Functional equations for Mahler measures of genus-one curves

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Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}. \end{aligned}$$

By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

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Examples in several variables

Smyth (1981)

- $$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$

- $$m(1 + x + y + z) = \frac{7}{2\pi^2}\zeta(3)$$

The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd (1998)

$$m(k) \stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

E_k elliptic curve, projective closure of

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0.$$

Deninger (1997)

L-functions ← Bloch-Beilinson's conjectures

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Rodriguez-Villegas (1997)
$k = 4\sqrt{2}$ (CM case)

$$m\left(4\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\sqrt{2}\right) = \mathrm{L}'(E_{4\sqrt{2}}, 0)$$

(By Bloch)

$k = 3\sqrt{2}$ (modular curve $X_0(24)$)

$$m\left(3\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2}\right) = q\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

(By Beilinson)

L. & Rogers (2007)
For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

Kurokawa & Ochiai (2005)
For $h \in \mathbb{R}^*$,

$$m(4h^2)+m\left(\frac{4}{h^2}\right)=2m\left(2\left(h+\frac{1}{h}\right)\right).$$

$h = \frac{1}{\sqrt{2}}$ in both equations, and using K-theory,

Corollary

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2}) = 4\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

Regulators and Mahler measures

By Jensen's formula,

$$m(k) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log \left| x + \frac{1}{x} + y + \frac{1}{y} + k \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_{\mathbb{T}^1} \eta(x, y),$$

$$\eta(x,y) := \log |x| \mathrm{d}\,\mathrm{i} \arg y - \log |y| \mathrm{d}\,\mathrm{i} \arg x$$

Regulator map (Beilinson, Bloch):

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R})$$

$$\{x, y\} \to \left\{ \gamma \to \int_\gamma \eta(x, y) \right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$.
Need integrality conditions, trivial tame symbols...

Computing the regulator

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$$

$$\mathbb{Z}[E(\mathbb{C})]^- = \mathbb{Z}[E(\mathbb{C})]/\sim \quad [-P] \sim -[P].$$

$$R_\tau : \mathbb{Z}[E(\mathbb{C})]^- \to \mathbb{C}.$$

R_τ is a Kronecker-Eisenstein series.

$$R_\tau = D_\tau - \mathrm{i}J_\tau$$

D_τ is the elliptic dilogarithm.

Proposition

$E/\mathbb{R}$ elliptic curve, x, y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, $\omega \in \Omega^1$

$$-\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau \left((x)^- * (y)\right)\right)$$

where Ω_0 is the real period and $\Omega = \int_\gamma \omega$.

Use results of Beilinson, Bloch, Deninger

$$(x) = \sum m_i (a_i), \qquad (y) = \sum n_j (b_j).$$

$$\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$$

$$(x)^- * (y) = \sum m_i n_j (a_i - b_j).$$

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Idea of Proof

Modular elliptic surface associated to $\Gamma_0(4)$

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

$$(x)^- * (y) = 8(P),$$

P torsion point of order 4.

$$P \equiv -\frac{1}{4} \mod \mathbb{Z} + \tau\mathbb{Z} \qquad k \in \mathbb{R}$$

$$\tau = \mathrm{i}y_\tau \qquad k \in \mathbb{R}, |k| > 4,$$

$$\tau = \frac{1}{2} + \mathrm{i}y_\tau \qquad k \in \mathbb{R}, |k| < 4$$

Understand cycle $[|x| = 1] \in H_1(E, \mathbb{Z})$

$$\Omega = \tau\Omega_0 \quad k \in \mathbb{R}$$

$$-\int_\gamma \eta(x, y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau\Omega_0} R_\tau\left((x)^- * (y)\right)\right)$$

$$m(k) = \frac{4}{\pi}\mathrm{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right), \quad k \in \mathbb{R}$$

Theorem

(Rodriguez-Villegas)

$$m(k) = \mathrm{Re}\left(\frac{16y_\mu}{\pi^2}\sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})}\right)$$

$$= \mathrm{Re}\left(-\pi \mathrm{i}\mu + 2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2\frac{q^n}{n}\right)$$

where $j(E_k) = j\left(-\frac{1}{4\mu}\right)$

$$q = \mathrm{e}^{2\pi \mathrm{i}\mu} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

and y_μ *is the imaginary part of* μ*.*

Functional equations

Functional equations of the regulator

$$J_{4\mu}\left(e^{2\pi i\mu}\right) = 2J_{2\mu}\left(e^{\pi i\mu}\right) + 2J_{2(\mu+1)}\left(e^{\frac{2\pi i(\mu+1)}{2}}\right)$$

$$\frac{1}{y_{4\mu}}J_{4\mu}\left(e^{2\pi i\mu}\right) = \frac{1}{y_{2\mu}}J_{2\mu}\left(e^{\pi i\mu}\right) + \frac{1}{y_{2\mu}}J_{2\mu}\left(-e^{\pi i\mu}\right)$$

$$q = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

Second degree modular equation, $|h| < 1$, $h \in \mathbb{R}$,

$$q^2\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(h^4\right).$$

$h \to \mathrm{i}h$

$$-q\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right).$$

Then the equation with J becomes

$$m\left(q\left(\left(\frac{2h}{1+h^2}\right)^2\right)\right)+m\left(q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right)\right)=m\left(q\left(h^4\right)\right).$$

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

Direct approach

Also some equations can be proved directly using isogenies:

$$\phi_1 : E_{2\left(h+\frac{1}{h}\right)} \to E_{4h^2}, \qquad \phi_2 : E_{2\left(h+\frac{1}{h}\right)} \to E_{\frac{4}{h^2}}.$$

$$\phi_1 : (X, Y) \to \left(\frac{X(h^2X+1)}{X+h^2}, -\frac{h^3Y\left(X^2+2h^2X+1\right)}{\left(X+h^2\right)^2} \right)$$

$$m\left(4h^2\right) = r_1\left(\{x_1, y_1\}\right) = \frac{1}{2\pi}\int_{|X_1|=1} \eta(x_1, y_1)$$

$$= \frac{1}{4\pi}\int_{|X|=1} \eta(x_1 \circ \phi_1, y_1 \circ \phi_1) = \frac{1}{2} r\left(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}\right)$$

Other families

- Hesse family

$$h(a^3) = m\left(x^3 + y^3 + 1 - \frac{3xy}{a}\right)$$

(studied by Rodriguez-Villegas 1997)

$$h(u^3) = \sum_{j=0}^{2} h\left(1 - \left(\frac{1-\xi_3^j u}{1+2\xi_3^j u}\right)^3\right) \qquad |u| \text{ small}$$

- More complicated equations for examples studied by Stienstra 2005:

$$m\left((x+1)(y+1)(x+y) - \frac{xy}{t}\right)$$

and Bertin 2004, Zagier $<$ 2005, and Stienstra 2005:

$$m\left((x+y+1)(x+1)(y+1) - \frac{xy}{t}\right)$$