Mahler measure and 3-hyperbolic volumes

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Mahler measure and Lehmer's question

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\text{Look at} \quad \frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n \to \infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma:

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$$= \log(1.176280818\ldots) = 0.162357612\ldots$$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

There exists $C > 0$, for all $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the polynomial above the best possible?

Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha| = \log \max\{1, |\alpha|\}$$

recovers one-variable case.

Some properties

- $m(P \cdot Q) = m(P) + m(Q)$

- $m(P) \geq 0$ if P has integral coefficients.

- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

  $$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

  where h is the logarithmic Weil height.

- Boyd & Lawton : $P \in \mathbb{C}[x_1, \dots, x_n]$

  $$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n}))$$

  $$= m(P(x_1, \dots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s}$$

$$\chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

Bloch − Wigner Dilogarithm

Dilogarithm:

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad x \in \mathbb{C}, \quad |x| < 1$$

$$\mathrm{Li}_2(x) := -\int_0^x \log(1-t)\frac{\mathrm{d}t}{t} \qquad x \in \mathbb{C} \setminus (1, \infty)$$

Bloch − Wigner Dilogarithm:

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$

Properties:

1. $D(\bar{x}) = -D(x)$ ($\Rightarrow D|_{\mathbb{R}} \equiv 0$)

2. $D(x) = -D\left(\frac{1}{x}\right) = -D(1-x)$

3. $-2\int_0^\theta \log|2\sin t|\mathrm{d}t = D(\mathrm{e}^{2\mathrm{i}\theta})$

4. Five-term relation

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0$$

Hyperbolic volumes

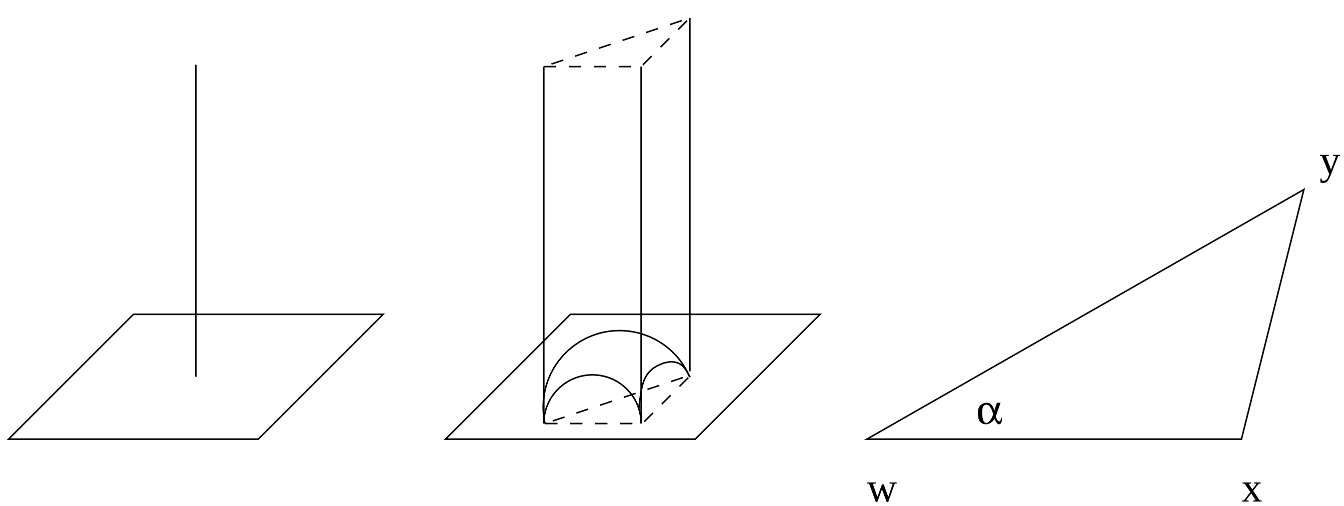
$$\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$$

Volume Element:

$$\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{z^3}$$

Ideal Tetrahedron:

vertices in $\partial\mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$

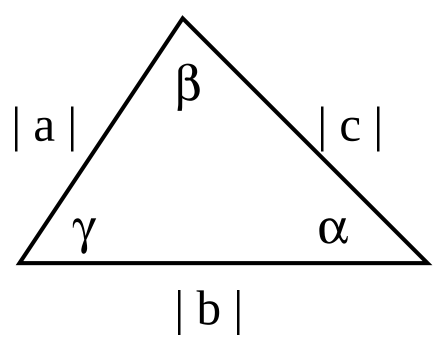


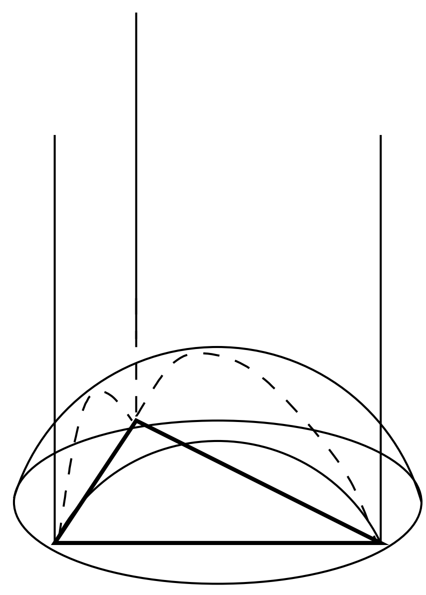
$$\mathrm{Vol}\left(\pi^*\left(\overset{\triangle}{wxy}\right)\right) = D\left(\frac{y-w}{x-w}\right) = D\left(\left|\frac{y-w}{x-w}\right| e^{i\alpha}\right)$$

Cassaigne and Maillot's example

$$\pi m(a+bx+cy) =$$

$$\begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$





Results

1. Cassaigne and Maillot(2000):

$$y = \frac{ax + b}{c}$$

2. Vandervelde(2003):

$$y = \frac{bx + d}{ax + c}$$

3. L(2004):

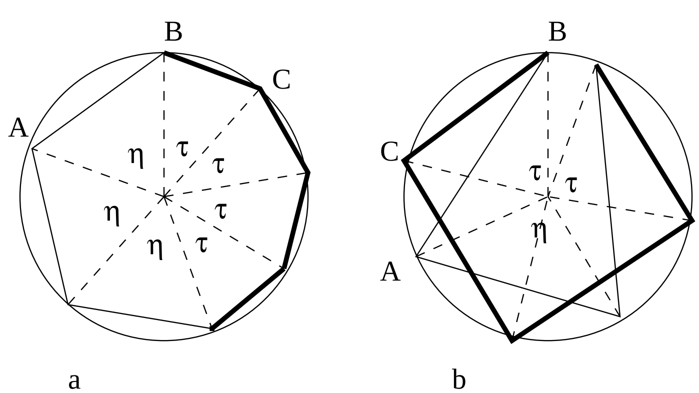
$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \ldots + 1}{t(x^{m-1} + \ldots + 1)}$$

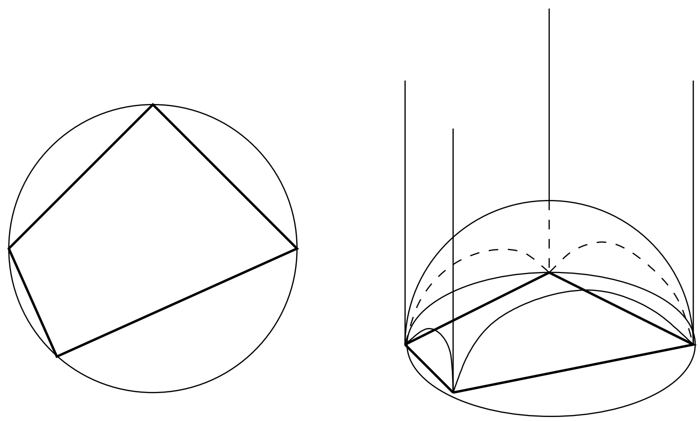
$$R_t(x, y) = t(x^m - 1)y - (x^n - 1)$$

P cyclic plane polygon is admissible of type (m, n) if

- P has m sides of length t (with $t \in \mathbb{R}_{>0}$) and n of length 1.

- Sides of length t wind around the center in the same direction and sides of length 1 wind around the center in the same direction, which may be opposite from the direction of the sides of length t.





Theorem 1

$$\pi m(R_t(x,y)) = \pi \log|t|$$

$$+\frac{2}{mn}\sum \epsilon_k \text{Vol}(\pi^*(P_k)) + \epsilon \sum_{k=1}^{N} (-1)^k \log|t| \arg \alpha_k$$

$\epsilon, \epsilon_k = \pm 1$, P_k are all the admissible polygons of type (m,n).

An example

$$y = \frac{x^3 - 1}{t(x^2 - 1)}$$

$$R_t(x, y) = t(x^2 - 1)y - (x^3 - 1)$$

$m(R_t) - \log|t|$

$$= \begin{cases} \frac{2}{2\cdot 3\cdot \pi}\left(\epsilon_1 \mathrm{Vol}(\pi^*(P_1)) + \epsilon_2 \mathrm{Vol}(\pi^*(P_2))\right) & \\ +\frac{\sigma_1 - \sigma_2}{\pi}\log|t| & 0 < t < \frac{3}{2} \\ & \\ \frac{2}{2\cdot 3\cdot \pi}\epsilon_1 \mathrm{Vol}(\pi^*(P_1)) + \frac{\sigma_1}{\pi}\log|t| & \frac{3}{2} \leq t \end{cases}$$

$$m(t(x^2-1)y-(x^3-1))$$

$$=\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2}\log|x^3-1-t(x^2-1)y|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

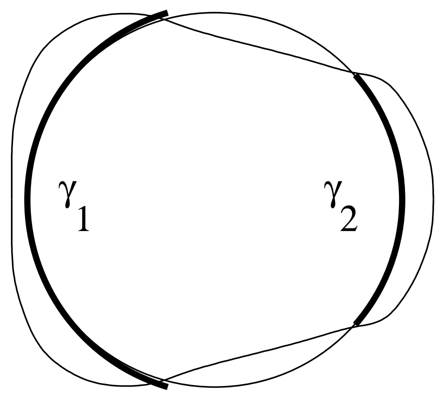
$$=\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1}\log|t(x^2-1)|\frac{\mathrm{d}x}{x}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2}\log\left|\frac{x^3-1}{t(x^2-1)}-y\right|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$=\log t+\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1}\log^+\left|\frac{x^3-1}{t(x^2-1)}\right|\frac{\mathrm{d}x}{x}$$

by Jensen's formula

$$=\frac{1}{2\pi \mathrm{i}}\int_{\gamma}\log\left|\frac{x^3-1}{t(x^2-1)}\right|\frac{\mathrm{d}x}{x}$$



Determine for which points we have

$$\left|\frac{x^3-1}{t(x^2-1)}\right| = 1$$

$$\frac{x^3-1}{t(x^2-1)} \cdot \frac{x^{-3}-1}{t(x^{-2}-1)} = 1$$

$$x^4 + (2-t^2)x^3 + (3-2t^2)x^2 + (2-t^2)x + 1 = 0$$

Roots α_1, α_1^{-1}, α_2, α_2^{-1}

$$\mathrm{Re}\,\alpha_1 = \frac{t^2 - 2 - t\sqrt{t^2+4}}{4} \quad \text{for} \quad 0 < t$$

$$\mathrm{Re}\,\alpha_2 = \frac{t^2 - 2 + t\sqrt{t^2+4}}{4} \quad \text{for} \quad 0 < t < \frac{3}{2}$$

Let $\sigma_i = \arg \alpha_i$, $\mathrm{Im}\,\alpha_i \geq 1$

$$\pi > \sigma_1 > \frac{2\pi}{3}$$

$$\frac{2\pi}{3} > \sigma_2 > 0$$

$$\int_\alpha^\beta \log|x^n - 1| \frac{\mathrm{d}x}{\mathrm{i}x} = \frac{D(\alpha^n) - D(\beta^n)}{n}$$

For $0 < t < \frac{3}{2}$

$$m(t(x^2-1)y-(x^3-1)) - \log t$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\gamma_1 \cup \gamma_2} \log \left| \frac{x^3-1}{t(x^2-1)} \right| \frac{\mathrm{d}x}{x}$$

$$= \frac{D(\alpha_1^{-3}) - D(\alpha_1^3) + D(\alpha_2^3) - D(\alpha_2^{-3})}{3(2\pi)}$$

$$-\frac{D(\alpha_1^{-2}) - D(\alpha_1^2) + D(\alpha_2^2) - D(\alpha_2^{-2})}{2(2\pi)}$$

$$-\frac{2(\sigma_1 - \sigma_2)}{2\pi} \log t$$

$$= \frac{3D(\alpha_1^2) - 2D(\alpha_1^3)}{6\pi} - \frac{3D(\alpha_2^2) - 2D(\alpha_2^3)}{6\pi}$$

$$-\frac{\sigma_1 - \sigma_2}{\pi} \log t$$

For $\frac{3}{2} \leq t$

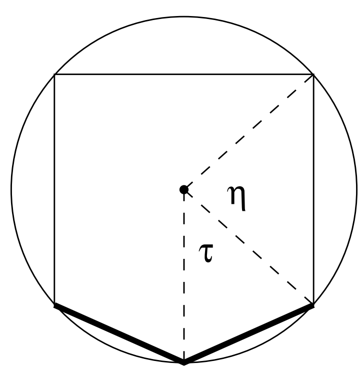
$$m(t(x^2-1)y-(x^3-1))-\log t$$

$$=\frac{3D(\alpha_1^2)-2D(\alpha_1^3)}{6\pi}-\frac{\sigma_1}{\pi}\log t$$

For α_1,

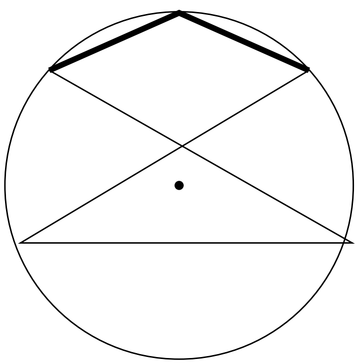
$$\pi > \sigma_1 > \frac{2\pi}{3} \quad \longrightarrow \quad \eta = 2\pi - 2\sigma_1, \quad \tau = 3\sigma_1 - 2\pi$$

$$3\eta + 2\tau = 2\pi$$



For α_2

$0 < t < \frac{1}{\sqrt{2}}$	$\frac{2\pi}{3} > \sigma_2 > \frac{\pi}{2}$	$\eta = 2\pi - 2\sigma_2$ $\tau = 2\pi - 3\sigma_2$	$3\eta - 2\tau = 2\pi$
$\frac{1}{\sqrt{2}} < t < \frac{2}{\sqrt{3}}$	$\frac{\pi}{2} > \sigma_2 > \frac{\pi}{3}$	$\eta = 2\sigma_2$ $\tau = 2\pi - 3\sigma_2$	$3\eta + 2\tau = 4\pi$
$\frac{2}{\sqrt{3}} < t < \frac{3}{2}$	$\frac{\pi}{3} > \sigma_2 > 0$	$\eta = 2\sigma_2$ $\tau = 3\sigma_2$	$3\eta - 2\tau = 0$



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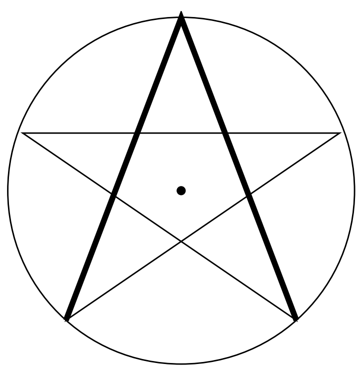

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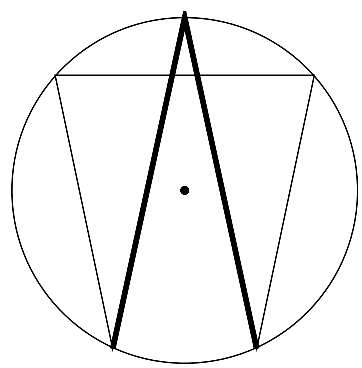

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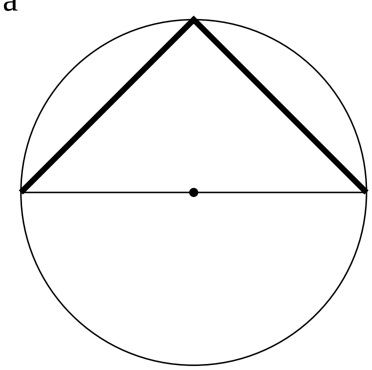

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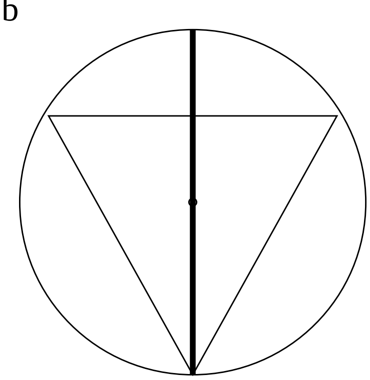

e

Analogies with the case of A-polynomials

M orientable, complete, one-cusped, hyperbolic manifold.

$$M = \bigcup_{j=1}^{k} \Delta(z_j)$$

- *Gluing equations*. ($r_{j,i}$, $r'_{j,i}$ integers depending on M).

$$\prod_{i=1}^{k} z_i^{r_{j,i}}(1-z_i)^{r'_{j,i}} = \pm 1 \qquad \text{for} \quad j = 1, \ldots, k$$

- *Completeness equations*. (l_i, l'_i, m_i, m'_i integers depending on M).

$$\prod_{i=1}^{k} z_i^{l_i}(1 - z_i)^{l'_i} = \pm 1$$

$$\prod_{i=1}^{k} z_i^{m_i}(1 - z_i)^{m'_i} = \pm 1$$

$\operatorname{Im} z_i > 0 \rightsquigarrow$ geometric solution

$$\mathrm{Vol}(M) = \sum_{j=1}^{k} D(z_j)$$

Boyd

$$\pi m(A) = \sum V_i$$

A is the A-polynomial.

$V_0 = \mathrm{Vol}(M)$ and the other V_i are pseudovolumes.

We have for $t = 1$

$$\pi m(R_1(x,y)) = \frac{2}{mn} \sum \epsilon_k \mathrm{Vol}(\pi^*(P_k))$$

Replace completeness relations by

$$\prod_{i=1}^{k} z_i^{l_i}(1 - z_i)^{l'_i} = x^2$$

$$\prod_{i=1}^{k} z_i^{m_i}(1 - z_i)^{m'_i} = y^2$$

"deformation parameters" x and y

A-polynomial is obtained by eliminating $z_1, \ldots, z_k$ from this and gluing equations.

$$U = \begin{pmatrix} l_1 & \cdots & l_k & l'_1 & \cdots & l'_k \\ m_1 & \cdots & m_k & m'_1 & \cdots & m'_k \\ r_{1,1} & \cdots & r_{1,k} & r'_{1,1} & \cdots & r'_{1,k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ r_{k,1} & \cdots & r_{k,k} & r'_{k,1} & \cdots & r'_{k,k} \end{pmatrix}$$

Theorem 2 *(Neumann–Zagier)*

$$U J_{2k} U^t = 2 \begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$J_{2p} = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}$$

$$\pi m(R_1(x,y)) = \frac{1}{mn}\sum \epsilon_k \mathrm{Vol}(\pi^*(P_k'))$$

$w = \mathrm{e}^{\mathrm{i}\eta}$ and $z = \mathrm{e}^{\mathrm{i}\tau}$

$$\left\{ \begin{array}{rcl} w_1^{\alpha} z_1^{\beta} & = & x^2 \\ w_1^{-mn(m+n)\alpha} z_1^{-mn(m+n)\beta}(1-w_1)^{2n}\dots(1-w_m)^{2n} & & \\ \cdot(1-z_1)^{-2m}\dots(1-z_n)^{-2m} & = & y^2 \\ w_1\dots w_m z_1 \dots z_n & = & 1 \\ w_1 w_2^{-1} & = & 1 \\ \vdots & & \\ w_1 w_m^{-1} & = & 1 \\ z_1 z_2^{-1} & = & 1 \\ \vdots & & \\ z_1 z_n^{-1} & = & 1 \end{array} \right.$$

for $n\alpha - m\beta = 1$, satisfies Neumann–Zagier.

One of the branches (the one with $w = x^{2n}, z = x^{-2m}$), is

$$y^2 = \left(\frac{x^n - x^{-n}}{x^m - x^{-m}}\right)^{2mn}$$

Consider

$$\tilde{R}(x, y) = (x^m - x^{-m})^{mn} y - (x^n - x^{-n})^{mn}$$

$$mn \cdot m(R_1) = m(\tilde{R})$$

Hence

$$\pi m(\tilde{R}(x, y)) = \sum \epsilon_k \mathrm{Vol}(\pi^*(P'_k))$$

Mahler measure of several-variable polynomials

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More examples in two variables

Boyd & Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathsf{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathsf{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathsf{L}'(A,0)$$

$$A: y^2 = x^3-44x+112$$

More examples in several variables

L (2003)

$$\pi^n m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)z\right)$$

$=$ combination of $\zeta(\text{odd})\,/\,\mathsf{L}(\chi_{-4},\text{even})$

$$\pi^n m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

$=$ combination of $\zeta(\text{odd})\,/\,\mathsf{L}(\chi_{-4},\text{even})$, polylogarithms

$$\pi^n m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)x\right.$$

$$\left.+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)\right)y\right)$$

$=$ combination of $\zeta(\text{odd})$

Examples

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$= 24\mathrm{L}(\chi_{-4},4)+\pi^2\mathrm{L}(\chi_{-4},2)$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)$$

$$= 62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$P_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number, $\mathrm{Li}_0(x) \equiv -\frac{1}{2}$,

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

P_k satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1}P_k(x) \qquad P_k(\bar{x}) = (-1)^{k-1}P_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0$$

Examples from the world of resultants

D'Andrea & L (2003).

- $m(\text{Res}_{\{0,m,n\}})$

$$= m(\text{Res}_t(x + yt^m + t^n, z + wt^m + t^n)) =$$

$$\frac{2}{\pi^2}(-mP_3(\varphi^n) - nP_3(-\varphi^m) + mP_3(\phi^n) + nP_3(\phi^m))$$

$$0 \le \varphi \le 1 \quad \text{root of} \quad x^n + x^{n-m} - 1 = 0$$

$$1 \le \phi \quad \text{root of} \quad x^n - x^{n-m} - 1 = 0$$

- $m(\text{Res}_{\{(0,0),(1,0),(0,1)\}}) = m\left(\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix}\right)$

$$= m((1-x)(1-y) - (1-z)(1-w)) = \frac{9\zeta(3)}{2\pi^2}$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F number field)

- L-function ($\mathsf{L}_X = \zeta_F$)

- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)

- Regulator map

  $$r : K \to \text{smooth differential forms}$$

  ($r = \log|\cdot|$)

Conjecture: special value of $\mathsf{L}_X \sim_{\mathbb{Q}^*} \int_\gamma r(\xi)$

(E.g. Dirichlet class number formula, F real quadratic, $\zeta_F'(0) \sim_{\mathbb{Q}^*} \log|\epsilon| \qquad \epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$P(x,y) = y + x - 1 \qquad C = \{P(x,y) = 0\}$$

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log|y + x - 1| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

by Jensen's equality:

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{\mathrm{d}x}{x}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \log|y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_{\gamma} \eta(x,y)$$

where

$$\gamma = C \cap \{|x| = 1, |y| \geq 1\}$$

and

$$\eta(x,y) = \log|x| \mathrm{d} \arg y - \log|y| \mathrm{d} \arg x$$

$$\mathrm{d}\eta(x,y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$$

- $\eta(x,y) = -\eta(y,x)$

- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$

$$\eta(x, 1-x) = \mathrm{d}D(x)$$

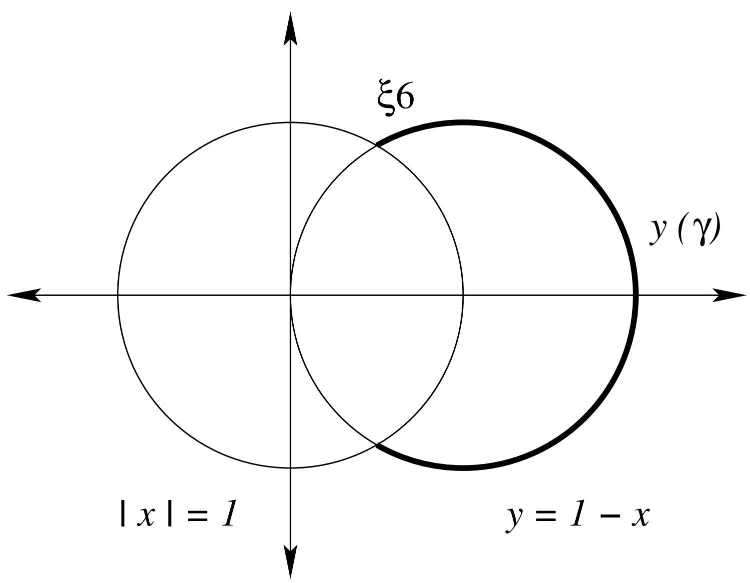
Use Stokes Theorem:

$$m(P) = -\frac{1}{2\pi} D(\partial\gamma)$$

$$x = \mathrm{e}^{2\pi \mathrm{i}\theta},$$

$$y(\gamma(\theta)) = 1 - \mathrm{e}^{2\pi \mathrm{i}\theta}, \quad \theta \in [1/6\,;5/6]$$

$$\partial\gamma = [\bar{\xi_6}] - [\xi_6]$$



$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi_6})$$

$$= 2D(\xi_6) = \frac{3\sqrt{3}}{2}\mathrm{L}(\chi_{-3}, 2)$$

In general,

$P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_\gamma \eta(x, y)$$

In $\bigwedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$,

$$x \wedge y = \sum_j r_j \;\; z_j \wedge (1 - z_j)$$

$\{x, y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$.

Then

$$\int_\gamma \eta(x, y) = \sum r_j \int_\gamma \eta(z_j, 1 - z_j) = \sum r_j D(z_j)|_{\partial\gamma}.$$

Big picture

$$\cdots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C, \partial\gamma) \to K_2(C) \to \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x,y)$ is exact, then $\{x,y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

  $\rightsquigarrow$ dilogarithms $\rightsquigarrow$ zeta function of a number field.

- $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(C)$. We have $\eta(x,y)$ is not exact.

  $\rightsquigarrow$ L-series of a curve.

We may get combinations of both situations.

The three-variable case

$$P(x,y,z) = (1-x)+(1-y)z \qquad S = \{P(x,y,z)=0\}$$

$$m(P) = m(1-y)+\frac{1}{(2\pi \mathrm{i})^3}\int_{\mathbb{T}^3}\log\left|z-\frac{1-x}{1-y}\right|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2}\log^+\left|\frac{1-x}{1-y}\right|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma}\log|z|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma}\eta(x,y,z)$$

$$\Gamma = S\cap\{|x|=|y|=1, |z|\geq 1\}$$

$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y|\,\mathrm{d}\log|z| - \mathrm{d}\arg y\,\mathrm{d}\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z|\,\mathrm{d}\log|x| - \mathrm{d}\arg z\,\mathrm{d}\arg x\right)$$

$$+\log|z|\left(\frac{1}{3}\mathrm{d}\log|x|\,\mathrm{d}\log|y| - \mathrm{d}\arg x\,\mathrm{d}\arg y\right)$$

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} \wedge \frac{\mathrm{d}z}{z}\right)$$

Theorem 3

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

Theorem 4

$$\omega(x,x) = \mathrm{d}P_3(x)$$

Want to apply Stokes' Theorem again.

Maillot: if $P \in \mathbb{R}[x,y,z]$,

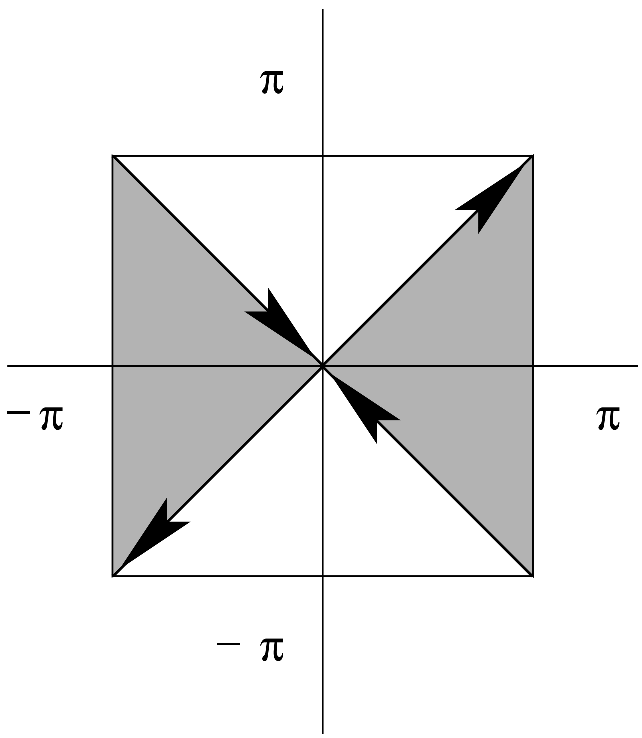
$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$

ω defined in

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$$

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m(P) = \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z)$$

In $\bigwedge^3(\mathbb{C}(S)^*)\otimes\mathbb{Q}$,

$$x\wedge y\wedge z = \sum r_i \ \ x_i \wedge (1-x_i)\wedge y_i$$

$\{x,y,z\} = 0$ in $K_3^M(\mathbb{C}(S))\otimes\mathbb{Q}$.

Then

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_\Gamma \eta(x_i, 1-x_i, y_i)$$

$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Need

$$\{x\}_2 \otimes y = \sum r_i \{x_i\}_2 \otimes x_i$$

in $(\mathbf{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$, where

$$\mathbf{B}_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\text{five term relation}$$

Then

$$\int_\gamma \omega(x,y) = \sum r_i \int_\gamma \omega(x_i, x_i) = \sum r_i \, P_3(x_i)|_{\partial\gamma}$$

The condition is $\{x_i\}_2 \otimes y_i$ is 0 "in" $K_4^{\{3\}}(\mathbb{C}(C))_{\mathbb{Q}}$.

Big picture II

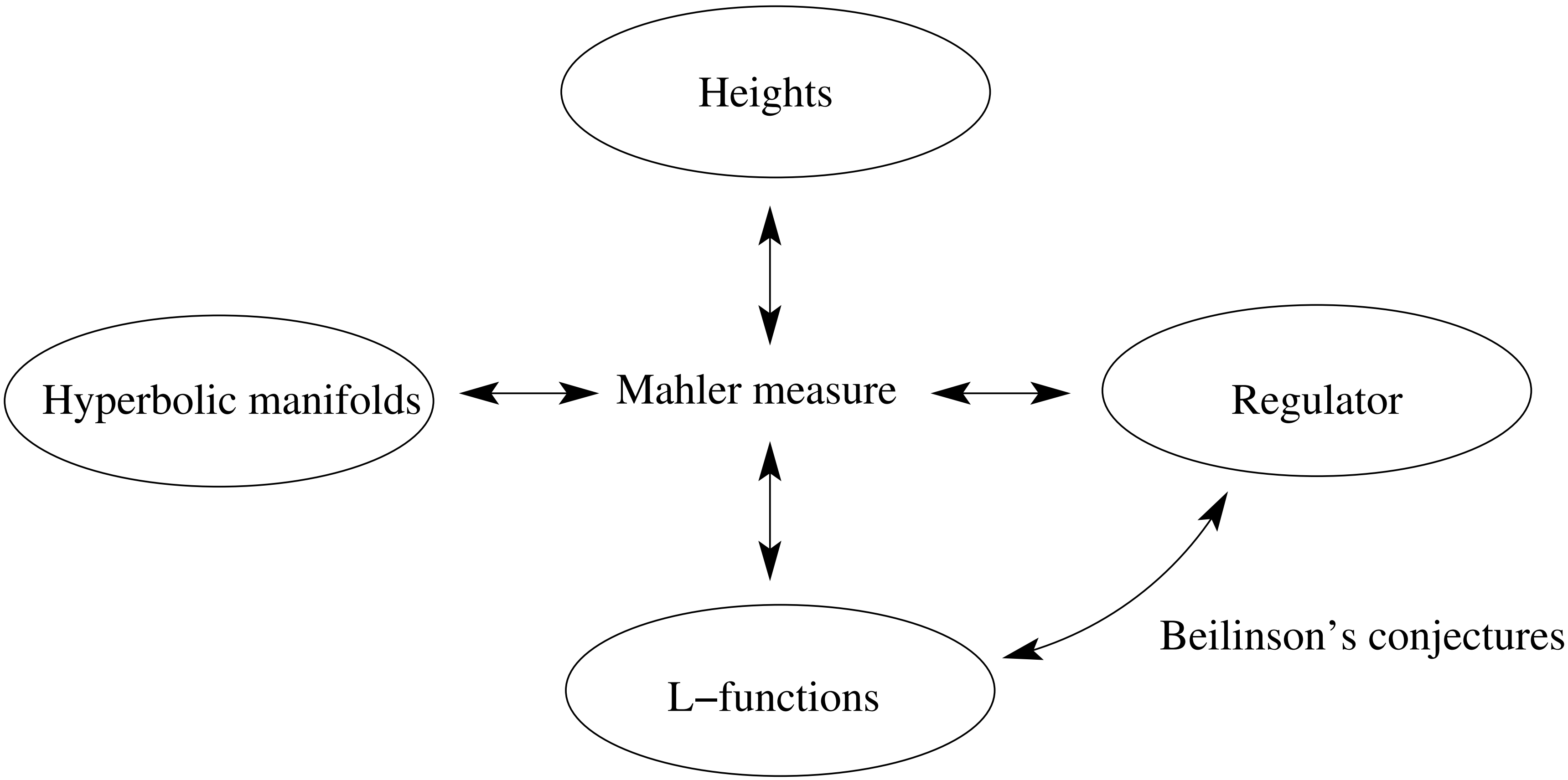
$$\cdots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \cdots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.


Heights
Hyperbolic manifolds
Mahler measure
Regulator
L−functions
Beilinson's conjectures