Condon's example

The last and most complex example was discovered numerically by Boyd and proved by Condon.

$$z = \frac{(1-y)(1+x)}{1-x}$$

$$\begin{aligned}
x \wedge y \wedge z &= x \wedge y \wedge (1-y) + x \wedge y \wedge (1+x) - x \wedge y \wedge (1-x) \\
&= y \wedge (1-y) \wedge x - (-x) \wedge (1+x) \wedge y + x \wedge (1-x) \wedge y
\end{aligned}$$

Then we need to consider

$$\Delta = [y]_2 \otimes x - [-x]_2 \otimes y + [x]_2 \otimes y$$

Under the condition,

$$\frac{(1-y)(1+x)(1-y^{-1})(1+x^{-1})}{(1-x)(1-x^{-1})} = 1$$

which may be written as

$$\left(\frac{1+x}{1-x}\right)^2 = \frac{y}{(1-y)^2}$$

We now use the five term relation,

$$[x]_2 + [-1]_2 + [1+x]_2 + \left[\frac{1-x}{1+x}\right]_2 + \left[\frac{2}{1+x}\right]_2 = 0$$

$$[x]_2 - [-x]_2 + \left[\frac{1-x}{1+x}\right]_2 - \left[\frac{x-1}{1+x}\right]_2 = 0$$

Then

$$\Delta = [y]_2 \otimes x - \left[\frac{1-x}{1+x}\right]_2 \otimes y + \left[\frac{x-1}{1+x}\right]_2 \otimes y$$

Write $y = z^2$ such that

$$\frac{1+x}{1-x} = \frac{z}{1-z^2} \tag{1}$$

Then

$$x = \frac{z^2+z-1}{-z^2+z+1}$$

Thus we may write,

$$\Delta = [z^2]_2 \otimes \frac{z^2+z-1}{-z^2+z+1} - \left[\frac{1-z^2}{z}\right]_2 \otimes z^2 + \left[\frac{z^2-1}{z}\right]_2 \otimes z^2$$

We will split the integration of Δ in two steps. In order to do that, write

$$\Delta_1 = [z^2]_2 \otimes \frac{z^2+z-1}{-z^2+z+1} \qquad \Delta_2 = 2[z-z^{-1}]_2 \otimes z - 2[z^{-1}-z]_2 \otimes z$$

then

$$\Delta = \Delta_1 + \Delta_2$$

We will first work with Δ_1. Let $\varphi = \frac{1+\sqrt{5}}{2}$, so $\varphi^2 - \varphi - 1 = 0$.

By the five-term relation,

$$[\varphi z]_2 + [(\varphi-1)z]_2 + [1-z^2]_2 + \left[\frac{1-\varphi z}{1-z^2}\right]_2 + \left[\frac{1-(\varphi-1)z}{1-z^2}\right]_2 = 0 \tag{2}$$

$$[-\varphi z]_2 + [(1-\varphi)z]_2 + [1-z^2]_2 + \left[\frac{1+\varphi z}{1-z^2}\right]_2 + \left[\frac{1+(\varphi-1)z}{1-z^2}\right]_2 = 0 \tag{3}$$

Now

$$\begin{aligned}
\Delta_1 &= [z^2]_2 \otimes \frac{z^2+z-1}{-z^2+z+1} = [z^2]_2 \otimes \frac{(\varphi z-1)((\varphi-1)z+1)}{(\varphi z+1)((\varphi-1)z-1)} \\
&= [z^2]_2 \otimes \frac{1-\varphi z}{1-(\varphi-1)z} - [z^2]_2 \otimes \frac{1+\varphi z}{1+(\varphi-1)z}
\end{aligned}$$

Apply the five-term relation (2) and (3):

$$\begin{aligned}
&= [\varphi z]_2 \otimes \frac{1-\varphi z}{1-(\varphi-1)z} + [(\varphi-1)z]_2 \otimes \frac{1-\varphi z}{1-(\varphi-1)z} \\
&+ \left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes \frac{1-\varphi z}{1-(\varphi-1)z} + \left[\frac{1-(\varphi-1)z}{1-z^2}\right]_2 \otimes \frac{1-\varphi z}{1-(\varphi-1)z} \\
&-[-\varphi z]_2 \otimes \frac{1+\varphi z}{1+(\varphi-1)z} - [(1-\varphi)z]_2 \otimes \frac{1+\varphi z}{1+(\varphi-1)z} \\
&- \left[\frac{1+\varphi z}{1-z^2}\right]_2 \otimes \frac{1+\varphi z}{1+(\varphi-1)z} - \left[\frac{1+(\varphi-1)z}{1-z^2}\right]_2 \otimes \frac{1+\varphi z}{1+(\varphi-1)z}
\end{aligned}$$

Then

$$\begin{aligned}
\Delta_1 &= -[1-\varphi z]_2 \otimes (1-\varphi z) + [1-(\varphi-1)z]_2 \otimes (1-(\varphi-1)z) \\
&+[1+\varphi z]_2 \otimes (1+\varphi z) - [1+(\varphi-1)z]_2 \otimes (1+(\varphi-1)z) \\
&-[\varphi z]_2 \otimes (1-(\varphi-1)z) + [-\varphi z]_2 \otimes (1+(\varphi-1)z) + [(\varphi-1)z]_2 \otimes (1-\varphi z) - [(1-\varphi)z]_2 \otimes (1+\varphi z) \\
&+ \left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes \frac{1-\varphi z}{1-(\varphi-1)z} + \left[\frac{1-(\varphi-1)z}{1-z^2}\right]_2 \otimes \frac{1-\varphi z}{1-(\varphi-1)z} \\
&- \left[\frac{1+\varphi z}{1-z^2}\right]_2 \otimes \frac{1+\varphi z}{1+(\varphi-1)z} - \left[\frac{1+(\varphi-1)z}{1-z^2}\right]_2 \otimes \frac{1+\varphi z}{1+(\varphi-1)z}
\end{aligned}$$

Now we will work with Δ_2. By the five-term relation,

$$[\varphi + z^{-1}]_2 + [1-(\varphi-1)z]_2 + [z-z^{-1}]_2 + \left[\frac{1+(\varphi-1)z}{1-z^2}\right]_2 + \left[\frac{(\varphi-1)z}{z-z^{-1}}\right]_2 = 0$$

$$[\varphi - z^{-1}]_2 + [1-(1-\varphi)z]_2 + [z^{-1}-z]_2 + \left[\frac{1-(\varphi-1)z}{1-z^2}\right]_2 + \left[\frac{(\varphi-1)z}{z-z^{-1}}\right]_2 = 0$$

$$[1+\varphi z]_2 + [z^{-1}-(\varphi-1)]_2 + [z-z^{-1}]_2 + \left[\frac{\varphi z}{z^{-1}-z}\right]_2 + \left[\frac{1-\varphi z}{1-z^2}\right]_2 = 0$$

$$[1-\varphi z]_2 + [-z^{-1}-(\varphi-1)]_2 + [z^{-1}-z]_2 + \left[\frac{\varphi z}{z^{-1}-z}\right]_2 + \left[\frac{1+\varphi z}{1-z^2}\right]_2 = 0$$

We have

$$\begin{aligned}
\Delta_2 &= 2\left[z-z^{-1}\right]_2 \otimes z - 2\left[z^{-1}-z\right]_2 \otimes z \\
&= [\varphi - z^{-1}]_2 \otimes z - [(1-\varphi)z]_2 \otimes z + \left[\frac{1-(\varphi-1)z}{1-z^2}\right]_2 \otimes z \\
&\quad -[\varphi + z^{-1}]_2 \otimes z + [(\varphi-1)z]_2 \otimes z - \left[\frac{1+(\varphi-1)z}{1-z^2}\right]_2 \otimes z \\
&\quad -[\varphi z]_2 \otimes z - [\varphi + z^{-1}]_2 \otimes z + \left[\frac{1+\varphi z}{1-z^2}\right]_2 \otimes z \\
&\quad [-\varphi z]_2 \otimes z + [\varphi - z^{-1}]_2 \otimes z - \left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes z
\end{aligned}$$

We will now use the fact that we know we will integrate in a zeta were $|z|=1$. Under those circumstances we have the following two identities:

$$[(\varphi-1)z]_2 \otimes z = [(\varphi-1)\bar{z}]_2 \otimes \bar{z} = [\varphi z]_2 \otimes z \tag{4}$$

$$[(1-\varphi)z]_2 \otimes z = [-\varphi z]_2 \otimes z \tag{5}$$

Both identities depend on the fact that $|z|=1$ since we are conjugating and using that $\bar{z} = z^{-1}$.

Thus

$$\begin{aligned}
\Delta_2 &= 2[\varphi - z^{-1}]_2 \otimes z - 2[\varphi + z^{-1}]_2 \otimes z \\
&+ \left[\frac{1-(\varphi-1)z}{1-z^2}\right]_2 \otimes z - \left[\frac{1+(\varphi-1)z}{1-z^2}\right]_2 \otimes z + \left[\frac{1+\varphi z}{1-z^2}\right]_2 \otimes z - \left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes z
\end{aligned}$$

We will add Δ_1 and Δ_2. But first, let us note

$$\begin{aligned}
\left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes \frac{1-\varphi z}{z(1-(\varphi-1)z)} &= \left[\frac{z^{-1}-\varphi}{z^{-1}-z}\right]_2 \otimes \frac{z^{-1}-\varphi}{z-\varphi} - \left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes (1-\varphi) \\
&= -\left[\frac{z^{-1}-z}{z^{-1}-\varphi}\right]_2 \otimes \frac{z^{-1}-\varphi}{z-\varphi} - \left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes (1-\varphi)
\end{aligned}$$

$$= -\left[\frac{z-\varphi}{z^{-1}-\varphi}\right]_2 \otimes \frac{z-\varphi}{z^{-1}-\varphi} - \left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes (\varphi-1)$$

Thus,

$$\begin{aligned}
\Delta &= -[1-\varphi z]_2 \otimes (1-\varphi z) + [1-(\varphi-1)z]_2 \otimes (1-(\varphi-1)z) \\
&\quad +[1+\varphi z]_2 \otimes (1+\varphi z) - [1+(\varphi-1)z]_2 \otimes (1+(\varphi-1)z) \\
&\quad -[\varphi z]_2 \otimes (1-(\varphi-1)z) + [-\varphi z]_2 \otimes (1+(\varphi-1)z) + [(\varphi-1)z]_2 \otimes (1-\varphi z) - [(1-\varphi)z]_2 \otimes (1+\varphi z) \\
&\quad - \left[\frac{z-\varphi}{z^{-1}-\varphi}\right]_2 \otimes \frac{z-\varphi}{z^{-1}-\varphi} + \left[\frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)}\right]_2 \otimes \frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)} \\
&\quad + \left[\frac{z+\varphi}{z^{-1}+\varphi}\right]_2 \otimes \frac{z+\varphi}{z^{-1}+\varphi} - \left[\frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)}\right]_2 \otimes \frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)} \\
&\quad - \left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes (\varphi-1) + \left[\frac{1-(\varphi-1)z}{1-z^2}\right]_2 \otimes \varphi \\
&\quad + \left[\frac{1+\varphi z}{1-z^2}\right]_2 \otimes (\varphi-1) - \left[\frac{1+(\varphi-1)z}{1-z^2}\right]_2 \otimes \varphi \\
&\quad +2[\varphi - z^{-1}]_2 \otimes z - 2[\varphi + z^{-1}]_2 \otimes z
\end{aligned}$$

Observe that

$$\begin{aligned}
&\left[\frac{1-\varphi z}{1-z^2}\right]_2 \otimes \varphi + \left[\frac{1-(\varphi-1)z}{1-z^2}\right]_2 \otimes \varphi - \left[\frac{1+\varphi z}{1-z^2}\right]_2 \otimes \varphi - \left[\frac{1+(\varphi-1)z}{1-z^2}\right]_2 \otimes \varphi \\
&= [-\varphi z]_2 \otimes \varphi + [(1-\varphi)z]_2 \otimes \varphi - [\varphi z]_2 \otimes \varphi - [(\varphi-1)z]_2 \otimes \varphi
\end{aligned}$$

by five-term relations (2) and (3)

Hence

$$\begin{aligned}
\Delta &= -[1-\varphi z]_2 \otimes (1-\varphi z) + [1-(\varphi-1)z]_2 \otimes (1-(\varphi-1)z) \\
&\quad +[1+\varphi z]_2 \otimes (1+\varphi z) - [1+(\varphi-1)z]_2 \otimes (1+(\varphi-1)z) \\
&\quad -[\varphi z]_2 \otimes (1-(\varphi-1)z) + [-\varphi z]_2 \otimes (1+(\varphi-1)z) + [(\varphi-1)z]_2 \otimes (1-\varphi z) - [(1-\varphi)z]_2 \otimes (1+\varphi z) \\
&\quad - \left[\frac{z-\varphi}{z^{-1}-\varphi}\right]_2 \otimes \frac{z-\varphi}{z^{-1}-\varphi} + \left[\frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)}\right]_2 \otimes \frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)} \\
&\quad + \left[\frac{z+\varphi}{z^{-1}+\varphi}\right]_2 \otimes \frac{z+\varphi}{z^{-1}+\varphi} - \left[\frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)}\right]_2 \otimes \frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)} \\
&\quad +[-\varphi z]_2 \otimes \varphi + [(1-\varphi)z]_2 \otimes \varphi - [\varphi z]_2 \otimes \varphi - [(\varphi-1)z]_2 \otimes \varphi \\
&\quad +2[\varphi - z^{-1}]_2 \otimes z - 2[\varphi + z^{-1}]_2 \otimes z
\end{aligned}$$

Next,

$$\begin{aligned}
\Delta &= -[1-\varphi z]_2 \otimes (1-\varphi z) + [1-(\varphi-1)z]_2 \otimes (1-(\varphi-1)z) \\
&\quad +[1+\varphi z]_2 \otimes (1+\varphi z) - [1+(\varphi-1)z]_2 \otimes (1+(\varphi-1)z)
\end{aligned}$$

$$\begin{aligned}
&- \left[\frac{z-\varphi}{z^{-1}-\varphi}\right]_2 \otimes \frac{z-\varphi}{z^{-1}-\varphi} + \left[\frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)}\right]_2 \otimes \frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)} \\
&+ \left[\frac{z+\varphi}{z^{-1}+\varphi}\right]_2 \otimes \frac{z+\varphi}{z^{-1}+\varphi} - \left[\frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)}\right]_2 \otimes \frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)} \\
&-[\varphi z]_2 \otimes (\varphi - z) + [-\varphi z]_2 \otimes (\varphi + z) + [(\varphi-1)z]_2 \otimes ((\varphi-1)-z) - [(1-\varphi)z]_2 \otimes ((\varphi-1)+z) \\
&+2[\varphi - z^{-1}]_2 \otimes z - 2[\varphi + z^{-1}]_2 \otimes z
\end{aligned}$$

Now observe that

$$[(\varphi-1)z]_2 \otimes ((\varphi-1)-z) = [(\varphi-1)z]_2 \otimes ((\varphi-1)z) + [(\varphi-1)z]_2 \otimes (\varphi - z^{-1})$$

Now conjugate the elements of the second term

$$= [(\varphi-1)z]_2 \otimes ((\varphi-1)z) + [(\varphi-1)z^{-1}]_2 \otimes (\varphi - z) = [(\varphi-1)z]_2 \otimes ((\varphi-1)z) - [\varphi z]_2 \otimes (\varphi - z)$$

Hence

$$\begin{aligned}
&-[\varphi z]_2 \otimes (\varphi - z) + [-\varphi z]_2 \otimes (\varphi + z) + [(\varphi-1)z]_2 \otimes ((\varphi-1)-z) - [(1-\varphi)z]_2 \otimes ((\varphi-1)+z) \\
&+2[\varphi - z^{-1}]_2 \otimes z - 2[\varphi + z^{-1}]_2 \otimes z \\
&= -2[\varphi z]_2 \otimes (\varphi - z) + 2[\varphi - z^{-1}]_2 \otimes z + 2[-\varphi z]_2 \otimes (\varphi + z) - 2[\varphi + z^{-1}]_2 \otimes z \\
&+[(\varphi-1)z]_2 \otimes ((\varphi-1)z) - [(1-\varphi)z]_2 \otimes ((1-\varphi)z)
\end{aligned}$$

We want to simplify the term $-[\varphi z]_2 \otimes (\varphi - z) + [\varphi - z^{-1}]_2 \otimes z$. On the one hand,

$$-[\varphi z]_2 \otimes (\varphi - z) = -[\varphi z]_2 \otimes z + [1-\varphi z]_2 \otimes (1-\varphi z^{-1})$$

On the other hand,

$$[\varphi - z^{-1}]_2 \otimes z = [\varphi - z]_2 \otimes z^{-1} = -[\varphi - z]_2 \otimes (\varphi - z) + [\varphi - z]_2 \otimes (1-\varphi z^{-1})$$

By the five term relation

$$[1-\varphi z]_2 + [1-\varphi]_2 + [\varphi - z]_2 - [1-\varphi+z^{-1}]_2 + [(\varphi-1)z]_2 = 0$$

but $[1-\varphi]_2$ correspond to zero in the differential since it is a constant real number. We get

$$\begin{aligned}
&[1-\varphi z]_2 \otimes (1-\varphi z^{-1}) + [\varphi - z]_2 \otimes (1-\varphi z^{-1}) = [1-\varphi+z^{-1}]_2 \otimes (1-\varphi z^{-1}) - [(\varphi-1)z]_2 \otimes (1-\varphi z^{-1}) \\
&= [1-\varphi+z^{-1}]_2 \otimes (1-\varphi+z^{-1}) - [1-\varphi+z^{-1}]_2 \otimes (1-\varphi) \\
&+[1-(\varphi-1)z]_2 \otimes (1-(\varphi-1)z) + [(\varphi-1)z]_2 \otimes ((\varphi-1)z)
\end{aligned}$$

Then

$$-[\varphi z]_2 \otimes (\varphi - z) + [\varphi - z^{-1}]_2 \otimes z = -[\varphi z]_2 \otimes z - [\varphi - z]_2 \otimes (\varphi - z) + [1-\varphi+z^{-1}]_2 \otimes (1-\varphi+z^{-1})$$

$$-[1-\varphi+z^{-1}]_2 \otimes (1-\varphi) + [1-(\varphi-1)z]_2 \otimes (1-(\varphi-1)z) + [(\varphi-1)z]_2 \otimes ((\varphi-1)z)$$

Analogously,

$$\begin{aligned}
&[-\varphi z]_2 \otimes (\varphi + z) - [\varphi + z^{-1}]_2 \otimes z = [-\varphi z]_2 \otimes z + [\varphi + z]_2 \otimes (\varphi + z) - [1-\varphi-z^{-1}]_2 \otimes (1-\varphi-z^{-1}) \\
&+[1-\varphi-z^{-1}]_2 \otimes (1-\varphi) - [1+(\varphi-1)z]_2 \otimes (1+(\varphi-1)z) - [(1-\varphi)z]_2 \otimes ((1-\varphi)z)
\end{aligned}$$

Then

$$\begin{aligned}
&-2[\varphi z]_2 \otimes (\varphi - z) + 2[\varphi - z^{-1}]_2 \otimes z + 2[-\varphi z]_2 \otimes (\varphi + z) - 2[\varphi + z^{-1}]_2 \otimes z \\
&+[(\varphi-1)z]_2 \otimes ((\varphi-1)z) - [(1-\varphi)z]_2 \otimes ((1-\varphi)z) \\
&= -2[\varphi z]_2 \otimes \varphi z + 2[\varphi z]_2 \otimes \varphi - 2[\varphi - z]_2 \otimes (\varphi - z) + 2[1-\varphi+z^{-1}]_2 \otimes (1-\varphi+z^{-1}) \\
&-2[1-\varphi+z^{-1}]_2 \otimes (1-\varphi) + 2[1-(\varphi-1)z]_2 \otimes (1-(\varphi-1)z) + 3[(\varphi-1)z]_2 \otimes ((\varphi-1)z) \\
&+2[-\varphi z]_2 \otimes (-\varphi z) - 2[-\varphi z]_2 \otimes \varphi + 2[\varphi + z]_2 \otimes (\varphi + z) - 2[1-\varphi-z^{-1}]_2 \otimes (1-\varphi-z^{-1}) \\
&+2[1-\varphi-z^{-1}]_2 \otimes (1-\varphi) - 2[1+(\varphi-1)z]_2 \otimes (1+(\varphi-1)z) - 3[(1-\varphi)z]_2 \otimes ((1-\varphi)z)
\end{aligned}$$

Next we will see that

$$[\varphi + z^{-1}]_2 \otimes \varphi - [\varphi - z^{-1}]_2 \otimes \varphi + [\varphi z]_2 \otimes \varphi - [-\varphi z]_2 \otimes \varphi$$

corresponds to zero in the differential.

Using that $|z|=1$, the differential is

$$\begin{aligned}
\frac{3\omega}{\log\varphi} &= \log|1-\varphi-z| \,\mathrm{d}\log|\varphi+z| - \log|\varphi+z| \,\mathrm{d}\log|1-\varphi-z| \\
&- \log|1-\varphi+z| \,\mathrm{d}\log|\varphi-z| + \log|\varphi-z| \,\mathrm{d}\log|1-\varphi+z| \\
&- \log\varphi \,\mathrm{d}\log|1-\varphi z| + \log\varphi \,\mathrm{d}\log|1+\varphi z|
\end{aligned}$$

$$\begin{aligned}
&= \log|1+\varphi z| \,\mathrm{d}\log|\varphi+z| - \log\varphi \,\mathrm{d}\log|\varphi+z| - \log|\varphi+z| \,\mathrm{d}\log|1+\varphi z| \\
&- \log|-1+\varphi z| \,\mathrm{d}\log|\varphi-z| + \log\varphi \,\mathrm{d}\log|\varphi-z| + \log|\varphi-z| \,\mathrm{d}\log|1-\varphi z| \\
&- \log\varphi \,\mathrm{d}\log|z^{-1}-\varphi| + \log\varphi \,\mathrm{d}\log|z^{-1}+\varphi|
\end{aligned}$$

$$\begin{aligned}
&= \log|1+\varphi z| \,\mathrm{d}\log|\varphi z^{-1}+1| - \log|\varphi z^{-1}+1| \,\mathrm{d}\log|1+\varphi z| \\
&- \log|-1+\varphi z| \,\mathrm{d}\log|\varphi z^{-1}-1| + \log|\varphi z^{-1}-1| \,\mathrm{d}\log|1-\varphi z|
\end{aligned}$$

$$= 0$$

Finally Δ integrates to

$$\Gamma = -[1-\varphi z]_3 + [1-(\varphi-1)z]_3 + [1+\varphi z]_3 - [1+(\varphi-1)z]_3$$

$$\begin{aligned}
&- \left[\frac{z-\varphi}{z^{-1}-\varphi}\right]_3 + \left[\frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)}\right]_3 + \left[\frac{z+\varphi}{z^{-1}+\varphi}\right]_3 - \left[\frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)}\right]_3 \\
&-2[\varphi z]_3 - 2[\varphi - z]_3 + 2[1-\varphi+z^{-1}]_3 + 2[1-(\varphi-1)z]_3 + 3[(\varphi-1)z]_3 \\
&+2[-\varphi z]_3 + 2[\varphi + z]_3 - 2[1-\varphi-z^{-1}]_3 - 2[1+(\varphi-1)z]_3 - 3[(1-\varphi)z]_3
\end{aligned}$$

$$\begin{aligned}
&= -[1-\varphi z]_3 + [1+\varphi z]_3 \\
&- \left[\frac{z-\varphi}{z^{-1}-\varphi}\right]_3 + \left[\frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)}\right]_3 + \left[\frac{z+\varphi}{z^{-1}+\varphi}\right]_3 - \left[\frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)}\right]_3 \\
&-2[\varphi z]_3 - 2[\varphi - z]_3 + 2[1-\varphi+z^{-1}]_3 + 3[1-(\varphi-1)z]_3 + 3[(\varphi-1)z]_3 \\
&+2[-\varphi z]_3 + 2[\varphi + z]_3 - 2[1-\varphi-z^{-1}]_3 - 3[1+(\varphi-1)z]_3 - 3[(1-\varphi)z]_3
\end{aligned}$$

Now use that

$$[x]_3 + [1-x]_3 + \left[1-\frac{1}{x}\right]_3 = [1]_3$$

$$\begin{aligned}
\Gamma &= -4[1-\varphi z]_3 + 4[1+\varphi z]_3 \\
&- \left[\frac{z-\varphi}{z^{-1}-\varphi}\right]_3 + \left[\frac{z-(\varphi-1)}{z^{-1}-(\varphi-1)}\right]_3 + \left[\frac{z+\varphi}{z^{-1}+\varphi}\right]_3 - \left[\frac{z+(\varphi-1)}{z^{-1}+(\varphi-1)}\right]_3 \\
&-2[\varphi z]_3 - 2[\varphi - z]_3 + 2[1-\varphi+z^{-1}]_3 + 2[-\varphi z]_3 + 2[\varphi + z]_3 - 2[1-\varphi-z^{-1}]_3
\end{aligned}$$

We need to check the integration path. If we let $x = \mathrm{e}^{2\mathrm{i}\alpha}$, with $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and $z = \mathrm{e}^{\mathrm{i}\beta}$, with $-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$, this is half the path, since there are two solutions for each z. Then the condition (1) translates into

$$\tan\alpha = \sin\beta$$

The boundaries of the above condition are meet when $\tan\alpha = \pm 1$. In other words, we need to integrate with z between -1 and 1 and multiply the final result by two.

Then

$$\begin{aligned}
4\pi^2 m(P) &= 2\Gamma\,\big|_{-1}^{1} \\
&= 2(2(-4P_3(1-\varphi) + 4P_3(1+\varphi) - 2P_3(\varphi) + 2P_3(-\varphi) - 2P_3(\varphi-1) + 2P_3(\varphi+1) \\
&\quad +2P_3(2-\varphi) - 2P_3(-\varphi))) \\
&= 4(-4P_3(1-\varphi) + 6P_3(1+\varphi) - 2P_3(\varphi) - 2P_3(\varphi-1) + 2P_3(2-\varphi))
\end{aligned}$$

Use that

$$[1-\varphi]_3 = [-\varphi]_3 \qquad [\varphi-1]_3 = [\varphi]_3$$

$$4\pi^2 m(P) = 24P_3(1+\varphi) - 16P_3(\varphi) - 16P_3(-\varphi) + 8P_3(2-\varphi)$$

Now we use

$$[\varphi-1]_3 + [2-\varphi]_3 + \left[1-\frac{1}{\varphi-1}\right]_3 = [1]_3$$

$$[\varphi]_3 + [2-\varphi]_3 + [-\varphi]_3 = [1]_3$$

And

$$[-\varphi]_3 + [1+\varphi]_3 + \left[1+\frac{1}{\varphi}\right]_3 = [1]_3$$

$$[-\varphi]_3 + [1+\varphi]_3 + [\varphi]_3 = [1]_3$$

Then

$$4\pi^2 m(P) = 32P_3(1) - 48P_3(\varphi) - 48P_3(-\varphi)$$

But

$$4[\varphi]_3 + 4[-\varphi]_3 = [\varphi^2]_3 = [\varphi+1]_3 = [1]_3 - [\varphi]_3 - [-\varphi]_3$$

then

$$[\varphi]_3 + [-\varphi]_3 = \frac{1}{5}[1]_3$$

Then

$$m(P) = \frac{28}{5\pi^2}\zeta(3)$$