Polylogarithms and Hyperbolic volumes

Matilde N. Lalín

University of British Columbia and PIMS,
Max-Planck-Institut für Mathematik,
University of Alberta

mlalin@math.ubc.ca

http://www.math.ubc.ca/~mlalin

July 2nd, 2007

The hyperbolic space

Beltrami's Half-space model:

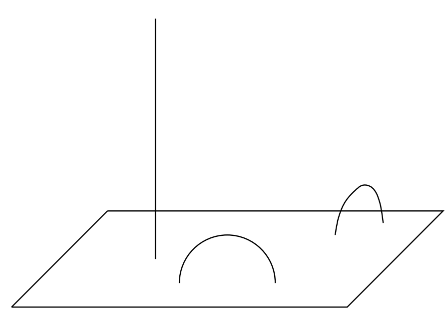
$$\mathbb{H}^n = \{(x_1, \ldots, x_{n-1}, x_n) \,|\, x_i \in \mathbb{R}, x_n > 0\},$$

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \ldots + \mathrm{d}x_n^2}{x_n^2},$$

$$\mathrm{d}V = \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_n}{x_n^n},$$

$$\partial\mathbb{H}^n = \{(x_1, \ldots, x_{n-1}, 0)\} \cup \infty.$$

Geodesics are given by vertical lines and semi-circles whose endpoints lie in $\{x_n = 0\}$ and intersect it orthogonally.



Orientation preserving isometries of $\mathbb{H}^2$

$$PSL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) \,\middle|\, ad - bc = 1 \right\} /\pm I.$$

$$z = x_1 + x_2 i \to \frac{az+b}{cz+d}.$$

Orientation preserving isometries of $\mathbb{H}^3$ is $PSL(2,\mathbb{C})$.

$$\mathbb{H}^3 = \{z = x_1 + x_2 i + x_3 j \mid x_3 > 0\},$$

subspace of quaternions ($i^2 = j^2 = k^2 = -1$, $ij = -ji = k$).

$$z \to (az+b)(cz+d)^{-1} = (az+b)(\bar{z}\bar{c}+\bar{d})|cz+d|^{-2}.$$

Poincaré: study of discrete groups of hyperbolic isometries.

Volumes in $\mathbb{H}^3$

Picard: fundamental domain for $PSL(2,\mathbb{Z}[i])$ in $\mathbb{H}^3$ has a finite volume.

Humbert extended this result.

Lobachevsky function:

$$л(\theta) = -\int_0^\theta \log|2\sin t| \mathrm{d}t.$$

$$л(\theta) = \frac{1}{2}\mathrm{Im}\left(\mathrm{Li}_2\left(e^{2i\theta}\right)\right),$$

where

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad |z| \le 1.$$

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-x)\frac{\mathrm{d}x}{x}.$$

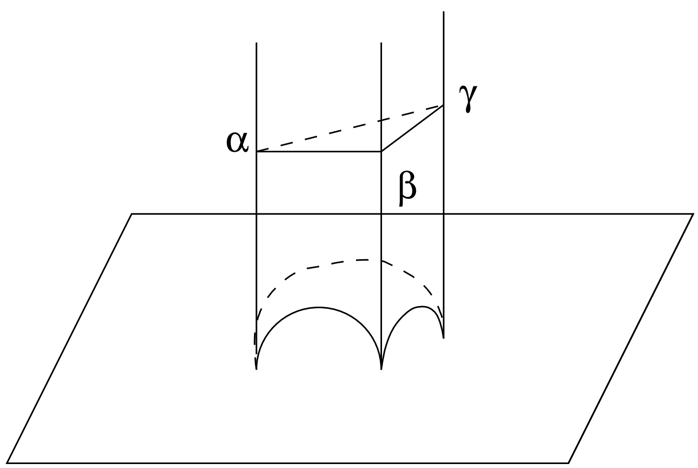
(multivalued) analytic continuation to $\mathbb{C}\backslash[1,\infty)$

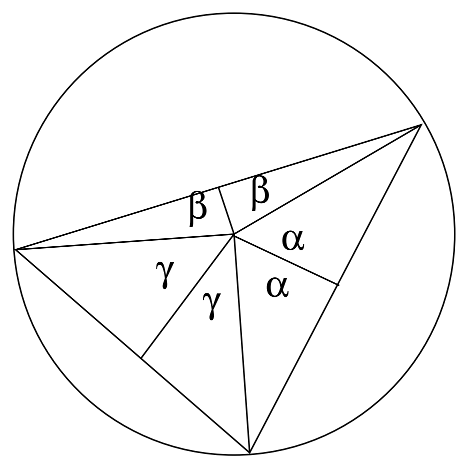
Let Δ be an ideal tetrahedron (vertices in $\partial\mathbb{H}^3$).

Theorem 1 *(Milnor, after Lobachevsky)*

The volume of an ideal tetrahedron with dihedral angles α, β, and γ is given by

$$\text{Vol}(\Delta) = \quad \text{л}(\alpha) + \quad \text{л}(\beta) + \quad \text{л}(\gamma).$$

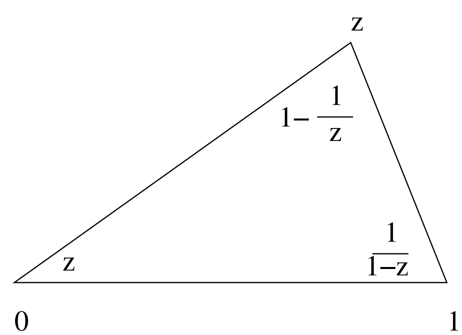




Move a vertex to ∞ and use baricentric subdivision to get six simplices with three right dihedral angles.

Triangle with angles α, β, γ, defined up to similarity.

Let $\Delta(z)$ be the tetrahedron determined up to transformations by any of $z, 1-\frac{1}{z}, \frac{1}{1-z}$.



If ideal vertices are z_1, z_2, z_3, z_4,

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

Bloch-Wigner dilogarithm

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log|z|\log(1-z)).$$

Continuous in $\mathbb{P}^1(\mathbb{C})$, real-analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$.

$$D(z) = -D(1-z) = -D\left(\frac{1}{z}\right) = -D(\bar{z}).$$

Five-term relation:

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0.$$

$$D(z) = \frac{1}{2}\left(D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-z^{-1}}{1-\bar{z}^{-1}}\right) + D\left(\frac{(1-z)^{-1}}{(1-\bar{z})^{-1}}\right)\right).$$

$$\mathrm{Vol}(\Delta(z)) = D(z).$$

Five points in $\partial\mathbb{H}^3 \cong \mathbb{P}^1(\mathbb{C})$, then the sum of the signed volumes of the five possible tetrahedra must be zero:

$$\sum_{i=0}^{5}(-1)^i \mathrm{Vol}([z_1 : \ldots : \widehat{z_i} : \ldots : z_5]) = 0.$$

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0.$$

Dedekind ζ-function

F number field

$[F:\mathbb{Q}] = n = r_1 + 2r_2$

$\tau_1, \ldots, \tau_{r_1}$ real embeddings

$\sigma_1, \ldots, \sigma_{r_2}$ a set of complex embeddings (one for each pair of conjugate embeddings).

Dedekind ζ-function

$$\zeta_F(s) = \sum_{\mathfrak{A}\,\text{ideal} \neq 0} \frac{1}{N(\mathfrak{A})^s}, \qquad \operatorname{Re} s > 1,$$

$N(\mathfrak{A}) = |\mathcal{O}_F/\mathfrak{A}|$ norm.

Euler product

$$\prod_{\mathfrak{P}\,\text{prime}} \frac{1}{1 - N(\mathfrak{P})^{-s}}.$$

Theorem 2 *(Dirichlet's class number formula)* $\zeta_F(s)$ *extends meromorphically to* $\mathbb{C}$ *with only one simple pole at* $s = 1$ *with*

$$\lim_{s \to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \operatorname{reg}_F}{\omega_F \sqrt{|D_F|}}.$$

where

- h_F *is the class number.*

- ω_F *is the number of roots of unity in* F.

- D_F *is the discriminant.*

- reg_F *is the regulator.*

Regulator

$\{u_1, \ldots, u_{r_1+r_2-1}\}$ basis for $\mathcal{O}_F^*$ modulo torsion

$L(u_i) :=$

$(\log|\tau_1 u_i|, \ldots, \log|\tau_{r_1} u_i|, 2\log|\sigma_1 u_i|, \ldots, 2\log|\sigma_{r_2-1} u_i|)$

reg_F is the determinant of the matrix.

= (up to a sign) the volume of fundamental domain for $L(\mathcal{O}_F^*)$.

Functional equation

$$\xi_F(s) = \xi_F(1-s),$$

$$\xi_F(s) = \left(\frac{|D_F|}{4^{r_2}\pi^n}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_F(s).$$

$$\lim_{s \to 0} s^{1-r_1-r_2} \zeta_F(s) = -\frac{h_F \,\mathrm{reg}_F}{\omega_F}.$$

Euler:

$$\zeta(2m) = \frac{(-1)^{m-1}(2\pi)^{2m}B_m}{2(2m)!}$$

Klingen , Siegel:

F is totally real ($r_2 = 0$),

$\zeta_F(2m) = r(m)\sqrt{|D_F|}\pi^{2mn}$ for $m > 0$.

where $r(m) \in \mathbb{Q}$.

Building manifolds

Bianchi, Humbert :

$F = \mathbb{Q}\left(\sqrt{-d}\right)$ $d \geq 1$ square-free

Γ a torsion-free subgroup of $PSL\,(2, \mathcal{O}_d)$,

$[PSL\,(2, \mathcal{O}_d) : \Gamma] < \infty$.

Then $\mathbb{H}^3/\Gamma$ is an oriented hyperbolic three-manifold.

Example: $d = 3$, $\mathcal{O}_3 = \mathbb{Z}[\omega]$, $\omega = \frac{-1+\sqrt{-3}}{2}$.

Riley: $[PSL\,(2, \mathcal{O}_3) : \Gamma] = 12$.

$\mathbb{H}^3/\Gamma$ diffeomorphic to $S^3 \setminus \mathrm{Fig} - 8$.

Theorem 3 *(Essentially Humbert)*

$$\textit{Vol}\left(\mathbb{H}^3/PSL(2,\mathcal{O}_d)\right) = \frac{D_d\sqrt{D_d}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-d})}(2).$$

$$D_d = \begin{cases} d & d \equiv 3 \bmod 4, \\ 4d & \text{otherwise.} \end{cases}$$

M hyperbolic 3-manifold

$$\text{Vol}(M) = \sum_{j=1}^{J} D(z_j).$$

$$\zeta_{\mathbb{Q}(\sqrt{-d})} = \frac{D_d\sqrt{D_d}}{2\pi^2}\sum_{j=1}^{J} D(z_j).$$

Example:

$$\mathrm{Vol}(S^3 \setminus \mathrm{Fig}-8) = 12\frac{3\sqrt{3}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)$$

$$= 3D\left(\mathrm{e}^{\frac{2i\pi}{3}}\right) = 2D\left(\mathrm{e}^{\frac{i\pi}{3}}\right).$$

Zagier:

$[F : \mathbb{Q}] = r_1 + 2$,

Γ group of units of an order in a quaternion algebra B over F that is ramified at all real places such that

$M = \mathbb{H}^3/\Gamma$ has volume

$$\sim_{\mathbb{Q}^*} \frac{\sqrt{|D_F|}}{\pi^{2(n-1)}}\zeta_F(2).$$

$[F : \mathbb{Q}] = r_1 + 2r_2$, $r_2 > 1$,

Zagier:

discrete subgroup Γ of $PSL(2, \mathbb{C})^{r_2}$ such that

$M = \left(\mathbb{H}^3\right)^{r_2} / \Gamma$ has volume

$$\sim_{\mathbb{Q}^*} \frac{\sqrt{|D_F|}}{\pi^{2(r_1+r_2)}} \zeta_F(2).$$

$$M = \bigcup \Delta(z_1) \times \ldots \times \Delta(z_{r_2})$$

The Bloch group

$$\mathrm{Vol}(M) = \sum_{j=1}^{J} D(z_j),$$

then

$$\sum_{j=1}^{J} z_j \wedge (1 - z_j) = 0 \in \bigwedge^2 \mathbb{C}^*.$$

$$\bigwedge^2 \mathbb{C}^* = \{x \wedge y \mid x \wedge x = 0,\ x_1 x_2 \wedge y = x_1 \wedge y + x_2 \wedge y\}$$

$\mathrm{Vol}(M) = D(\xi_M)$, where $\xi_M \in \mathcal{A}(\bar{\mathbb{Q}})$, and

$$\mathcal{A}(F) = \left\{ \sum n_i [z_i] \in \mathbb{Z}[F] \,\middle|\, \sum n_i z_i \wedge (1 - z_i) = 0 \right\}.$$

Let

$$\mathcal{C}(F) = \left\{ [x] + [1 - xy] + [y] + \left[\frac{1-y}{1-xy}\right] + \left[\frac{1-x}{1-xy}\right] \right| \\ x, y \in F, xy \neq 1 \right\},$$

Bloch group is

$$\mathcal{B}(F) = \mathcal{A}(F)/\mathcal{C}(F).$$

$D : \mathcal{B}(\mathbb{C}) \to \mathbb{R}$ well-defined function,

$\mathrm{Vol}(M) = D(\xi_M)$ for some $\xi_M \in \mathcal{B}(\bar{\mathbb{Q}})$, independently of the triangulation.

Then

$\zeta_F(2) = \sqrt{|D_F|}\pi^{2(n-1)} D(\xi_M)$ for $r_2 = 1$.

The K-theory connection

R ring. $K_0(R), K_1(R) = R^*$, etc

Borel:

$K_n(F) \otimes \mathbb{Q}$ free abelian and

$$\dim_{\mathbb{Q}}(K_n(F) \otimes \mathbb{Q}) = \begin{cases} 0 & n \geq 2 \text{ even}, \\ r_1 + r_2 & n \equiv 1 \bmod 4, \\ r_2 & n \equiv 3 \bmod 4. \end{cases}$$

Let $n_+ = r_1 + r_2$ and $n_- = r_2$.

The regulator map

$$\text{reg}_m : K_{2m-1}(\mathbb{C}) \to \mathbb{R},$$

is such that

$$K_{2m-1}(F) \to K_{2m-1}(\mathbb{R})^{r_1} \times K_{2m-1}(\mathbb{C})^{r_2} \to \mathbb{R}^{n_\mp}.$$

$K_{2m-1}(F)$/torsion goes to a cocompact lattice of $\mathbb{R}^{n_\mp}$.

Covolume is a rational multiple of $\sqrt{|D_F|}\zeta_F(m)/\pi^{kn_\pm}$.

$m = 2$, Bloch and Suslin: $\mathcal{B}(F)$ is "essentially" $K_3(F)$, and D is the regulator.

$$\begin{array}{ccc} K_3(F) & \xrightarrow{\text{reg}_2} & \mathbb{R}^{r_2} \\ \phi_F \uparrow & \nearrow_{(D\circ\sigma_1,\ldots,D\circ\sigma_{r_2})} & \\ \mathcal{B}(F) & & \end{array}$$

Theorem 4 *For a number field* $[F:\mathbb{Q}] = r_1 + 2r_2$*,*

- $\mathcal{B}(F)$ *is finitely generated of rank* r_2*.*

- $\xi_1, \ldots \xi_{r_2}$ $\mathbb{Q}$*-basis of* $\mathcal{B}(F) \otimes \mathbb{Q}$*. Then*

$$\zeta_F(2) \sim_{\mathbb{Q}^*} \sqrt{|D_F|}\pi^{2(r_1+r_2)} \det \left\{ D\left(\sigma_i\left(\xi_j\right)\right)\right\}_{1\le i,j\le r_2}.$$

Zagier's conjecture

Generalize to values $\zeta_F(m)$.

k-polylogarithm

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \qquad |z| \leq 1.$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

$$\mathcal{L}_k(z) = \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} \log^j |z| \mathrm{Li}_{k-j}(z) \right),$$

$\mathrm{Re}_k = \mathrm{Re}$ or Im depending on whether k is odd or even, and B_j is the jth Bernoulli number.

It is continuous in $\mathbb{P}^1(\mathbb{C})$, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

$$\mathcal{L}_k(z) = (-1)^{k-1} \mathcal{L}_k\left(\frac{1}{z}\right).$$

Generalized Bloch groups

$$\mathcal{B}_k(F) = \mathcal{A}_k(F)/\mathcal{C}_k(F).$$

$\mathcal{C}_k(F)$ set of functional equations of the kth polylogarithm and $\mathcal{A}_k(F)$ is the set of allowed elements.

$$\mathcal{A}_k(F) := \Big\{\xi \in \mathbb{Z}[F] \mid \iota_\phi(\xi) \in \mathcal{C}_{k-1}(F)\ \forall\phi \in \mathrm{Hom}(F^*, \mathbb{Z})\Big\}$$

where $\iota_\phi(\sum n_i[x_i]) = \sum n_i\phi(x_i)[x_i]$.

$$\mathcal{C}_k(F) := \{\xi \in \mathcal{A}_k(F) \,|\, \mathcal{L}_k(\sigma(\xi)) = 0\ \forall\sigma \in \Sigma_F\}\,.$$

Conjecture 5 *Let F be a number field. Let $n_+ = r_1 + r_2$, $n_- = r_2$, and $\mp = (-1)^{k-1}$. Then*

- *$\mathcal{B}_k(F)$ is finitely generated of rank $n_\mp$.*

- *$\xi_1, \ldots \xi_{n_\mp}$ $\mathbb{Q}$-basis of $\mathcal{B}_k(F) \otimes \mathbb{Q}$. Then*

  $$\zeta_F(k) \sim_{\mathbb{Q}^*} \sqrt{|D_F|}\pi^{kn_\pm} \det \left\{ \mathcal{L}_k \left(\sigma_i \left(\xi_j \right) \right) \right\}_{1 \le i,j \le n_\mp}.$$

Example

$F = \mathbb{Q}(\sqrt{5})$, $r_1 = 2, r_2 = 0$.

$\left\{[1], \left[\frac{1+\sqrt{5}}{2}\right]\right\} \in \mathcal{A}_3(F)$, basis for $\mathcal{B}_3(F)$.

$$\left| \begin{matrix} \mathcal{L}_3(1) & \mathcal{L}_3(1) \\ \mathcal{L}_3\left(\frac{1+\sqrt{5}}{2}\right) & \mathcal{L}_3\left(\frac{1-\sqrt{5}}{2}\right) \end{matrix} \right|$$

$$= \left| \begin{matrix} \zeta(3) & \zeta(3) \\ \frac{1}{10}\zeta(3) + \frac{25}{48}\sqrt{5}L(3,\chi_5) & \frac{1}{10}\zeta(3) - \frac{25}{48}\sqrt{5}L(3,\chi_5) \end{matrix} \right|$$

$$= -\frac{25}{24}\sqrt{5}\zeta(3)L(3,\chi_5) = -\frac{25}{24}\sqrt{5}\zeta_F(3).$$

More K-theory

Expectation:

- $K_{2m-1}(F)$ and $\mathcal{B}_m(F)$ should be isomorphic.
- The regulator map should be given by $\mathcal{L}_m$.

de Jeu, Beilinson–Deligne: there is a map

$$K_{2m-1}(F) \to \mathcal{B}_m(F).$$

Goncharov: map is surjective for $m = 3$.

Goncharov

$$G_F(k) : \mathcal{G}_k(F) \xrightarrow{\partial} \mathcal{G}_{k-1} \otimes F^* \xrightarrow{\partial} \mathcal{G}_{k-2} \otimes \wedge^2 F^* \xrightarrow{\partial} \dots$$

$$\dots \xrightarrow{\partial} \mathcal{G}_2 \otimes \wedge^{k-2} F^* \xrightarrow{\partial} \wedge^k F^*$$

$\mathcal{G}_k(F) = \mathbb{Z}[F]/\mathcal{C}_k(F)$

$\partial[x] \otimes x_1 \wedge \dots \wedge x_l = [x] \otimes x \wedge x_1 \wedge \dots \wedge x_l.$

$$H^1(G_F(k)) \simeq \mathcal{B}_k(F).$$

Goncharov conjectures

$$H^i(G_F(k) \otimes \mathbb{Q}) \simeq gr_k^\gamma K_{2k-i}(F) \otimes \mathbb{Q}.$$

Volumes in higher dimensions

Orthoscheme in $\mathbb{H}^n$: simplex bounded by hyperplanes $H_0, \ldots, H_n$ such that

$$H_i \perp H_j \qquad |i-j| > 1$$

Theorem 6 *(Schläfli's formula) $R \subset \mathbb{H}^n$ is an orthoscheme,*

$$\mathrm{dVol}_n(R) = \frac{1}{n-1} \sum_{j=1}^{n} \mathrm{Vol}_{n-2}(F_j) \mathrm{d}\alpha_j.$$

where $F_j = R \cap H_{j-1} \cap H_j$, α_j is the angle attached at F_j, and $\mathrm{Vol}_0(F_j) = 1$.

Vol $2m$-simplex $\rightsquigarrow$ Vol dimension $2m-1$ and lower.

dimension 5: sum of trilogarithmic expressions (Böhn, Müller, Kellerhals, and Goncharov).

Goncharov:

M $(2m-1)$-dimensional hyperbolic manifold of finite volume.

Theorem 7 *There is a* $\gamma_M \in K_{2m-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}$ *such that*

$$\mathit{Vol}(M) = \mathrm{reg}_m(\gamma_M).$$

Conjecture 8 *There is a* $\xi_M \in \mathcal{B}_m(\bar{\mathbb{Q}}) \otimes \bar{\mathbb{Q}}^*$ *such that*

$$\mathit{Vol}(M) = \mathcal{L}_m(\xi_M).$$

Goncharov: true for dimension 5.