THE MAHLER MEASURE OF A THREE-VARIABLE FAMILY AND AN APPLICATION TO THE BOYD–LAWTON FORMULA

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Abstract. We prove a formula relating the Mahler measure of an infinite family of three-variable polynomials to a combination of the Riemann zeta function at $s = 3$ and special values of the Bloch–Wigner dilogarithm by evaluating a regulator. The evaluation requires two different applications of Jensen's formula and analyzing the integral in two different planes (as opposed to only one plane as usually). The degrees of the monomials involving one of the variables is allowed to vary freely, leading to an interesting application of the Boyd–Lawton formula.

1. Introduction

The (logarithmic) Mahler measure of a non-zero rational function $P \in \mathbb{C}(x_1, \dots, x_n)$ is defined by

$$\mathrm{m}(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where the integration is taken over the unit torus $\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\}$ with respect to the Haar measure.

This construction originated for one-variable polynomials in the search for large prime numbers (for example, see Lehmer's work [Leh33]) and was later extended to multi-variable polynomials by Mahler [Mah62] in applications to classical heights of polynomials. Eventually Mahler measure was found to yield special values of functions of number theoretic significance, such as the Riemann zeta function and other L-functions. The first examples of these relationships were given by Smyth [Smy81, Boy81b]

$$\mathrm{m}(x + y + 1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2),$$

$$\mathrm{m}(x + y + z + 1) = \frac{7}{2\pi^2} \zeta(3),$$

where $L(\chi_{-3}, s)$ is the Dirichlet L-function in the character of conductor 3 and $\zeta(s)$ is the Riemann zeta function.

Deninger explained the appearance of L-functions in some Mahler measure formulas in terms of Beilinson's conjectures via relationships with regulators in [Den97]. (Additional insight into this direction can be found in the works of Boyd [Boy98] and Rodriguez-Villegas [RV99].) We see from this point of view that the Riemann zeta function and the L-functions

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which appear in Mahler measure formulas come from special values of polylogarithms arising from regulators. The cases involving the Riemann zeta function and Dirichlet L-functions have been linked to particular applications of the Borel regulator and evaluations of polylogarithms in algebraic numbers [BRV02, BRVD03, Lal07, Lal08].

Our goal is to employ the regulator to prove the following result.

Theorem 1. *Let a, b be coprime positive integers. Then, if b is odd,*

$$\begin{aligned}
&\mathrm{m}\big(x^{a+b}+1+(x^a+1)y+(x^b-1)z\big)\\
=&\left(4-\frac{1}{ab(a+b)}\right)\frac{3\zeta(3)}{4\pi^2}-\frac{a+b}{ab\pi^2}\sum_{1\le k\le \frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a+b}\right)\right)\left(\frac{(2k-1)\pi}{a+b}-\pi\right)\\
&+\frac{a}{b(a+b)\pi^2}\sum_{1\le k\le\frac{a+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a}\right)\right)\left(\frac{(2k-1)\pi}{a}-\pi\right)\\
&+\frac{b}{a(a+b)\pi^2}\sum_{1\le k\le\frac{b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{b}\right)\right)\left(\frac{(2k-1)\pi}{b}-\pi\right).
\end{aligned}$$

If b is even,

$$\begin{aligned}
&\mathrm{m}\big(x^{a+b}+1+(x^a+1)y+(x^b-1)z\big)\\
=&\left(3+\frac{1}{ab(a+b)}\right)\frac{\zeta(3)}{\pi^2}-\frac{a+b}{ab\pi^2}\sum_{1\le k\le \frac{a+b}{2}} D\left(\exp\left(\frac{2k\pi i}{a+b}\right)\right)\left(\frac{2k\pi}{a+b}-\pi\right)\\
&+\frac{a}{b(a+b)\pi^2}\sum_{1\le k\le\frac{a}{2}} D\left(\exp\left(\frac{2k\pi i}{a}\right)\right)\left(\frac{2k\pi}{a}-\pi\right)\\
&+\frac{b}{a(a+b)\pi^2}\sum_{1\le k\le\frac{b}{2}} D\left(\exp\left(\frac{2k\pi i}{b}\right)\right)\left(\frac{2k\pi}{b}-\pi\right).
\end{aligned}$$

In the above formulas, $D(z)$ denotes the Bloch–Wigner dilogarithm defined by equation (5).

The idea to study this formula arose from some conversations with David Boyd and Mathew Rogers. Boyd [Boy06] made a systematic study of numerical examples of the form $p_0(x)+p_1(x)y+p_2(x)z$ with $p_0(x), p_1(x)$, and $p_2(x)$ given by products of cyclotomic polynomials, and Rogers discovered the particular case of the above formula when $b=1$.

We remark that Theorem 1 gives the Mahler measure of a whole polynomial family with a fixed number of three variables and unbounded degree. To our knowledge, the only previous result of such nature is the following formula due to D'Andrea and Lalín (Theorem 6 in [DL07], see also Theorem 9 in [Lal07]).

$$(1)\qquad \mathrm{m}\left(z-\frac{(1-x)^a(1-y)^b}{(1-xy)^{a+b}}\right)=\frac{2b}{\pi^2}(\mathscr{L}_3(\phi_2^a)-\mathscr{L}_3(-\phi_1^a))+\frac{2a}{\pi^2}(\mathscr{L}_3(\phi_1^b)-\mathscr{L}_3(-\phi_2^b)),$$

where ϕ_1 is the root of $x^{a+b}+x^b-1=0$ which lies in the interval $[0,1]$, ϕ_2 is the root of $x^{a+b}-x^b-1=0$ which lies in $[1,\infty)$, and $\mathscr{L}_3(z)$ is the trilogarithm given by equation (8).

However, equation (1) differs from Theorem 1 in two fundamental ways. Firstly, the variation of exponents in (1) affects linear terms (of the form $1-w$) and ultimately the coefficients of the rational function vary as a, b vary, while the variation of exponents in Theorem 1 only affects the powers of x but not the coefficients. This leads to an interesting application of Boyd–Lawton limit formula. More precisely,

$$\begin{aligned}\lim_{\substack{a\to\infty\\ b\to\infty}} \mathrm{m}\big(x^{a+b}+1+(x^a+1)y+(x^b-1)z\big) =& \mathrm{m}(xw+1+(w+1)y+(x-1)z)\\ =& \frac{9}{2\pi^2}\zeta(3),\end{aligned}$$

as proven in ([DL07], Theorem 7). Moreover, as Boyd–Lawton formula predicts, the same limit is valid if we fix a and let b go to infinity, or vice versa.

Secondly, the proof of Theorem 1 presents an innovative technical aspect regarding the integration of the regulator. Usually the first step in the integration consists of an application of Jensen's formula that eliminates one variable. The rest of the work is done with the remaining two variables, in a variety defined by a construction proposed by Maillot [Mai03] based on a idea of Darboux [Dar75]. Concretely, we normally complete the computation by integrating in a curve defined in the xy-plane. However, in this case, due to the complexity of the singularities involved, it does not suffice with eliminating one variable, and we have to consider another point of view and eliminate a different variable. More precisely, we have to both integrate in a curve defined in the xy-plane and in another curve defined in the zx-plane. This introduces interesting new considerations on orientation and repetition of terms.

This paper is organized as follows. In Section 2 we define and present some results on polylogarithms and explain the regulator integration. The integral over the plane xy is treated in Section 3, while the one over the plane zx is treated in Section 4. We combine the relevant terms coming from both integrals in Section 5 to obtain the main result. In Section 6 we discuss the Boyd–Lawton limit for this formula, while we explore avenues for future research in Section 7.

2. Preliminaries on polylogarithms and the regulator integration

Here we proceed to present the necessary background on polylogarithms and the algebraic integration of the regulator that leads to the Mahler measure. This section follows the theory presented in [Lal07], and was built from previous works such as [Den97, RV99, BRV02].

2.1. **Polylogarithms.** The classical nth polylogarithm function is defined by the power series

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad z \in \mathbb{C}, \quad |z| < 1.$$

This function can be continued analytically to $\mathbb{C}\setminus(1,\infty)$ by means of the following integral

$$\mathrm{Li}_1(z) = -\log(1-z), \qquad \mathrm{Li}_{n+1}(z) = \int_0^z \mathrm{Li}_n(x)\frac{dx}{x}.$$

Li_n satisfies some functional equations such as

$$\mathrm{Li}_n(z) = r^{n-1} \sum_{w^r=z} \mathrm{Li}_n(w), \qquad |z| \le 1. \tag{2}$$

In this paper we will work with Zagier's modification of the polylogarithm [Zag91].

$$\mathscr{L}_n(z) := \mathrm{Re}_n\left(\sum_{j=0}^{n-1}\frac{2^jB_j}{j!}\log^j|z|\mathrm{Li}_{n-j}(z)\right), \tag{3}$$

where B_j is the jth Bernoulli number and Re_k denotes Re or Im, depending on whether n is odd or even. This function is one-valued, continuous in $\mathbb{P}^1(\mathbb{C})$, and real analytic in $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$. Notice that in particular, if $|z|=1$, then $\mathscr{L}_n(z)=\mathrm{Re}_n(\mathrm{Li}_n(z))$.

$\mathscr{L}_n$ satisfies cleaner functional equations than Li_n, such as

$$\mathscr{L}_n\left(\frac{1}{z}\right)=(-1)^{n-1}\mathscr{L}_n(z), \qquad \mathscr{L}_n(\bar{z})=(-1)^{n-1}\mathscr{L}_n(z), \tag{4}$$

where $\bar{z}$ denotes the complex conjugate of z.

For $n=2$, equation (3) becomes the Bloch–Wigner dilogarithm,

$$D(z)=\mathrm{Im}(\mathrm{Li}_2(z)-\log|z|\mathrm{Li}_1(z))=\mathrm{Im}(\mathrm{Li}_2(z))+\log|z|\arg(1-z), \tag{5}$$

which satisfies the five-term relation

$$D(x)+D(y)+D(1-xy)+D\left(\frac{1-x}{1-xy}\right)+D\left(\frac{1-y}{1-xy}\right)=0. \tag{6}$$

A particularly useful case of the above equation (setting $x=z$ and $y=1$) is

$$D(1-z)=-D(z). \tag{7}$$

Similarly, the first equation in (4) can be deduced for $D=\mathscr{L}_2$ by setting $x=z=\frac{1}{y}$ in (6).

Analogously, by setting $n=3$ in (3), we obtain the trilogarithm

$$\mathscr{L}_3(z)=\mathrm{Re}\left(\mathrm{Li}_3(z)-\log|z|\mathrm{Li}_2(z)+\frac{1}{3}\log^2|z|\mathrm{Li}_1(z)\right). \tag{8}$$

We close this subsection by defining the Borel group, a structure where we can naturally evaluate the dilogarithm. For F a field, let

$$\mathcal{A}_2(F):=\left\{\sum_j n_j\{z_j\}\in\mathbb{Z}[F]\,\middle|\,\sum_j n_j(z_j\wedge(1-z_j))=0\right\},$$

where the corresponding term in the sum is omitted if $z_j=0,1$. Let

$$\mathcal{C}_2(F):=\left\{\{x\}+\{y\}+\{1-xy\}+\left\{\frac{1-x}{1-xy}\right\}+\left\{\frac{1-y}{1-xy}\right\}\,\middle|\,x,y\in F^*,xy\neq 1\right\}.$$

Then $\mathcal{C}_2(F)\subset\mathcal{A}_2(F)$ and we can define the Borel group by

$$\mathcal{B}_2(F):=\mathcal{A}_2(F)/\mathcal{C}_2(F).$$

Thus, if we extend the Bloch–Wigner dilogarithm to $\mathbb{Z}[\mathbb{C}]$ by linearity, D is naturally defined in $\mathcal{B}_2(\mathbb{C})$. We denote by $\{z\}_2$ the element of $\mathcal{B}_2(F)$ coming from $\{z\}\in\mathbb{Z}[F]$. The subindex 2 indicates that we work with the dilogarithm. It is possible to extend the definition of the Borel groups to $\mathscr{L}_n$. For more details, see [Zag91, Gon05].

2.2. **The regulator integral.** Let $P \in \mathbb{C}[x,y,z]$. Then we may write

$$\begin{aligned} P(x,y,z) =& a_d(x,y)z^d + \cdots + a_0(x,y) \\ =& a_d(x,y)\prod_{j=1}^{d}(z-\alpha_j(x,y)), \end{aligned}$$

where we view the roots $\alpha_j(x,y)$ as algebraic functions.

Let $P^*(x,y) = a_d(x,y)$. By applying Jensen's formula (with respect to the variable z) to the definition, we obtain

$$\mathrm{m}(P) = \mathrm{m}(P^*) + \frac{1}{(2\pi i)^2}\sum_{j=1}^{d}\int_{|x|=|y|=1}\log^+|\alpha_j(x,y)|\frac{dx}{x}\frac{dy}{y}, \tag{9}$$

where $\log^+ s = \log s$ if $s > 1$ and 0 otherwise.

Recall that, while the argument $\arg z$ is a multivalued function, $d\arg z$ is well defined as

$$d\arg z = \mathrm{Im}\,\frac{dz}{z}.$$

Let

$$\Gamma = \{P(x,y,z) = 0\} \cap \{|x| = |y| = 1, |z| \geq 1\},$$

and

$$\begin{aligned} \eta(x,y,z) :=& \log|x|\left(\frac{1}{3}d\log|y| \wedge d\log|z| - d\arg y \wedge d\arg z\right) \\ &+ \log|y|\left(\frac{1}{3}d\log|z| \wedge d\log|x| - d\arg z \wedge d\arg x\right) \\ &+ \log|z|\left(\frac{1}{3}d\log|x| \wedge d\log|y| - d\arg x \wedge d\arg y\right). \end{aligned}$$

When $d = 1$, formula (9) may be rewritten as

$$\mathrm{m}(P) = \mathrm{m}(P^*) - \frac{1}{4\pi^2}\int_\Gamma \eta(x,y,z). \tag{10}$$

The differential form $\eta(x,y,z)$ is defined on the surface $S = \{P(x,y,z) = 0\}$ minus the set Z of poles and zeros of x, y, and z.

Since $d\eta(x,y,z) = \mathrm{Re}\left(\frac{dx}{x} \wedge \frac{dy}{y} \wedge \frac{dz}{z}\right)$, we conclude that η in closed in $S \setminus Z$. For the rest of this paper, we will focus on the case in which η is also exact, which will imply that we can proceed with the integration by means of Stokes' theorem. The exactness of η may happen, for example, due to the following identity

$$\eta(x, 1-x, y) = d\,\omega(x,y),$$

where

$$\omega(x,y) := -D(x)d\arg y + \frac{1}{3}\log|y|(\log|1-x|d\log|x| - \log|x|d\log|1-x|).$$

Thus, in order to apply Stokes' theorem, we could require that

$$x \wedge y \wedge z = \sum_j r_j x_j \wedge (1-x_j) \wedge y_j \tag{11}$$

in $\bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$ for η to be exact.

In this case,

$$\int_\Gamma \eta(x,y,z) = \sum_j r_j \int_\Gamma \eta(x_j, 1-x_j, y_j) = \sum_j r_j \int_{\partial\Gamma} \omega(x_j, y_j),$$

where

$$\partial\Gamma = \{P(x,y,z) = 0\} \cap \{|x| = |y| = |z| = 1\}.$$

At first sight the set $\partial\Gamma$ seems to have trivial boundary. However, $\partial\Gamma$ as described above may contain singularities which may give rise to a boundary when desingularized. We proceed to change our point of view. Assume that $P \in \mathbb{R}[x,y,z]$ and is nonreciprocal (this condition is true for the polynomials we consider in this work); then,

$$P(x,y,z) = P(\bar{x}, \bar{y}, \bar{z}).$$

This property, together with the condition $|x| = |y| = |z| = 1$, allows us to write

$$\partial\Gamma = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = |z| = 1\}.$$

This argument was proposed by Maillot [Mai03], after an idea of Darboux [Dar75]. Observe that we are integrating now on a path $\{|x| = |y| = |z| = 1\}$ inside the curve

$$C = \{\mathrm{Res}_z(P(x,y,z), P(x^{-1}, y^{-1}, z^{-1})) = 0\}.$$

In order to easily compute

$$\int_{\partial\Gamma} \omega(x,y),$$

we could also focus on the case when ω, defined in this new curve C, is exact. In fact, we have

$$\omega(x,x) = d\mathscr{L}_3(x).$$

The condition for ω to be exact is not as easily established as in the preceding cases because ω is not multiplicative in the first variable. In fact, the first variable behaves as the argument for the dilogarithm; in other words, the transformations are ruled by the five-term relation, with the first variable belonging to the Borel group. We may express the condition we need as

$$\{x\}_2 \otimes y = \sum_j r_j \{x_j\}_2 \otimes x_j \tag{12}$$

in $(\mathcal{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$. Finally, we have

$$\int_\gamma \omega(x,y) = \sum_j r_j \; \mathscr{L}_3(x_j)|_{\partial\gamma},$$

where $\gamma = C \cap \mathbb{T}^2$.

In conclusion, our strategy for evaluating the Mahler measure of $x^{a+b} + 1 + (x^a+1)y + (x^b-1)z$ will consist of solving equation (11) first, and then equation (12) when appropriate.

3. The treatment over the xy-plane

In this section we perform the integral by eliminating the variable z. As explained in the previous section, we proceed to solve equation (11) first. Then we work on determining the integration path $\partial\Gamma$. The final integration will be done by solving equation (12) in some cases that will lead to terms involving the Riemann zeta function, and by direct evaluation in other cases that will lead to terms involving the dilogarithm.

3.1. **The initial integrand.** Our first goal is to solve the corresponding equation (11). Since we are working on $S=\{P(x,y,z)=0\}$, we assume that

$$z=-\frac{x^{a+b}+1+(x^a+1)y}{x^b-1} \tag{13}$$

and proceed.

Remark 2. *In all our computations involving the wedge product we can ignore terms of the form* $(\pm 1)\wedge\alpha\wedge\beta$ *since they lead to* $\eta=0$.

We obtain

$$\begin{aligned} x\wedge y\wedge z =& x\wedge y\wedge \frac{x^{a+b}+1+(x^a+1)y}{x^b-1}\\ =& x\wedge y\wedge \frac{x^{a+b}+1}{x^b-1}+x\wedge y\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right). \end{aligned} \tag{14}$$

The first term in (14) equals

$$x\wedge y\wedge\frac{x^{a+b}+1}{x^b-1}=-\frac{1}{a+b}x^{a+b}\wedge\left(1+x^{a+b}\right)\wedge y+\frac{1}{b}x^b\wedge\left(1-x^b\right)\wedge y,$$

while the second term gives

$$\begin{aligned} x\wedge y\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right)=&\frac{(x^a+1)y}{x^{a+b}+1}\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right)\wedge x - x\wedge\frac{x^a+1}{x^{a+b}+1}\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right)\\ =&\frac{(x^a+1)y}{x^{a+b}+1}\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right)\wedge x-\frac{1}{a}x^a\wedge(1+x^a)\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right)\\ &+\frac{1}{a+b}x^{a+b}\wedge(1+x^{a+b})\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right). \end{aligned}$$

Combining everything, we obtain

$$\begin{aligned} x\wedge y\wedge z=&\frac{1}{b}x^b\wedge\left(1-x^b\right)\wedge y-\frac{1}{a}x^a\wedge(1+x^a)\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right)\\ &+\frac{1}{a+b}x^{a+b}\wedge(1+x^{a+b})\wedge\left(\frac{1}{y}+\frac{x^a+1}{x^{a+b}+1}\right)\\ &+\frac{(x^a+1)y}{x^{a+b}+1}\wedge\left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right)\wedge x. \end{aligned}$$

Therefore, the goal is to integrate $\omega(\Delta)$ in $\partial\Gamma$, where

$$
\begin{aligned}
\Delta =&\frac{1}{b}\{x^b\}_2 \otimes y - \frac{1}{a}\{-x^a\}_2 \otimes \left(1+\frac{(x^a+1)y}{x^{a+b}+1}\right) + \frac{1}{a+b}\{-x^{a+b}\}_2 \otimes \left(\frac{1}{y}+\frac{x^a+1}{x^{a+b}+1}\right)\\
&+\left\{-\frac{(x^a+1)y}{x^{a+b}+1}\right\}_2 \otimes x.
\end{aligned}
\tag{15}
$$

3.2. **The integration path.** Our goal in this subsection is to describe the integration path $\partial\Gamma$. By applying Maillot's trick, the integration path is given by

$$
\left(\frac{x^{a+b}+1+(x^a+1)y}{x^b-1}\right)\left(\frac{x^{-a-b}+1+(x^{-a}+1)y^{-1}}{x^{-b}-1}\right)=1,
$$

which reduces to

$$
(x^a+1)(x^{a+b}+1)(y+1)(y+x^b)=0. \tag{16}
$$

The above set does not take into account singularities resulting from having zero divided by zero in the rational fraction (13) expressing z. This essentially amounts to considering the extra condition $x^b=1$. As we will see later in Section 4, the integration of this path contributes to the final calculation.

We can write the integration path (16) as

$$
\partial\Gamma_{xy}=\bigcup_{j=1}^{4} T_j,
$$

where

$$
\begin{aligned}
T_1 &= \left\{(x,y,z)\in\mathbb{C}^3 \middle| |x|=1, y=-1, z=-\frac{x^{a+b}+1+(x^a+1)y}{x^b-1}\right\},\\
T_2 &= \left\{(x,y,z)\in\mathbb{C}^3 \middle| x^{a+b}=-1, |y|=1, z=-\frac{x^{a+b}+1+(x^a+1)y}{x^b-1}\right\},\\
T_3 &= \left\{(x,y,z)\in\mathbb{C}^3 \middle| x^{a}=-1, |y|=1, z=-\frac{x^{a+b}+1+(x^a+1)y}{x^b-1}\right\},\\
T_4 &= \left\{(x,y,z)\in\mathbb{C}^3 \middle| |x|=|y|=1, y=-x^b, z=-\frac{x^{a+b}+1+(x^a+1)y}{x^b-1}\right\}.
\end{aligned}
$$

Representing the paths in the plane of $\arg x$ and $\arg y$, we have the following arguments to consider.

Set	$\arg x$	$\arg y$
T_1	$(-\pi, \pi]$	π
T_2	$\frac{(2k-1)\pi}{a+b}, -\frac{a+b-1}{2} < k \leq \frac{a+b+1}{2}$	$(-\pi, \pi]$
T_3	$\frac{(2k-1)\pi}{a}, -\frac{a-1}{2} < k \leq \frac{a+1}{2}$	$(-\pi, \pi]$
T_4	$(-\pi, \pi]$	$(b \arg x + \pi) \bmod 2\pi$

Table 1: Arguments of x and y according to the corresponding path in $\partial\Gamma_{xy}$.

Remark 3. *The notation* $\alpha \bmod 2\pi$ *denotes a number* $-\pi < \beta \leq \pi$ *such that* $\beta \equiv \alpha \bmod 2\pi$.

We have illustrated the integration paths for $a = 3$ and $b = 4$ in Figure 1. Notice that the paths are all contained in the square $[-\pi, \pi] \times [-\pi, \pi]$. It is useful to understand the picture with periodic boundary conditions.

It is not immediately obvious how to properly describe the exact paths of integration in T_1 and in T_4, since this requires to write the fractions $\frac{(2u-1)\pi}{a}$ and $\frac{(2v-1)\pi}{a+b}$ in an increasing order. More precisely, the set of end points of the paths in T_1 is

$$E = E_1 \cup E_2,$$

where

$$E_1 = \left\{ \arg x = \frac{2k\pi}{b} \,\middle|\, -\frac{b}{2} < k \leq \frac{b}{2} \right\}$$

corresponds to the first coordinates of the points where T_1 and T_4 intersect, and

$$E_2 = \left\{ \arg x = \frac{(2k-1)\pi}{a} \,\middle|\, -\frac{a-1}{2} < k \leq \frac{a+1}{2} \right\}$$

corresponds to the first coordinates of the points where T_1 and T_3 intersect. Also, the origin points of the paths on T_1 form the set

$$F = \left\{ \arg x = \frac{(2k-1)\pi}{a+b} \,\middle|\, -\frac{a+b-1}{2} < k \leq \frac{a+b+1}{2} \right\},$$

which corresponds to the first coordinates of the points where T_1 intersects T_2.

When considering the paths in $T_1 \cup T_4$ with respect to their first coordinate $\arg x$, the paths in T_4 have the opposite orientation to the ones in T_1.

The following result tells us that we can not have two consecutive points of F, as they are always separated by points of E_1 or E_2. It also tells us that these sets are essentially disjoint, with the possibly exception at the boundary corresponding to $\pm\pi$.

Proposition 4. *Let* $a, b, u, v \in \mathbb{Z}$*,* $a, b > 0$*.*

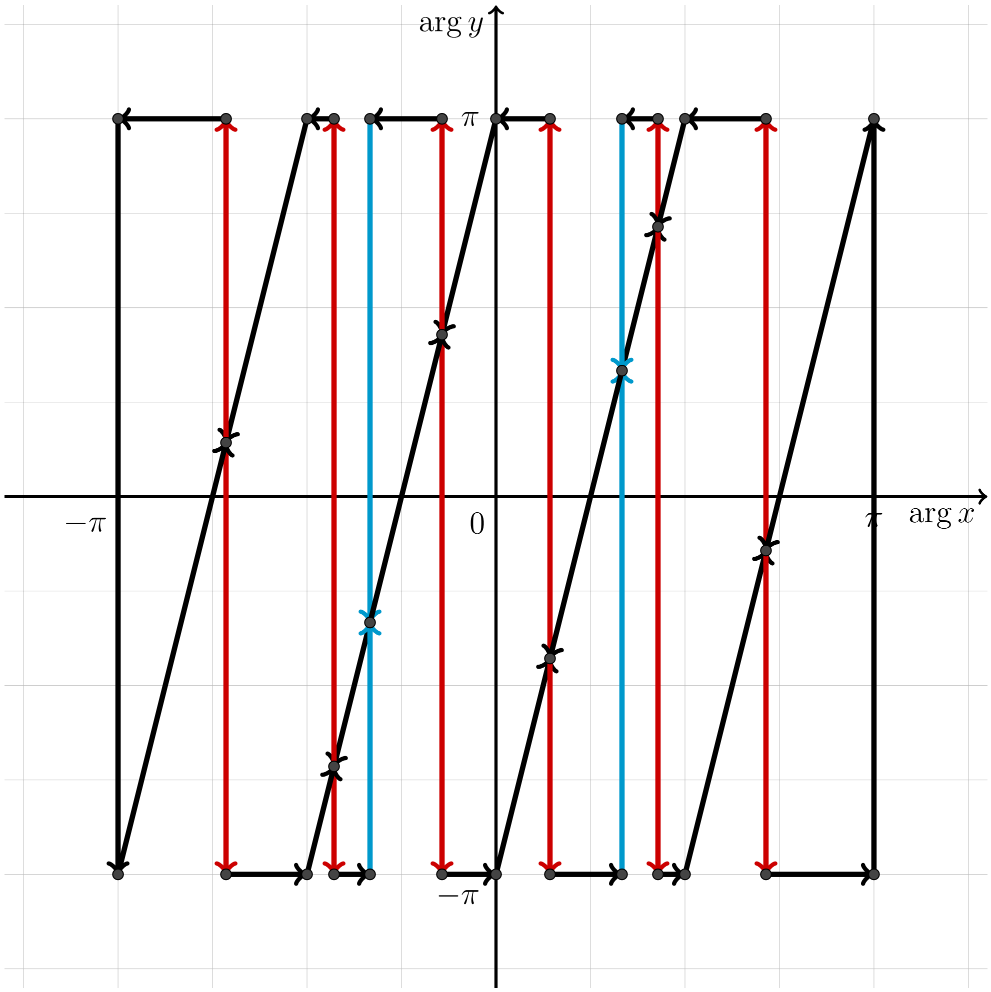

FIGURE 1. An illustration of the integration path when $a = 3$ and $b = 4$ in the plance xy. The vectors in red are multiples of $\frac{\pi}{7}$ and the ones in blue are multiples of $\frac{\pi}{3}$.

(i) If

$$\frac{2u-1}{a} < \frac{2v-1}{a+b} < \frac{2v+1}{a+b} < \frac{2u+1}{a},$$

then

$$\frac{2v-1}{a+b} < \frac{2(v-u)}{b} < \frac{2v+1}{a+b}.$$

(ii) If

$$\frac{2u}{b} < \frac{2v-1}{a+b} < \frac{2v+1}{a+b} < \frac{2u+2}{b},$$

then

$$\frac{2v-1}{a+b} < \frac{2(v-u)-1}{a} < \frac{2v+1}{a+b}.$$

(iii) If, in addition, a, b are coprime, $-\frac{a-1}{2} < u < \frac{a+1}{2}$ and $-\frac{b}{2} < v < \frac{b}{2}$, we have

$$\min\left\{\frac{2u-1}{a}, \frac{2v}{b}\right\} < \frac{2(u+v)-1}{a+b} < \max\left\{\frac{2u-1}{a}, \frac{2v}{b}\right\}$$

Proof. From $\frac{2u-1}{a} < \frac{2v-1}{a+b}$, we see that

$$(2u-1)(a+b) < (2v-1)a.$$

Therefore

$$(2u-1)b < 2(v-u)a.$$

By adding $2(v-u)b$ on both sides, we obtain

$$(2v-1)b < 2(v-u)(a+b),$$

and the left-hand side inequality in (i) follows. The other inequality in (i) and those of (ii) can be proven in a similar way.

Finally, (iii) is a consequence of the fact that $\frac{2u-1}{a} \neq \frac{2v}{b}$, which is true since $(a, b) = 1$. $\square$

Proposition 4 implies that there is a precise shuffle between the elements of $E_1 \cup E_2$ and those of F, in the sense that they alternate. Each element of F is placed between two elements of $E_1 \cup E_2$ and vice versa.

Accordingly, for $-\frac{a+b-1}{2} < k \leq \frac{a+b+1}{2}$, we define

$$k_\ell = \max\left\{\max\left\{\frac{2u}{b}\,\middle|\,\frac{2u}{b} \leq \frac{2k-1}{a+b}\right\}, \max\left\{\frac{2v-1}{a}\,\middle|\,\frac{2v-1}{a} \leq \frac{2k-1}{a+b}\right\}\right\},$$

and similarly

$$k_u = \min\left\{\min\left\{\frac{2u}{b}\,\middle|\,\frac{2u}{b} \geq \frac{2k-1}{a+b}\right\}, \min\left\{\frac{2v-1}{a}\,\middle|\,\frac{2v-1}{a} \geq \frac{2k-1}{a+b}\right\}\right\}.$$

By Stokes's theorem, we orient the paths in the following way.

Set	$\arg x$		$\arg y$
T_1	$\frac{(2k-1)\pi}{a+b} \to k_\ell \pi$, $\frac{(2k-1)\pi}{a+b} \to k_u \pi$.	$-\frac{a+b-1}{2} < k \leq \frac{a+b+1}{2}$	π
T_2	$\frac{(2k-1)\pi}{a+b}$	$-\frac{a+b-1}{2} < k \leq \frac{a+b+1}{2}$	$\left(\frac{(2k-1)b\pi}{a+b} + \pi\right) \bmod 2\pi \to \pi$ $\left(\frac{(2k-1)b\pi}{a+b} + \pi\right) \bmod 2\pi \to -\pi$
T_3	$\frac{(2k-1)\pi}{a}$	$-\frac{a-1}{2} < k \leq \frac{a+1}{2}$	$\pi \to \left(\frac{(2k-1)b\pi}{a} + \pi\right) \bmod 2\pi$ $-\pi \to \left(\frac{(2k-1)b\pi}{a} + \pi\right) \bmod 2\pi$
T_4	$k_\ell \pi \to \frac{(2k-1)\pi}{a+b}$, $k_u \pi \to \frac{(2k-1)\pi}{a+b}$.	$-\frac{a+b-1}{2} < k \leq \frac{a+b+1}{2}$	$(b \arg x + \pi) \bmod 2\pi$

Table 2: Orientation of the integral over the path $\partial\Gamma_{xy}$

Remark 5. *In the case where b is even, then a is necessarily odd, and both T_2 and T_3 contribute with a path at $x = \pi$. In T_2, this path is $\pi \to -\pi$ while in the case of T_3, it is $-\pi \to \pi$.*

3.3. **Restricting the integrand to $\partial\Gamma_{xy}$.** Our goal here is to restrict Δ to each component T_j of $\partial\Gamma_{xy}$. We will use the notation

$$\Delta_j := \Delta|_{T_j}.$$

We will consider each Δ_j and express as many terms as possible as the terms in equation (12).

Remark 6. *In all our computations we can ignore terms of the form $\{\pm 1\}_2 \otimes \alpha$ and $\{\alpha\}_2 \otimes (\pm 1)$ since they lead to $\omega = 0$. We proceed similarly for $\{0\}_2 \otimes \alpha$ and $\{\alpha\}_2 \otimes 0$ as long as $|\alpha| = 1$. In addition, we apply identities (4) and (7) often, namely $\{\alpha\}_2 = -\left\{\frac{1}{\alpha}\right\}_2 = -\{1-\alpha\}_2$.*

Recall that in T_1 we have $y = -1$. Therefore equation (15) becomes

$$\Delta_1 = -\frac{1}{a}\{-x^a\}_2 \otimes \frac{x^a(x^b-1)}{x^{a+b}+1} + \frac{1}{a+b}\{-x^{a+b}\}_2 \otimes \frac{x^a(1-x^b)}{x^{a+b}+1} + \left\{\frac{x^a+1}{x^{a+b}+1}\right\}_2 \otimes x.$$

In T_2 we have $x^{a+b} = -1$. Thus we have

$$\begin{aligned}\Delta_2 =& \frac{1}{b}\{x^b\}_2 \otimes y - \frac{1}{a}\{-x^a\}_2 \otimes y + \frac{1}{a}\{-x^a\}_2 \otimes (x^a+1)\\ =& \frac{1}{b}\{x^b\}_2 \otimes y - \frac{1}{a}\left\{\frac{1}{x^b}\right\}_2 \otimes y + \frac{1}{a}\{-x^a\}_2 \otimes (x^a+1)\\ =& \frac{a+b}{ab}\{x^b\}_2 \otimes y - \frac{1}{a}\{x^a+1\}_2 \otimes (x^a+1).\end{aligned}$$

In T_3 we have $x^a = -1$ and this leads to

$$\Delta_3 = \frac{a}{(a+b)b}\{x^b\}_2 \otimes y.$$

Over T_4 we have $y = -x^b$ and

$$\begin{aligned}\Delta_4 =& \frac{1}{b}\{x^b\}_2 \otimes x^b - \frac{1}{a}\{-x^a\}_2 \otimes \frac{1-x^b}{x^{a+b}+1} + \frac{1}{a+b}\{-x^{a+b}\}_2 \otimes \frac{x^b-1}{x^b(x^{a+b}+1)}\\ &+ \left\{\frac{x^b(x^a+1)}{x^{a+b}+1}\right\}_2 \otimes x.\end{aligned}$$

3.4. **Integrating $\omega(\Delta)$.** Here we proceed to integrate $\omega(\Delta)$ over T_j. We consider three cases. We take T_1 and T_4 together, since they can be parametrized with the same path over the first coordinate $\arg x$.

3.4.1. *Integration over $T_1 \cup T_4$.* We have

$$\begin{aligned}\int_{T_1\cup T_4} \omega(\Delta) =& \int_{T_1} \omega(\Delta_1) + \int_{T_4} \omega(\Delta_4)\\ =& \int_{T_1} \omega(\Delta_1 - \Delta_4).\end{aligned}$$

We find

$$\begin{aligned}\Delta_1 - \Delta_4 =& -\frac{1}{a}\{-x^a\}_2 \otimes x^a - \frac{1}{b}\{x^b\}_2 \otimes x^b + \frac{1}{a+b}\{-x^{a+b}\}_2 \otimes x^{a+b}\\ &+ \left\{\frac{x^a+1}{x^{a+b}+1}\right\}_2 \otimes x - \left\{\frac{x^b(x^a+1)}{x^{a+b}+1}\right\}_2 \otimes x.\end{aligned} \tag{17}$$

Notice that the five-term relation for the dilogarithm (6) implies

$$\{-x^a\}_2 + \{x^b\}_2 + \{1+x^{a+b}\}_2 + \left\{\frac{x^a+1}{x^{a+b}+1}\right\}_2 + \left\{\frac{1-x^b}{x^{a+b}+1}\right\}_2 = 0,$$

and (7) gives

$$\left\{\frac{1-x^b}{x^{a+b}+1}\right\}_2 = -\left\{1 - \frac{1-x^b}{x^{a+b}+1}\right\}_2 = -\left\{\frac{x^b(x^a+1)}{x^{a+b}+1}\right\}_2.$$

Combining this with equation (17), we obtain

$$\begin{aligned}\Delta_1 - \Delta_4 =& -\frac{1}{a}\{-x^a\}_2 \otimes x^a - \frac{1}{b}\{x^b\}_2 \otimes x^b + \frac{1}{a+b}\{-x^{a+b}\}_2 \otimes x^{a+b} \\ & - \{-x^a\}_2 \otimes x - \frac{1}{b}\{x^b\}_2 \otimes x^b - \{1+x^{a+b}\}_2 \otimes x \\ =& -\frac{2}{a}\{-x^a\}_2 \otimes x^a - \frac{2}{b}\{x^b\}_2 \otimes x^b + \frac{2}{a+b}\{-x^{a+b}\}_2 \otimes x^{a+b}.\end{aligned}$$

Evaluating on ω, this leads to

$$\begin{aligned}\int_{T_1\cup T_4} \omega(\Delta) =& \left(-\frac{2}{a}\mathscr{L}_3(-x^a) - \frac{2}{b}\mathscr{L}_3(x^b) + \frac{2}{a+b}\mathscr{L}_3(-x^{a+b})\right)\Big|_{\partial T_1} \\ =& 2\left(\sum_{x\in E_1\cup E_2} - \sum_{x\in F}\right)\left(-\frac{2}{a}\mathscr{L}_3(-x^a) - \frac{2}{b}\mathscr{L}_3(x^b) + \frac{2}{a+b}\mathscr{L}_3(-x^{a+b})\right),\end{aligned}$$

where the factor of 2 appears because each origin and final point is counted exactly twice in the segments.

We will use identity (2) repeatedly. For example, the term $\sum_{x\in E_1}\mathscr{L}_3(-x^a)$ corresponds to the sum of Li_3 over all the b-roots of $(-1)^b$. Indeed, $(-x^a)^b = (-1)^b$ for $x \in E_1$, and since $(a,b) = 1$, summing $\mathscr{L}_3(-x^a)$ over E_1 is the same as summing $\mathscr{L}_3(-x)$ over E_1. Finally $\mathscr{L}_3(z) = \operatorname{Re}\mathrm{Li}_3(z)$ for $|z| = 1$. A similar reasoning for each term leads to

$$\begin{aligned}\int_{T_1\cup T_4} \omega(\Delta) =& 2\left(-\frac{2}{ab^2}\mathrm{Li}_3((-1)^b) - 2\mathrm{Li}_3(1) + \frac{2}{(a+b)b^2}\mathrm{Li}_3((-1)^b)\right) \\ & + 2\left(-2\mathrm{Li}_3(1) - \frac{2}{ba^2}\mathrm{Li}_3((-1)^b) + \frac{2}{(a+b)a^2}\mathrm{Li}_3((-1)^b)\right) \\ & - 2\left(-\frac{2}{a(a+b)^2}\mathrm{Li}_3((-1)^b) - \frac{2}{b(a+b)^2}\mathrm{Li}_3((-1)^b) + 2\mathrm{Li}_3(1)\right) \\ =& -12\zeta(3) - \frac{4}{ab(a+b)}\mathrm{Li}_3((-1)^b).\end{aligned}$$

3.4.2. *Integration over* T_2. Evaluating on ω, the integration over T_2 leads to

$$\int_{T_2} \omega(\Delta_2) = -\frac{a+b}{ab} \sum_{\substack{x=\exp\left(\frac{(2k-1)\pi i}{a+b}\right) \\ -\frac{a+b-1}{2} < k \le \frac{a+b+1}{2}}} D(x^b)\int d\arg y,$$

since the term coming from $\{x^a+1\}_2 \otimes (x^a+1)$ is independent of y and constant in each component of T_3 and therefore its integral equals 0. Thus

$$\int_{T_2} \omega(\Delta_2) = \frac{2(a+b)}{ab} \sum_{-\frac{a+b-1}{2} < k \le \frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)b\pi i}{a+b}\right)\right)\left[\left(\frac{(2k-1)b\pi}{a+b} + \pi\right) \bmod 2\pi\right].$$

Since $(b, a+b) = 1$, we observe that the sum above is taking place over exactly all the $a+b-$th roots of $(-1)^b$.

When b is odd, the formula becomes

$$\begin{aligned}\int_{T_2}\omega(\Delta_2) =&\frac{2(a+b)}{ab}\sum_{-\frac{a+b-1}{2}<k\leq\frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a+b}\right)\right)\left[\left(\frac{(2k-1)\pi}{a+b}+\pi\right)\bmod 2\pi\right]\\
=&\frac{2(a+b)}{ab}\sum_{1\leq k\leq\frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a+b}\right)\right)\left(\frac{(2k-1)\pi}{a+b}-\pi\right)\\
&+\frac{2(a+b)}{ab}\sum_{1\leq k\leq\frac{a+b+1}{2}} D\left(\exp\left(-\frac{(2k-1)\pi i}{a+b}\right)\right)\left(-\frac{(2k-1)\pi}{a+b}+\pi\right)\\
=&\frac{4(a+b)}{ab}\sum_{1\leq k\leq\frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a+b}\right)\right)\left(\frac{(2k-1)\pi}{a+b}-\pi\right).\end{aligned}$$

When b is even, we have

$$\begin{aligned}\int_{T_2}\omega(\Delta_2) =&\frac{2(a+b)}{ab}\sum_{-\frac{a+b}{2}<k\leq\frac{a+b}{2}} D\left(\exp\left(\frac{2k\pi i}{a+b}\right)\right)\left[\left(\frac{2k\pi}{a+b}+\pi\right)\bmod 2\pi\right]\\
=&\frac{2(a+b)}{ab}\sum_{1\leq k\leq\frac{a+b}{2}} D\left(\exp\left(\frac{2k\pi i}{a+b}\right)\right)\left(\frac{2k\pi}{a+b}-\pi\right)\\
&+\frac{2(a+b)}{ab}\sum_{1\leq k\leq\frac{a+b}{2}} D\left(\exp\left(-\frac{2k\pi i}{a+b}\right)\right)\left(-\frac{2k\pi}{a+b}+\pi\right)\\
=&\frac{4(a+b)}{ab}\sum_{1\leq k\leq\frac{a+b}{2}} D\left(\exp\left(\frac{2k\pi i}{a+b}\right)\right)\left(\frac{2k\pi}{a+b}-\pi\right),\end{aligned}$$

where we have used that $D(\pm 1)=0$.

3.4.3. *Integration over T_3.* Evaluating on ω, this integral leads to

$$\begin{aligned}\int_{T_3}\omega(\Delta_3) =& -\frac{a}{(a+b)b}\sum_{\substack{x=\exp\left(\frac{(2k-1)\pi i}{a}\right)\\ -\frac{a-1}{2}<k\leq\frac{a+1}{2}}} D(x^b)\int d\arg y\\
=&-\frac{2a}{(a+b)b}\sum_{-\frac{a-1}{2}<k\leq\frac{a+1}{2}} D\left(\exp\left(\frac{(2k-1)b\pi i}{a}\right)\right)\left[\left(\frac{(2k-1)b\pi}{a}+\pi\right)\bmod 2\pi\right].\end{aligned}$$

As in the case of T_2, we observe that since $(b,a)=1$, the sum above is taking place over exactly all the $a-$th roots of $(-1)^b$.

When b is odd,

$$\begin{aligned}\int_{T_3}\omega(\Delta_3) =& -\frac{2a}{(a+b)b}\sum_{-\frac{a-1}{2}<k\leq\frac{a+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a}\right)\right)\left[\left(\frac{(2k-1)\pi}{a}+\pi\right)\bmod 2\pi\right]\\
=&-\frac{4a}{(a+b)b}\sum_{1\leq k\leq\frac{a+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a}\right)\right)\left(\frac{(2k-1)\pi}{a}-\pi\right).\end{aligned}$$

When b is even,

$$\int_{T_3}\omega(\Delta_3) = -\frac{2a}{(a+b)b}\sum_{-\frac{a}{2}<k\leq\frac{a}{2}} D\left(\exp\left(\frac{2k\pi i}{a}\right)\right)\left[\left(\frac{2k\pi}{a}+\pi\right) \bmod 2\pi\right]$$
$$= -\frac{4a}{(a+b)b}\sum_{1\leq k\leq\frac{a}{2}} D\left(\exp\left(\frac{2k\pi i}{a}\right)\right)\left(\frac{2k\pi}{a}-\pi\right),$$

where we have used that $D(\pm 1)=0$.

4. The treatment over the zx-plane

The analysis from the previous section does not consider the singularities of $-\frac{x^{a+b}+1+(x^a+1)y}{x^b-1}$. Indeed, we have singularities when $x^b-1=0$ and $y=-\frac{x^{a+b}+1}{x^a+1}$. In sum, we need to consider the set

$$\left\{(x,y,z)\in\mathbb{C}^3 \middle| x^b=1, y=-1, |z|=1\right\}.$$

In order to study what happens when integrating in this path, we need to analyze it in the plane zx. In this section we will proceed to make a full study analogous to the previous section. Most of the terms that we will recover here were already present when we did the analysis over the plane xy, and therefore we will not count these terms again.

Before starting with this analysis, we must reflect on the orientation. In this section we eliminate the variable y as oppose to eliminating the variable z as we did in the previous section. Looking at the formula for η and very particularly at equation (10), we see that the term contributing to the Mahler measure corresponds to zx and not xz.

4.1. **The initial integrand.** We start by expressing Δ in terms of x and z. Proceeding similarly as in the xy plane case, we obtain

$$\begin{aligned}
x\wedge y\wedge z =& x\wedge \frac{x^{a+b}+1+\left(x^b-1\right)z}{x^a+1}\wedge z\\
=& x\wedge\frac{x^{a+b}+1}{x^a+1}\wedge z + x\wedge\left(1+\frac{(x^b-1)z}{x^{a+b}+1}\right)\wedge z\\
=& x\wedge\left(x^{a+b}+1\right)\wedge z - x\wedge(x^a+1)\wedge z + x\wedge\left(1+\frac{(x^b-1)z}{x^{a+b}+1}\right)\wedge\left(\frac{(x^b-1)z}{x^{a+b}+1}\right)\\
&- x\wedge\left(1+\frac{(x^b-1)z}{x^{a+b}+1}\right)\wedge\left(x^b-1\right) + x\wedge\left(1+\frac{(x^b-1)z}{x^{a+b}+1}\right)\wedge\left(x^{a+b}+1\right).
\end{aligned}$$

In sum, this results in

(18)
$$\Delta' = -\frac{1}{a}\{-x^a\}_2\otimes z + \frac{1}{b}\{x^b\}_2\otimes\left(1+\frac{(x^b-1)z}{x^{a+b}+1}\right) - \frac{1}{a+b}\{-x^{a+b}\}_2\otimes\left(\frac{1}{z}+\frac{x^b-1}{x^{a+b}+1}\right)$$
$$-\left\{-\frac{(x^b-1)z}{x^{a+b}+1}\right\}_2\otimes x.$$

4.2. **The integration path.** The integration boundaries are to be found in the same way as the plane xy. The condition

$$\left(\frac{x^{a+b}+1+\left(x^b-1\right)z}{x^a+1}\right)\left(\frac{x^{-a-b}+1+\left(x^{-b}-1\right)z^{-1}}{x^{-a}+1}\right)=1$$

reduces to

$$(z-1)(x^a+z)(x^{a+b}+1)(x^b-1)=0.$$

We remark that all the above conditions were considered in the xy-plane in equation (16), except for the condition $x^b=1$. Indeed, $z=1$ corresponds to $(x^a+1)(y+x^b)=0$, $x^a+z=0$ corresponds to $(x^a+1)(y+1)=0$, and $x^{a+b}+1=0$ is already a factor in (16). As before, we write

$$\partial\Gamma_{zx}=\bigcup_{j=1}^{4}T'_j,$$

where

$$T'_1=\left\{(x,y,z)\in\mathbb{C}^3\middle||x|=1, z=1, y=-\frac{x^{a+b}+1+\left(x^b-1\right)z}{x^a+1}\right\},$$

$$T'_2=\left\{(x,y,z)\in\mathbb{C}^3\middle|x^{a+b}=-1, |z|=1, y=-\frac{x^{a+b}+1+\left(x^b-1\right)z}{x^a+1}\right\},$$

$$T'_3=\left\{(x,y,z)\in\mathbb{C}^3\middle|x^{b}=1, |z|=1, y=-\frac{x^{a+b}+1+\left(x^b-1\right)z}{x^a+1}\right\},$$

$$T'_4=\left\{(x,y,z)\in\mathbb{C}^3\middle||x|=|z|=1, z=-x^a, y=-\frac{x^{a+b}+1+\left(x^b-1\right)z}{x^a+1}\right\},$$

We have illustrated the integration paths for $a=3$ and $b=4$ in Figure 2. In this case, it is more natural to represent $0\le\arg z\le 2\pi$ and $-\pi\le\arg x\le\pi$ with the understanding, as before, that we are working with periodic boundary conditions.

By doing an analysis similar to the one we used to produce Table 2, we conclude that the paths can be described as follows.

Set	$\arg z$	$\arg x$	
T'_1	2π	$k_\ell\pi\to\frac{(2k-1)\pi}{a+b}$, $k_u\pi\to\frac{(2k-1)\pi}{a+b}$.	$-\frac{a+b-1}{2}<k\le\frac{a+b+1}{2}$
T'_2	$2\pi\to\left(\frac{(2k-1)a\pi}{a+b}\right)\bmod 2\pi+\pi$ $0\to\left(\frac{(2k-1)a\pi}{a+b}\right)\bmod 2\pi+\pi$	$\frac{(2k-1)\pi}{a+b}$	$-\frac{a+b-1}{2}<k\le\frac{a+b+1}{2}$

T_3'	$\left(\frac{2ka\pi}{b}\right) \bmod 2\pi + \pi \to 2\pi$ $\left(\frac{2ka\pi}{b}\right) \bmod 2\pi + \pi \to 0$	$\frac{2k\pi}{b}$	$-\frac{b}{2} < k \leq \frac{b}{2}$
T_4'	$(a \arg x) \bmod 2\pi + \pi$	$\frac{(2k-1)\pi}{a+b} \to k_\ell \pi,$ $\frac{(2k-1)\pi}{a+b} \to k_u \pi.$	$-\frac{a+b-1}{2} < k \leq \frac{a+b+1}{2}$

Table 3: Orientation of the integral over the path $\partial\Gamma_{zx}$

4.3. **Restricting the integrand to $\partial\Gamma_{zx}$ and integrating $\omega(\Delta')$.** As before, we aim at restricting Δ' to each component of $\partial\Gamma_{zx}$ in order to proceed with the integration over each component. We write

$$\Delta_j' = \Delta'|_{T_j'}.$$

We expect that the integral over $T_1' \cup T_4'$ yield the same coefficient over $\zeta(3)$ as the integral over $T_1 \cup T_4$ did before. Note that

$$\Delta_1' = \frac{1}{b}\{x^b\}_2 \otimes \left(\frac{x^{a+b}+x^b}{x^{a+b}+1}\right) - \frac{1}{a+b}\{-x^{a+b}\}_2 \otimes \left(\frac{x^{a+b}+x^b}{x^{a+b}+1}\right) - \left\{-\frac{x^b-1}{x^{a+b}+1}\right\} \otimes x,$$

and

$$\begin{aligned}\Delta_4' = & -\frac{1}{a}\{-x^a\}_2 \otimes (-x^a) + \frac{1}{b}\{x^b\}_2 \otimes \left(\frac{1+x^a}{x^{a+b}+1}\right) - \frac{1}{a+b}\{-x^{a+b}\}_2 \otimes \left(\frac{1+x^{-a}}{x^{a+b}+1}\right) \\ & - \left\{\frac{x^{a+b}-x^a}{x^{a+b}+1}\right\} \otimes x.\end{aligned}$$

Combining the above terms,

$$\begin{aligned}\Delta_1' - \Delta_4' = & \frac{1}{a}\{-x^a\}_2 \otimes x^a + \frac{1}{b}\{x^b\}_2 \otimes x^b - \frac{1}{a+b}\{-x^{a+b}\}_2 \otimes x^{a+b} \\ & - \left\{-\frac{x^b-1}{x^{a+b}+1}\right\} \otimes x + \left\{\frac{x^{a+b}-x^a}{x^{a+b}+1}\right\} \otimes x \\ = & \frac{2}{a}\{-x^a\}_2 \otimes x^a + \frac{2}{b}\{x^b\}_2 \otimes x^b - \frac{2}{a+b}\{-x^{a+b}\}_2 \otimes x^{a+b},\end{aligned}$$

where we have used, as before, the five-term relation (6).

In addition, we also obtain,

$$\begin{aligned}\Delta_2' = & \frac{a+b}{ab}\{x^b\}_2 \otimes z - \frac{1}{b}\{1-x^b\}_2 \otimes \left(1-x^b\right), \\ \Delta_3' = & -\frac{b}{a(a+b)}\{-x^a\}_2 \otimes z.\end{aligned}$$

4.3.1. *Integration over $T_1' \cup T_4'$.* Given the relative reversal in the direction of the integration path in $T_1' \cup T_4'$ compared to $T_1 \cup T_4$, we recover

$$\int_{T_1' \cup T_4'} \omega(\Delta') = \int_{T_1 \cup T_4} \omega(\Delta).$$

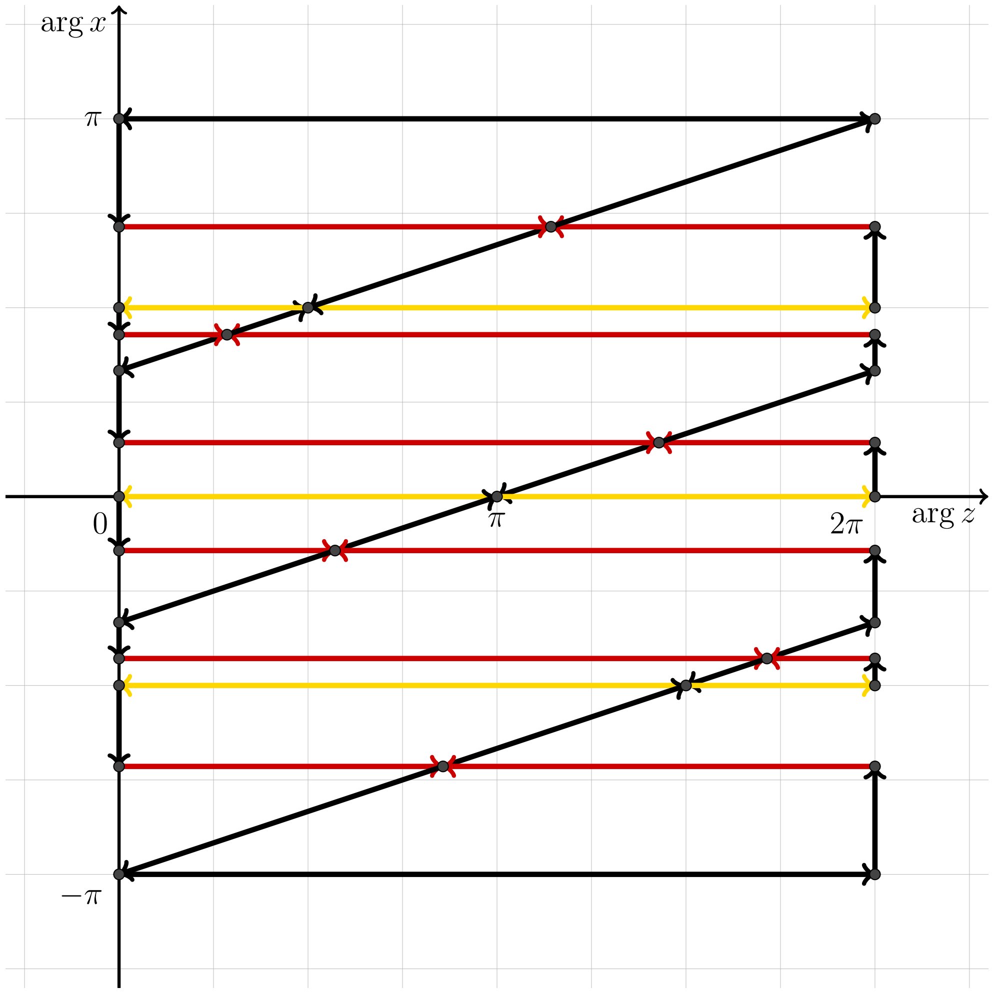

FIGURE 2. An illustration of the integration path when $a = 3$ and $b = 4$ in the plane zx. The vectors in red are multiples of $\frac{\pi}{7}$ and the ones in yellow are multiples of $\frac{\pi}{4}$.

4.3.2. *Integration over T_2'.* Evaluating on ω, this integral leads to

$$\begin{aligned}
\int_{T_2'} \omega(\Delta_2') &= -\frac{a+b}{ab} \sum_{\substack{x=\exp\left(\frac{(2k-1)\pi i}{a+b}\right) \\ -\frac{a+b-1}{2}<k\leq\frac{a+b+1}{2}}} D(x^b) \int d\arg z \\
&= -\frac{2(a+b)}{ab} \sum_{-\frac{a+b-1}{2}<k\leq\frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)b\pi i}{a+b}\right)\right)\left[\left(\frac{a(2k-1)\pi}{a+b}\right) \bmod 2\pi\right] \\
&= -\frac{2(a+b)}{ab} \sum_{-\frac{a+b-1}{2}<k\leq\frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)b\pi i}{a+b}\right)\right)\left[\left(\frac{(a+b-b)(2k-1)\pi}{a+b}\right) \bmod 2\pi\right] \\
&= -\frac{2(a+b)}{ab} \sum_{-\frac{a+b-1}{2}<k\leq\frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)b\pi i}{a+b}\right)\right)\left[\left(\pi-\frac{(2k-1)b\pi}{a+b}\right) \bmod 2\pi\right] \\
&= \int_{T_2} \omega(\Delta_2).
\end{aligned}$$

4.3.3. *Integration over T_3'.* This is the only term that we can not compare to the integration of Δ in the plane xy. Indeed, evaluating on ω, this integral leads to

$$\begin{aligned}
\int_{T_3'} \omega(\Delta_3') =& \frac{b}{a(a+b)} \sum_{\substack{x=\exp\left(\frac{2k\pi i}{b}\right)\\ -\frac{b}{2}<k\leq\frac{b}{2}}} D(-x^a) \int d\arg z \\
=& -\frac{2b}{a(a+b)} \sum_{-\frac{b}{2}<k\leq\frac{b}{2}} D\Big(-\exp\left(\frac{2ka\pi i}{b}\right)\Big) \left[\left(\frac{a2k\pi}{b}\right) \bmod 2\pi\right] \\
=& -\frac{2b}{a(a+b)} \sum_{-\frac{b}{2}<k\leq\frac{b}{2}} D\Big(\exp\left(\frac{2ka\pi i}{b}-\pi i\right)\Big) \left[\left(\left(\frac{a2k\pi}{b}-\pi\right)+\pi\right) \bmod 2\pi\right].
\end{aligned}$$

As in the case of T_2, we observe that since $(b,a)=1$, the sum above is taking place over exactly all the $b-$th roots of $(-1)^b$. Then this sum depends again on the parity of b.

For b odd we have

$$\begin{aligned}
\int_{T_3'} \omega(\Delta_3') =& -\frac{2b}{a(a+b)} \sum_{-\frac{b-1}{2}<k\leq\frac{b+1}{2}} D\Big(\exp\left(\frac{(2k-1)\pi i}{b}\right)\Big) \left[\left(\frac{(2k-1)\pi}{b}+\pi\right) \bmod 2\pi\right] \\
=& -\frac{4b}{a(a+b)} \sum_{1\leq k\leq\frac{b+1}{2}} D\Big(\exp\left(\frac{(2k-1)\pi i}{b}\right)\Big)\Big(\frac{(2k-1)\pi}{b}-\pi\Big).
\end{aligned}$$

For b even we have

$$\begin{aligned}
\int_{T_3'} \omega(\Delta_3') =& -\frac{2b}{a(a+b)} \sum_{-\frac{b}{2}<k\leq\frac{b}{2}} D\Big(\exp\left(\frac{2k\pi i}{b}\right)\Big) \left[\left(\frac{2k\pi}{b}+\pi\right) \bmod 2\pi\right] \\
=& -\frac{4b}{a(a+b)} \sum_{1\leq k\leq\frac{b}{2}} D\Big(\exp\left(\frac{2k\pi i}{b}\right)\Big)\Big(\frac{2k\pi}{b}-\pi\Big).
\end{aligned}$$

5. Conclusion of the proof

In this section we combine the results of Sections 3 and 4. Notice that the integral over $T_1 \cup T_4$ corresponds to the integral over $T_1' \cup T_4'$, and similarly for the integrals over T_2 and T_2'. However, the integrals over T_3 and T_3' correspond to different contributions. More precisely, we can write that

$$\int_{\partial\Gamma} \omega = \sum_{j=1}^{4} \int_{T_j} \omega + \int_{T_3'} \omega.$$

Putting everything together, we obtain, for b odd,

$$\begin{aligned}\int_{\partial\Gamma}\omega =& -3\left(4-\frac{1}{ab(a+b)}\right)\zeta(3)\\ &+\frac{4(a+b)}{ab}\sum_{1\le k\le \frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a+b}\right)\right)\left(\frac{(2k-1)\pi}{a+b}-\pi\right)\\ &-\frac{4a}{(a+b)b}\sum_{1\le k\le \frac{a+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a}\right)\right)\left(\frac{(2k-1)\pi}{a}-\pi\right)\\ &-\frac{4b}{a(a+b)}\sum_{1\le k\le \frac{b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{b}\right)\right)\left(\frac{(2k-1)\pi}{b}-\pi\right).\end{aligned}$$

For b even

$$\begin{aligned}\int_{\partial\Gamma}\omega =& -4\left(3+\frac{1}{ab(a+b)}\right)\zeta(3)\\ &+\frac{4(a+b)}{ab}\sum_{1\le k\le \frac{a+b}{2}} D\left(\exp\left(\frac{2k\pi i}{a+b}\right)\right)\left(\frac{2k\pi}{a+b}-\pi\right)\\ &-\frac{4a}{(a+b)b}\sum_{1\le k\le \frac{a}{2}} D\left(\exp\left(\frac{2k\pi i}{a}\right)\right)\left(\frac{2k\pi}{a}-\pi\right)\\ &-\frac{4b}{a(a+b)}\sum_{1\le k\le \frac{b}{2}} D\left(\exp\left(\frac{2k\pi i}{b}\right)\right)\left(\frac{2k\pi}{b}-\pi\right).\end{aligned}$$

Once we know $\int_{\partial\Gamma}\omega$, we can obtain the Mahler measure by using formula (10). In our case, $P^* = x^b - 1$ or $P^* = x^a + 1$, and therefore $\mathrm{m}(P^*)$ has no contribution to the final result.

6. An application of Boyd–Lawton Theorem

In this section we discuss the result of Theorem 1 in the light of Boyd–Lawton formula. More specifically, the following statement is true.

Theorem 7. [Boy81b, Boy81a, Law83] *Let $P(x_1,\dots,x_n)\in\mathbb{C}[x_1,\dots,x_n]$ and $\mathbf{r}=(r_1,\dots,r_n)$, $r_j\in\mathbb{Z}_{>0}$. Define $P_{\mathbf{r}}(x)$ as*

$$P_{\mathbf{r}}(x) = P(x^{r_1},\dots,x^{r_n}),$$

and let

$$q(\mathbf{r}) = \min\left\{H(\mathbf{t}) :\ \mathbf{t}=(t_1,\dots,t_n)\in\mathbb{Z}^n,\ \mathbf{t}\neq(0,\dots,0), \sum_{j=1}^n t_j r_j = 0\right\},$$

where $H(\mathbf{t}) = \max\{|t_j| :\ 1\le j\le n\}$. Then

$$\lim_{q(\mathbf{r})\to\infty}\mathrm{m}(P_{\mathbf{r}}) = \mathrm{m}(P).$$

More precisely, the above theorem says that if we replace $x_1,\dots,x_n$ by powers of a fixed variable x and let the exponents go to infinity independently from each other, then the Mahler measure of the one-variable polynomials obtained in this way approaches the Mahler measure of the original polynomial. In fact, more is true. We may choose to leave one of the exponents fixed. For example, we may choose to fix r_1, and only let $r_2, \dots, r_n$ go to infinity independently. This formula has been used in applications to polynomials of small Mahler measure, see for example [BM05].

Theorem 1 provides an example of a variation of this result. More precisely, we will prove the following.

Proposition 8. *We have that*

$$\begin{aligned}
&\lim_{\substack{a\to\infty\\ b\to\infty}} \mathrm{m}\big(x^{a+b}+1+(x^a+1)y+(x^b-1)z\big)\\
=&\lim_{a\to\infty} \mathrm{m}\big(x^{a+b}+1+(x^a+1)y+(x^b-1)z\big)\\
=&\lim_{b\to\infty} \mathrm{m}\big(x^{a+b}+1+(x^a+1)y+(x^b-1)z\big)\\
=&\mathrm{m}(xw+1+(w+1)y+(x-1)z)=\frac{9}{2\pi^2}\zeta(3),
\end{aligned}$$

where the first limit is taken with a and b independent of each other.

Here we remark that

$$\begin{aligned}
\mathrm{m}(xw+1+(w+1)y+(x-1)z) =&\mathrm{m}((1+w)(1+y)-(1-x)(w+z))\\
=&\mathrm{m}((1-w)(1-y)-(1-x)(1-z)),
\end{aligned}$$

and the Mahler measure of this last polynomial was computed in Theorem 7 of [DL07].

It is clear that the term involving $\zeta(3)$ in the formulas of Theorem 1 approaches $\frac{3}{\pi^2}\zeta(3)$ as either a or b go to infinity. Thus the proposition will be proven if we can prove the following.

Lemma 9. *We have that*

$$\begin{aligned}
&\lim_{a\to\infty}\Bigg[-\frac{a+b}{ab}\sum_{1\le k\le \frac{a+b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a+b}\right)\right)\left(\frac{(2k-1)\pi}{a+b}-\pi\right)\\
&\quad+\frac{a}{b(a+b)}\sum_{1\le k\le \frac{a+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{a}\right)\right)\left(\frac{(2k-1)\pi}{a}-\pi\right)\\
&\quad+\frac{b}{a(a+b)}\sum_{1\le k\le \frac{b+1}{2}} D\left(\exp\left(\frac{(2k-1)\pi i}{b}\right)\right)\left(\frac{(2k-1)\pi}{b}-\pi\right)\Bigg]\\
=&\frac{3}{2}\zeta(3),
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\lim_{a\to\infty}\Bigg[-\frac{a+b}{ab}\sum_{1\le k\le\frac{a+b}{2}} D\Big(\exp\Big(\frac{2k\pi i}{a+b}\Big)\Big)\Big(\frac{2k\pi}{a+b}-\pi\Big)\\
&\quad+\frac{a}{b(a+b)}\sum_{1\le k\le\frac{a}{2}} D\Big(\exp\Big(\frac{2k\pi i}{a}\Big)\Big)\Big(\frac{2k\pi}{a}-\pi\Big)\\
&\quad+\frac{b}{a(a+b)}\sum_{1\le k\le\frac{b}{2}} D\Big(\exp\Big(\frac{2k\pi i}{b}\Big)\Big)\Big(\frac{2k\pi}{b}-\pi\Big)\Bigg]\\
&=\frac{3}{2}\zeta(3).
\end{aligned}\tag{20}$$

Remark that the above formulas are independent of b. Thus, we may choose to take the limit when b goes to infinity *in addition* to the limit when a goes to infinity, and it does not change the result. Due to symmetry, the same formulas apply if we take the limit when b goes to infinity *instead* of the limit when a goes to infinity. Thus, Lemma 9 suffices to conclude the three limits in the statement of Proposition 8.

Proof of Lemma 9. We prove the first limit (19), the second limit (20) is proven similarly. First notice that

$$\lim_{a\to\infty}\frac{b}{a(a+b)}\sum_{1\le k\le\frac{b+1}{2}} D\Big(\exp\Big(\frac{(2k-1)\pi i}{b}\Big)\Big)\Big(\frac{(2k-1)\pi}{b}-\pi\Big)=0,\tag{21}$$

since the sum is independent of a and therefore constant.

Recall that $D(\exp(\theta i)) = \operatorname{Im}\operatorname{Li}_2(\exp(\theta i))$ and we also have

$$\frac{d}{d\theta}\operatorname{Li}_n(\exp(\theta i)) = i\operatorname{Li}_{n-1}(\exp(\theta i)).$$

By applying Riemann's summation,

$$\begin{aligned}
&\lim_{a\to\infty}\Bigg[-\frac{a+b}{ab}\sum_{1\le k\le\frac{a+b+1}{2}} D\Big(\exp\Big(\frac{(2k-1)\pi i}{a+b}\Big)\Big)\Big(\frac{(2k-1)\pi}{a+b}-\pi\Big)\\
&\quad+\frac{a}{b(a+b)}\sum_{1\le k\le\frac{a+1}{2}} D\Big(\exp\Big(\frac{(2k-1)\pi i}{a}\Big)\Big)\Big(\frac{(2k-1)\pi}{a}-\pi\Big)\Bigg]\\
&=\frac{1}{\pi}\lim_{a\to\infty}\Big(\frac{a^2}{2b(a+b)}-\frac{(a+b)^2}{2ab}\Big)\int_0^\pi \theta\operatorname{Im}\operatorname{Li}_2(\exp(\theta i))d\theta\\
&\quad-\lim_{a\to\infty}\Big(\frac{a^2}{2b(a+b)}-\frac{(a+b)^2}{2ab}\Big)\int_0^\pi \operatorname{Im}\operatorname{Li}_2(\exp(\theta i))d\theta.
\end{aligned}$$

We apply integration by parts and the above equals

$$
\begin{aligned}
&-\frac{3}{2\pi}\left(\theta\,\mathrm{Im}\,(-i\mathrm{Li}_3(\exp(\theta i)))|_0^\pi-\int_0^\pi \mathrm{Im}\,(-i\mathrm{Li}_3(\exp(\theta i)))d\theta\right)+\frac{3}{2}\,\mathrm{Im}\,(-i\mathrm{Li}_3(\exp(\theta i)))|_0^\pi\\
=&\frac{3}{2}\mathrm{Li}_3(-1)+\frac{3}{2\pi}\,\mathrm{Im}(\mathrm{Li}_4(1)-\mathrm{Li}_4(-1))+\frac{3}{2}(\mathrm{Li}_3(1)-\mathrm{Li}_3(-1))\\
=&\frac{3}{2}\zeta(3).
\end{aligned}
$$

Combining this with equation (21), we obtain the result.

□

7. Conclusion

As we have seen, the result of Theorem 1, in addition to being interesting on its own, provides a nontrivial example of Boyd–Lawton formula. We remark that in this case, we obtain the Mahler measure of a four-variable polynomial as limit of Mahler measures of three-variable polynomials. However, $(1-w)(1-y)-(1-x)(1-z)$ has zero constant coefficient, and therefore it behaves like a three-variable polynomial from the Mahler measure point of view. In fact, its Mahler measure formula has a single term with a trilogarithm, namely, a polylogarithm of weight 3, which is typical of a three-variable polynomial. It would be more interesting to find an example where the limiting polynomial behaves like a four-variable polynomial from the Mahler measure point of view.

The Mahler measures of two-variable polynomials of the form $(1-x^a)y+(1-x^b)$ were extensively studied in [BRV02]. (In fact, the authors considered more generally products of terms of the form $(1-x^a)$.) Their formulas are very similar to the dilogarithm terms in our main result, as they are also given as linear combinations of dilogarithms evaluated in roots of unity. Our results can be interpreted as a three-variable version of some of the results from [BRV02]. It would be interesting to see if this parallel extends to other examples without imposing that one exponent is the sum of the other two, or examples with more variables.

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