mathcal{M} \\ f(0) \neq 0}} \Xi(f) u^{\deg(f)} = \prod_{\substack{P \in \mathcal{P} \\ P(0) \neq 0}} \left(1 - \Xi(P) u^{\deg(P)}\right)^{-1}, \quad |u| < 1/q,$$

which is a polynomial of degree $d(\Xi)$. We have

$$\mathcal{L}(u,\Xi) = (1-u)\det\left(I - uq^{1/2}\Theta_\Xi\right)$$

with $\Theta_\Xi \in \mathrm{U}(d(\Xi)-1)$.

In [Kat17, Theorem 5.1], Katz proved that for $q \to \infty$ the set of Frobenius classes

$$\{\Theta_\Xi \,:\, \Xi \text{ primitive super-even } \pmod{S^k}\}$$

becomes uniformly distributed in $\mathrm{Sp}(2\kappa - 2)$ provided that $2\kappa \geq 8$, and that the same holds for $2\kappa = 6$ in odd characteristic and for $2\kappa = 4$ provided that the characteristic is coprime to 10.

We can also consider the twists $\Xi\chi_2$, where Ξ is a non-trivial super-even character. The associated L-function is

$$\mathcal{L}(u,\Xi\chi_2) = \sum_{\substack{f \in \mathcal{M} \\ f(0) \neq 0}} \Xi(f)\chi_2(f) u^{\deg(f)} = \prod_{\substack{P \in \mathcal{P} \\ P(0) \neq 0}} \left(1 - \Xi(P)\chi_2(P) u^{\deg(P)}\right)^{-1}, \quad |u| < 1/q.$$

Again, this is a polynomial of degree $d(\Xi)$. We have

$$\mathcal{L}(u,\Xi\chi_2) = \det\left(I - uq^{1/2}\Theta_{\Xi\chi_2}\right),$$

with $\Theta_{\Xi\chi_2} \in \mathrm{U}(d(\Xi))$.

Katz [Kat17, Theorem 7.1] showed that in odd characteristic, for $q \to \infty$ the set of Frobenius classes

$$\{\Theta_{\Xi\chi_2} \,:\, \Xi \text{ primitive super-even } \pmod{S^k}\}$$

becomes uniformly distributed in $\mathrm{O}(2\kappa - 1)$ if $2\kappa \geq 6$, and that the same holds for $2\kappa = 4$ if the characteristic is coprime to 5.

9.1. **The averages.** We proceed to study the following sums

$$\mathcal{N}^S_{\mathfrak{g},k,N}(v) = \sum_{\substack{f \in \mathcal{M}_N \\ f(0) \neq 0 \\ U(f) \in \mathrm{Sect}(v,k)}} \mathfrak{g}(f) \qquad \text{and} \qquad \mathcal{N}^O_{\mathfrak{g},k,N}(v) = \sum_{\substack{f \in \mathcal{M}_N \\ f(0) \neq 0 \\ U(f) \in \mathrm{Sect}(v,k)}} \mathfrak{g}(f)\left(\frac{1+\chi_2(f)}{2}\right).$$

Remark that $\mathcal{N}^S_{\mathfrak{g},k,N}(v)$ represents the average over the polynomials whose argument is restricted to certain sector on the unit circle. Meanwhile, $\mathcal{N}^O_{d_\ell,k,N}(v)$ represents an analogue average with the extra condition that the constant coefficient is a nonzero square in $\mathbb{F}_q$.

We set

$$\mathcal{M}_0(N;\Xi\mathfrak{g}) = \sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}} \Xi(f)\mathfrak{g}(f), \qquad \text{and} \qquad \mathcal{M}_0(N;\Xi\chi_2\mathfrak{g}) = \sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}} \Xi(f)\chi_2(f)\mathfrak{g}(f),$$

where the subindex 0 indicates that we perform the sum over $f(0)\neq 0$.

Lemma 3.1 becomes the following statement in this setting.

Lemma 9.3. *We have for $N \leq \ell d(\Xi)$,*

$$\mathcal{M}_0(N;\Xi\chi_2 d_\ell) = (-1)^N q^{\frac{N}{2}} \sum_{\substack{j_1+\cdots+j_\ell=N\\ 0\leq j_1,\ldots j_\ell\leq d(\Xi)}} \mathrm{Sc}_{j_1}(\Theta_{\Xi\chi_2})\cdots \mathrm{Sc}_{j_\ell}(\Theta_{\Xi\chi_2})$$

and $\mathcal{M}_0(N;\Xi\chi_2 d_\ell)=0$ otherwise.

We have for $N\leq \ell(d(\Xi)-1)$,

$$\mathcal{M}_0(N;\Xi d_\ell) = (-1)^N q^{\frac{N}{2}} \sum_{\substack{j_1+\cdots+j_\ell=N\\ 0\leq j_1,\ldots,j_\ell\leq d(\Xi)-1}} \mathrm{Sc}_{j_1}(\Theta_{\Xi})\cdots \mathrm{Sc}_{j_\ell}(\Theta_{\Xi}) + O_{\ell,k}\left(q^{\frac{N-1}{2}}\right).$$

If $\ell(d(\Xi)-1) < N \leq \ell d(\Xi)$,

$$|\mathcal{M}_0(N;\Xi d_\ell)| \ll_{\ell,k} q^{\frac{N-1}{2}}.$$

Finally, if $\ell d(\Xi) < N$, $\mathcal{M}_0(N;\Xi d_\ell)=0$.

Lemma 9.4. *Let λ be real. As $q\to\infty$, we have that*

$$\mathcal{M}_0(N;\chi_2^2 d_\lambda) = \sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}} d_\lambda(f) = q^N\binom{\lambda+N-1}{N} + O_{\lambda,N}(q^{N-1}) \tag{20}$$

$$\mathcal{M}_0(N;\chi_2 d_\lambda) = \begin{cases} 0 & N>0,\\ 1 & N=0.\end{cases} \tag{21}$$

Proof. Equation (20) follows from a particular case of Lemma 3.2, while Equation (21) follows from Equation (37) in [KL22b]. □

We start our analysis by looking at the mean value of $\mathcal{N}^S_{\mathfrak{g},k,N}$ averaged over all the directions of $v\in\mathbb{S}^1_k$.

Lemma 9.5. *As $q\to\infty$,*

$$\begin{aligned}\frac{1}{q^\kappa}\sum_{v\in\mathbb{S}^1_k}\mathcal{N}^S_{\mathfrak{g},k,N}(v) =& \frac{1}{q^\kappa}\sum_{m_1+2m_2+\cdots+Nm_N=N}\ \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ f_1f_2\cdots f_N(0)\neq 0}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)\\ =& q^{N-\kappa}\binom{\mathfrak{d}_1+N-1}{N}\left(1+O\left(q^{-1}\right)\right).\end{aligned}$$

Proof. As before, we apply (8) and obtain

$$\sum_{v\in\mathbb{S}_k^1}\mathcal{N}^S_{\mathfrak{g},k,N}(v)=\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}}\mathfrak{g}(f)=\sum_{m_1+2m_2+\cdots+Nm_N=N}\ \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ f_1f_2\cdots f_N(0)\neq 0}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N).$$

The rest of the argument proceeds as in the proof of Lemma 6.2. □

Lemma 9.6. *As* $q\to\infty$,

$$\begin{aligned}\frac{1}{q^\kappa}\sum_{v\in\mathbb{S}_k^1}\mathcal{N}^O_{\mathfrak{g},k,N}(v)=&\frac{1}{2q^\kappa}\sum_{m_1+2m_2+\cdots+Nm_N=N}\ \sum_{\substack{f_1\in\mathcal{M}_{m_1}\\ \cdots\\ f_N\in\mathcal{M}_{m_N}\\ f_1f_2\cdots f_N(0)\neq 0}} d_{h_1}(f_1)d_{h_2}(f_2)\cdots d_{h_N}(f_N)+O\left(q^{\frac{N}{2}-\kappa}\right)\\ =&\frac{q^{N-\kappa}}{2}\binom{\mathfrak{d}_1+N-1}{N}\left(1+O\left(q^{-1}\right)\right).\end{aligned}$$

Proof. As before, we have

$$\frac{1}{q^\kappa}\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}}\mathfrak{g}(f)\left(\frac{1+\chi_2(f)}{2}\right)=\frac{1}{2q^\kappa}\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}}\mathfrak{g}(f)+\frac{1}{2q^\kappa}\mathcal{M}_0(N;\chi_2\mathfrak{g}).$$

By equation (8) and Lemma 9.4, we have that, as $q\to\infty$,

$$\mathcal{M}_0(N;\chi_2\mathfrak{g})=\sum_{m_1+2m_2+\cdots+Nm_N=N}\prod_{j=1}^N\mathcal{M}_0(m_j;\chi_2^jd_{h_j})=O\left(q^{\frac{N}{2}}\right),\tag{22}$$

since the only way to obtain a maximal contribution for powers of q is to take $m_2=\frac{N}{2}$ and $m_1=m_3=\cdots=m_N=0$. We conclude by combining with the result of Lemma 9.5.

□

9.2. **The variances.** Our first step to understand the variance is to obtain formulas for $\mathcal{N}^S_{\mathfrak{g},k,N}(v)$ and $\mathcal{N}^O_{\mathfrak{g},k,N}(v)$ in terms of the super-even characters Ξ. By Proposition 9.2 and the orthogonality relations, we find, for $f\in\mathcal{M}_N$,

$$\frac{1}{q^\kappa}\sum_{\Xi\text{ super-even}\,(\mathrm{mod}\,S^k)}\overline{\Xi(v)}\Xi(f)=\begin{cases}1 & U(f)\in\mathrm{Sect}(v;k),\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\mathcal{N}^S_{\mathfrak{g},k,N}(v)=\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0\\ U(f)\in\mathrm{Sect}(v,k)}}\mathfrak{g}(f)=\frac{1}{q^\kappa}\sum_{\Xi\text{ super-even}\,(\mathrm{mod}\,S^k)}\overline{\Xi(v)}\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}}\Xi(f)\mathfrak{g}(f).$$

The contribution from the trivial character Ξ_0 is precisely

$$\frac{1}{q^\kappa}\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}}\mathfrak{g}(f)=\langle\mathcal{N}^S_{\mathfrak{g},k,N}\rangle.$$

Thus

$$\mathcal{N}^S_{\mathfrak{g},k,N}(v)-\langle\mathcal{N}^S_{\mathfrak{g},k,N}\rangle =\frac{1}{q^\kappa}\sum_{\substack{\Xi \text{ super-even} \pmod{S^k}\\ \Xi\neq\Xi_0}}\overline{\Xi(v)}\sum_{\substack{f\in\mathcal{M}_N\\ f(0)\neq 0}}\Xi(f)\mathfrak{g}(f)$$

$$=\frac{1}{q^\kappa}\sum_{\substack{\Xi \text{ super-even} \pmod{S^k}\\ \Xi\neq\Xi_0}}\overline{\Xi(v)}\mathcal{M}_0(N;\Xi\mathfrak{g}). \tag{23}$$

We define the variance as

$$\operatorname{Var}(\mathcal{N}^S_{\mathfrak{g},k,N}) :=\frac{1}{q^\kappa}\sum_{v\in\mathbb{S}^1_k}\left|\mathcal{N}^S_{\mathfrak{g},k,N}(v)-\langle\mathcal{N}^S_{\mathfrak{g},k,N}\rangle\right|^2. \tag{24}$$

Theorem 9.7. *Assume that $\mathfrak{d}_1$ is a positive integer and that $N\leq\mathfrak{d}_1(2\kappa-2)$ with $\kappa=\left\lfloor\frac{k}{2}\right\rfloor$. As $q\to\infty$,*

$$\operatorname{Var}(\mathcal{N}^S_{\mathfrak{g},k,N})\sim\frac{q^N}{q^\kappa}\int_{\mathrm{Sp}(2\kappa-2)}\Bigg(\sum_{\substack{j_1+\cdots+j_\ell=N\\ 0\leq j_1,\ldots,j_\ell\leq 2\kappa-2}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_\ell}(U)\Bigg)^2 dU.$$

Proof. By applying the orthogonality relations (19) to equations (23) and (24), we obtain

$$\begin{aligned}
\operatorname{Var}(\mathcal{N}^S_{\mathfrak{g},k,N}) =&\frac{1}{q^\kappa}\sum_{v\in\mathbb{S}^1_k}\frac{1}{q^{2\kappa}}\sum_{\substack{\Xi_1,\Xi_2 \text{ super-even} \pmod{S^k}\\ \Xi_1,\Xi_2\neq\Xi_0}}\overline{\Xi_1(v)}\mathcal{M}_0(N;\Xi_1\mathfrak{g})\Xi_2(v)\overline{\mathcal{M}_0(N;\Xi_2\mathfrak{g})}\\
=&\frac{1}{q^{2\kappa}}\sum_{\substack{\Xi_1,\Xi_2 \text{ super-even} \pmod{S^k}\\ \Xi_1,\Xi_2\neq\Xi_0}}\mathcal{M}_0(N;\Xi_1\mathfrak{g})\overline{\mathcal{M}_0(N;\Xi_2\mathfrak{g})}\frac{1}{q^\kappa}\sum_{v\in\mathbb{S}^1_k}\overline{\Xi_1(v)}\Xi_2(v)\\
=&\frac{1}{q^{2\kappa}}\sum_{\substack{\Xi \text{ super-even} \pmod{S^k}\\ \Xi\neq\Xi_0}}|\mathcal{M}_0(N;\Xi\mathfrak{g})|^2\\
=&\frac{1}{q^{2\kappa}}\sum_{\substack{m_1+2m_2+\cdots+Nm_N=N\\ n_1+2n_2+\cdots+Nn_N=N}}\sum_{\substack{\Xi \text{ super-even} \pmod{S^k}\\ \Xi\neq\Xi_0}}\prod_{j=1}^N\mathcal{M}_0(m_j;\Xi^jd_{h_j})\overline{\mathcal{M}_0(n_j;\Xi^jd_{h_j})},
\end{aligned}$$

where we have applied (8). Now Proposition 3.3 implies

$$\begin{aligned}
\operatorname{Var}(\mathcal{N}^S_{\mathfrak{g},k,N}) =&\frac{1}{q^{2\kappa}}\sum_{\substack{m_1+2m_2=N\\ n_1+2n_2=N}}\sum_{\substack{\Xi \text{ super-even} \pmod{S^k}\\ \Xi\neq\Xi_0}}\mathcal{M}_0(m_1;\Xi d_{h_1})\overline{\mathcal{M}_0(n_1;\Xi d_{h_1})}\mathcal{M}_0(m_2;\Xi^2d_{h_2})\overline{\mathcal{M}_0(n_2;\Xi^2d_{h_2})}\\
&+O\left(q^{N-\frac{1}{6}-\kappa}\right).
\end{aligned}$$

As in Subsection 4.2, we can consider the case of $m_1=n_1=N$ separately from the rest, and show that the rest contributes to an error term $O\left(q^{N-\frac{1}{2}-\kappa}\right)$. Following [KL22a, Theorem

5.6], this leads to

$$\begin{aligned}\mathrm{Var}(\mathcal{N}^S_{\mathfrak{g},k,N}) =& \frac{1}{q^{2\kappa}} \sum_{\substack{\Xi \text{ super-even} \\ \text{primitive} \pmod{S^k} \\ \Xi \neq \Xi_0}} |\mathcal{M}_0(N;\Xi d_{h_1})|^2 + O\left(q^{N-\frac{1}{6}-\kappa}\right) \\ \sim& \frac{q^N}{q^\kappa} \int_{\mathrm{Sp}(2\kappa-2)} \left(\sum_{\substack{j_1+\cdots+j_\ell=N \\ 0\leq j_1,\ldots,j_\ell\leq 2\kappa-2}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_\ell}(U)\right)^2 dU,\end{aligned}$$

by the equidistribution theorem due to Katz [Kat17, Theorem 5.1]. □

We now turn our attention to $\mathcal{N}^O_{\mathfrak{g},k,N}(v)$. This sum has two kinds of fluctuations away from its average: one is the fluctuations of the function $\mathfrak{g}$ itself in sectors, and the other is the fluctuations coming from the twist by the character χ_2. We are interested in isolating the latter fluctuations, which are more subtle, so we will consider the following "average" of $\mathcal{N}^O_{\mathfrak{g},k,N}$, defined as:

$$\langle \mathcal{N}^O_{\mathfrak{g},k,N}(v)\rangle_S := \frac{1}{2q^\kappa} \sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} \mathfrak{g}(f),$$

and we will compute the variance

$$\mathrm{Var}_S(\mathcal{N}^O_{\mathfrak{g},k,N}) := \frac{1}{q^\kappa}\sum_{v\in\mathbb{S}^1_k} \left|\mathcal{N}^O_{\mathfrak{g},k,N}(v) - \langle \mathcal{N}^O_{\mathfrak{g},k,N}(v)\rangle_S\right|^2. \tag{25}$$

We start by splitting the sum in terms of the character χ_2:

$$\mathcal{N}^O_{\mathfrak{g},k,N}(v) = \sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} \mathfrak{g}(f)\left(\frac{1+\chi_2(f)}{2}\right) = \frac{1}{2}\sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} \mathfrak{g}(f) + \frac{1}{2}\sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} \chi_2(f)\mathfrak{g}(f).$$

Using the super-even characters to impose the circle sector condition, we get

$$\begin{aligned}\mathcal{N}^O_{\mathfrak{g},k,N}(v) - \langle \mathcal{N}^O_{\mathfrak{g},k,N}(v)\rangle_S =& \frac{1}{2}\sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0 \\ U(f)\in \mathrm{Sect}(v,k)}} \chi_2(f)\mathfrak{g}(f) \\ =& \frac{1}{2q^\kappa} \sum_{\Xi \text{ super-even} \pmod{S^k}} \overline{\Xi(v)} \sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0}} \Xi(f)\chi_2(f)\mathfrak{g}(f) \\ =& \frac{1}{2q^\kappa} \sum_{\Xi \text{ super-even} \pmod{S^k}} \overline{\Xi(v)}\mathcal{M}_0(N;\Xi\chi_2\mathfrak{g}). \qquad (26)\end{aligned}$$

We are now ready to compute the variance.

Theorem 9.8. *Assume that $\mathfrak{d}_1$ is a positive integer and that $N \leq \mathfrak{d}_1(2\kappa-1)$ with $\kappa = \left\lfloor \frac{k}{2} \right\rfloor \geq 3$. As $q \to \infty$,*

$$\mathrm{Var}_S(\mathcal{N}^O_{\mathfrak{g},k,N}) \sim \frac{q^N}{4q^\kappa} \int_{\mathrm{O}(2\kappa-1)} \left(\sum_{\substack{j_1+\cdots+j_\ell=N \\ 0\leq j_1,\dots,j_\ell \leq 2\kappa-1}} \mathrm{Sc}_{j_1}(U)\cdots \mathrm{Sc}_{j_\ell}(U) \right)^2 dU.$$

Proof. By combining equations (26) and (25), we obtain

$$\begin{aligned}
\mathrm{Var}_S(\mathcal{N}^O_{\mathfrak{g},k,N}) =& \frac{1}{q^\kappa} \sum_{v\in \mathbb{S}^1_k} \frac{1}{4q^{2\kappa}} \sum_{\Xi_1,\Xi_2 \text{ super-even} \pmod{S^k}} \overline{\Xi_1(v)} \mathcal{M}_0(N;\Xi_1\chi_2\mathfrak{g}) \Xi_2(v) \overline{\mathcal{M}_0(N;\Xi_2\chi_2\mathfrak{g})} \\
=& \frac{1}{4q^{2\kappa}} \sum_{\Xi_1,\Xi_2 \text{ super-even} \pmod{S^k}} \mathcal{M}_0(N;\Xi_1\chi_2\mathfrak{g}) \overline{\mathcal{M}_0(N;\Xi_2\chi_2\mathfrak{g})} \frac{1}{q^\kappa} \sum_{v\in\mathbb{S}^1_k} \overline{\Xi_1(v)}\Xi_2(v) \\
=& \frac{1}{4q^{2\kappa}} |\mathcal{M}_0(N;\Xi_0\chi_2\mathfrak{g})|^2 + \frac{1}{4q^{2\kappa}} \sum_{\substack{\Xi \text{ super-even} \pmod{S^k} \\ \Xi\neq\Xi_0}} |\mathcal{M}_0(N;\Xi\chi_2\mathfrak{g})|^2,
\end{aligned}$$

where we have applied the orthogonal relations (19).

As in equation (22), we have that

$$\begin{aligned}
\sum_{\substack{f\in\mathcal{M}_N \\ f(0)\neq 0}} \chi_2(f)\mathfrak{g}(f) =& \sum_{m_1+2m_2+\cdots+Nm_N=N} \prod_{j=1}^N \mathcal{M}_0(m_j;\chi_2^j d_{h_j}) \\
=& \begin{cases} q^{\frac{N}{2}} \binom{h_2+N/2-1}{N/2} \left(1+O\left(q^{-1}\right)\right) & 2 \mid N, \\ 0 & 2\nmid N, \end{cases}
\end{aligned}$$

by Lemma 9.4, since the only way to obtain a maximal contribution for powers of q is for $m_2 = \frac{N}{2}$ and $m_1 = m_3 = \cdots = m_N = 0$. Moreover, as soon as one $m_j \neq 0$ with j odd, we obtain 0.

On the other hand,

$$\frac{1}{4q^{2\kappa}} \sum_{\substack{\Xi \text{ super-even} \pmod{S^k} \\ \Xi\neq\Xi_0}} |\mathcal{M}_0(N;\Xi\chi_2\mathfrak{g})|^2 = \frac{1}{4q^{2\kappa}} \sum_{\substack{\Xi \text{ super-even} \pmod{S^k} \\ \Xi\neq\Xi_0}} |\mathcal{M}_0(N;\Xi\chi_2 d_{h_1})|^2 + O\left(q^{N-\frac{1}{6}-\kappa}\right),$$

which follows exactly as in the proof of Theorem 9.7.

Putting all of this together,

$$\mathrm{Var}_S(\mathcal{N}^O_{\mathfrak{g},k,N}) = \delta_{2|N} \frac{q^N}{4q^{2\kappa}} \binom{h_2+N/2-1}{N/2}^2 + \frac{1}{4q^{2\kappa}} \sum_{\substack{\Xi \text{ super-even} \pmod{S^k} \\ \Xi\neq\Xi_0}} |\mathcal{M}_0(N;\Xi\chi_2 d_{h_1})|^2 + O\left(q^{N-\frac{1}{6}-\kappa}\right).$$

We remark that the first term above is of size $O\left(q^{N-2\kappa}\right)$, while Lemma 9.3 implies the second term is of size $O\left(q^{N-\kappa}\right)$. Therefore, the first term is part of the error term. Following [KL22b, Theorem 6.5], this leads to

$$\mathrm{Var}_S(\mathcal{N}^O_{\mathfrak{g},k,N}) \sim \frac{q^N}{4q^\kappa}\int_{\mathrm{O}(2\kappa-1)}\left(\sum_{\substack{j_1+\cdots+j_\ell=N\\0\le j_1,\ldots,j_\ell\le 2\kappa-1}}\mathrm{Sc}_{j_1}(U)\cdots\mathrm{Sc}_{j_\ell}(U)\right)^2 dU,$$

which follows from applying [Kat17, Theorem 7.1].

□

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