ARITHMETICS, INTERRUPTED

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Abstract. Adventures in mathematical research and homeschooling in the time of COVID-19.

A positive integer n is said to be congruent if it is equal to the area of a right triangle with rational sides. It is known that n is congruent if and only if the equation

$$E_n : y^2 = x^3 - n^2 x$$

has a solution $(x_0, y_0) \in \mathbb{Q}^2$ with $y_0 \neq 0$. The equation E_n corresponds to an elliptic curve, and having a solution with $y_0 \neq 0$ is equivalent to say that there are infinitely many rational solutions or, more precisely, that the elliptic curve has positive rank. The answer to this question is ultimately related to the Birch and Swinnerton-Dyer conjecture. These ideas have been extended to the study of various geometric objects such as Heron triangles (see for example Goins and Maddox [GM06]).

A hyperbolic triangle with area A, angles α, β, γ and sides a, b, c is a hyperbolic Heron triangle if

$$e^a, e^b, e^c \in \mathbb{Q} \text{ and } e^{i\alpha}, e^{i\beta}, e^{i\gamma}, e^{iA} \in \mathbb{Q}(i).$$

In collaboration with Mila [LM21] we show that hyperbolic Heron triangles with fixed area and a fixed angle are parametrized by an elliptic curve family of the form

$$E_{n,u} : y^2 = x(x-n)(x-n(u^2+1)),$$

where u depends on the angle and n depends on both the area and the angle. This family has always a point of infinite order, which implies that we can find infinitely many hyperbolic Heron triangles for most choices of the area and angle. In particular, the congruent number problem has always infinitely many solutions in this setting!

Oldest son wants to discuss the following problem: A "follower year" is a year where the number is written as two consecutive years. For example, 78 *and* 2021 *are follower years. Noélie sums the numbers that represent the follower years between* 12 *and* 2021 *included. What is her result?* [AQdJMA21]

The (logarithmic) Mahler measure of a non-zero polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ is defined by

$$\mathrm{m}(P) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \log |P(e^{i\theta_1}, \dots, e^{i\theta_n})| d\theta_1 \dots d\theta_n.$$

For the one-variable case, this integral can be expressed in terms of the roots of the polynomial that lie outside the unit circle. In favorable cases, the Mahler measure of several-variable polynomials yields formulas involving special values of arithmetically interesting formulas, such as the Riemann zeta function or L-functions of elliptic curves.

Jointly with Gu [GL21] we prove a formula that expresses the Mahler measure of $x^{a+b} + 1 + (x^a+1)y + (x^b-1)z \in \mathbb{C}[x, y, z]$ in terms of the Riemann zeta function value $\zeta(3)$ and a combination of Bloch–Wigner dilogarithms. The proof involves some new ideas regarding the integration of the regulator. This formula is a first on its kind for a three-variable family of polynomials, and it leads to an interesting application of the Boyd–Lawton formula! [Boy81, Law83]

Youngest son wants to discuss the following problem: "Pablo needs a stick to play his favorite sport. Karen needs a ball to play her favorite sport. Imad needs a basket to practise his throws. Binesi is tall, it is easy for them to send the ball to the other side. Lakeisha hates golf and gymnastics. Xinchun doesn't practise a team sports. Complete the following table with the favorite sport of each kid." [Rap].

	soccer	wheelchair basketball	golf	baseball	gymnastics	volleyball
Binesi						
Imad						
Karen						
Lakeisha						
Pablo						
Xinchun						

A Dirichlet character modulo k is a multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ such that $\chi(n) = \chi(n+k)$ for all n, $\chi(n) = 0$ iff $(n,k) > 0$, and $\chi(mn) = \chi(m)\chi(n)$ for all integers m, n. The Dirichlet L-series corresponding to χ is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where s is a complex variable. It converges for $\mathrm{Re}(s) > 1$, and it has a meromorphic continuation to the whole complex plane, the Dirichlet L-function, which satisfies a functional equation relating the value at s with the value at $1-s$. A famous conjecture of Chowla predicts that $L(\frac{1}{2},\chi) \neq 0$. The value $s = \frac{1}{2}$ is special (and difficult to estimate) because is the central value of the functional equation. The most studied case is the quadratic case, when $\chi^2(n) = 1$ or 0 for all n. An important breakthrough came when Soundararajan [Sou00] proved that at least 87.5% of the quadratic Dirichlet L-functions do not vanish at $s = \frac{1}{2}$. For cubic characters (when $\chi^3(n) = 1$ or 0 for all n) this is harder to analyze. A variant of this problem consists of studying Dirichlet L-functions over function fields, namely, constructed from the series that sums over monic polynomials with coefficients in a finite field of q elements $\mathbb{F}_q$:

$$L(s,\chi) = \sum_{f \in \mathbb{F}_q[T]} \frac{\chi(f)}{q^{s \deg(f)}}.$$

In this case Donepudi and Li [DL20] proved that there is some vanishing for cubic L-functions, but the proportion that they found still goes to zero when the number of characters goes to infinity. In [DFL20], we prove jointly with David and Florea that the proportion of nonvanishing is of at least $0.4718e^{-e^{182}}$. Now, this is a very small number, but positive!

Oldest son needs help making a model of a paper plane for a big school project due soon. He also needs to make a paper model of a turbine. And a model of a wing. And he has to interview a pilot. I also got a contact of an aircraft engineer whose son goes to the same school. Am I supposed to arrange the zoom call? Will the teacher do it?

Over number fields, the k-th divisor function $d_k(n)$ gives the number of ways of writing a positive integer as a product of k positive integers. It arises as coefficients of the k-power of the Riemann zeta function

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

The sum of the divisor function up to a positive real x is asymptotic to a polynomial in $\log x$:

$$\sum_{n \leq x} d_k(n) \sim x P_{k-1}(\log x),$$

where $P_{k-1}(T)$ is certain polynomial of degree $k-1$ (see [Tit86, Chapter XII]). There is interest in studying the distribution of the error term $\Delta_k(x) = \sum_{n \leq x} d_k(n) - xP_{k-1}(\log x)$ and in particular its variance. In [KRRGR18] Keating, Rodgers, Rodity-Gershon, and Rudnick make conjectures on these distributions by studying the analogue problems over the function field (again, polynomials over $\mathbb{F}_q[T]$). When $q \to \infty$, the distributions considered in [KRRGR18] can be expressed in terms of integrals in random matrix theory, in particular over the set of unitary matrices. With Kuperberg [KL] we are considering similar problems, where the distribution takes places over the symplectic unitary matrices. It is exciting to compare these results with the already known unitary case!

Youngest son needs help with dictation, conjugating the French verb "aimer": J'aime, tu aimes, il/elle aime, nous aimons, vous aimez, ils/elles aiment... je t'aime, je vous aime, même en pandémie! [1]

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Personal bio: I am an Argentinean-Canadian Professor of Mathematics and a Mother, who is passionate about research in Number Theory and related areas. I became a mathematician by participating in Mathematical Olympiads and I love solving problems, and sharing this excitement with students and with my children. COVID-19 has disrupted my life in many ways, and I am looking forward to regaining my usual nerdy normalcy.

References

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[1] I love, you love, he/she/it loves, we love, you love, they love... I love you, I love you all, even in pandemic times!

Answers to kid's questions : 17990

Binesi - volleyball, Imad - wheelchair basketball, Karen - soccer, Lakeisha - baseball, Pablo - golf, Xinchun - gymnastics

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