Some aspects of Mahler Measure

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1. Mahler Measure and Lehmer's question

Looking for large primes, Pierce [16] proposed the following idea in 1918.
Consider $P \in \mathbb{Z}[x]$ monic, and write

$$P(x) = \prod_i (x - \alpha_i).$$

Then, let us look at

$$\Delta_n = \prod_i (\alpha_i^n - 1).$$

The α_i are integers over $\mathbb{Z}$. By applying Galois theory, it is easy to see that $\Delta_n \in \mathbb{Z}$. Note that if $P(x) = x - 2$, we get the sequence $\Delta_n = 2^n - 1$. Thus, we recover the example of Mersenne numbers. The idea is to look for primes among the factors of Δ_n. The prime divisors of such integers must satisfy some congruence conditions that are quite restrictive, hence they are easier to factorize than a randomly given number. Moreover, one can show that $\Delta_m | \Delta_n$ if $m|n$. Then we may look at the numbers

$$\frac{\Delta_p}{\Delta_1} \qquad p \text{ prime}.$$

In order to minimize the number of trial divisions, the sequence Δ_n should grow slowly. Lehmer [15] studied $\frac{\Delta_{n+1}}{\Delta_n}$, observed that

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

and suggested the following definition.

Definition 1 *Given $P \in \mathbb{C}[x]$, such that*

$$P(x) = a \prod_i (x - \alpha_i),$$

define the Mahler measure [2] *of P as*

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}. \tag{1}$$

The logarithmic Mahler measure is defined as[3]

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|. \tag{2}$$

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[2] The name Mahler came later after the person who successfully extended this definition to the several-variable case.

[3] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

When does $M(P)=1$ for $P \in \mathbb{Z}[x]$? We have

Lemma 2 *(Kronecker) Let $P=\prod_i (x-\alpha_i) \in \mathbb{Z}[x]$, if $|\alpha_i| \leq 1$, then the α_i are zero or roots of the unity.*

By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $M(P)=1$ if and only if P is the product of powers of x and cyclotomic polynomials. This statement characterizes integral polynomials whose Mahler measure is 1.

Lehmer found the example

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1) = \log(1.176280818\ldots) = 0.162357612\ldots$$

and asked the following (Lehmer's question, 1933):

Is there a constant $C>1$ such that for every polynomial $P \in \mathbb{Z}[x]$ with $M(P)>1$, then $M(P) \geq C$?

Lehmer's question remains open nowadays. His 10-degree polynomial remains the best possible result.

The use of this polynomial has led to the discovery of the prime number $1,794,327,140,357$ but bigger primes were discovered with the use of other polynomials.

2. Mahler Measure in several variables

Definition 3 *For $P \in \mathbb{C}[x_1,\ldots,x_n]$, the (logarithmic)* Mahler measure *is defined by*

$$\begin{aligned} m(P) &:= \int_0^1 \cdots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1},\ldots,\mathrm{e}^{2\pi \mathrm{i}\theta_n})| \,\mathrm{d}\theta_1 \ldots \,\mathrm{d}\theta_n & (3)\\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1,\ldots,x_n)| \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}. & (4) \end{aligned}$$

It is possible to prove that this integral is not singular and that $m(P)$ always exists.

Because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \,\mathrm{d}\theta = \log^+ |\alpha|. \qquad (5)$$

We recover the formula for the one-variable case.

3. Some properties

Proposition 4 *For $P, Q \in \mathbb{C}[x_1,\ldots,x_n]$*

$$m(P \cdot Q) = m(P) + m(Q). \qquad (6)$$

It is also true that $m(P) \geq 0$ if P has integral coefficients.

Mahler measure is related to heights. Indeed, if α is an algebraic number, and P_α is its minimal polynomial over $\mathbb{Q}$, then

$$m(P_\alpha) = [\mathbb{Q}(\alpha):\mathbb{Q}]\, h(\alpha),$$

where h is the logarithmic Weil height. This identity also extends to several-variable polynomials and heights in hypersurfaces.

Let us also mention the following result:

Theorem 5 *(Boyd–Lawton) For* $P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, \dots, x_n)). \tag{7}$$

In particular Lehmer's problem in several variables reduces to the one-variable case.

4. Examples

For one-variable polynomials, the Mahler measure has to do with the roots of the polynomial. However, it is very hard to compute explicit formulas for examples in several variables. The first and simplest ones were computed by Smyth:

- Smyth [18]

$$m(x+y+1) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1), \tag{8}$$

  where

$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{if} \quad n \equiv 0 \bmod 3 \end{cases}$$

- Smyth [1]

$$m(x+y+z+1) = \frac{7}{2\pi^2} \zeta(3). \tag{9}$$

- Boyd– Rodriguez-Villegas [17]

$$\begin{aligned} m\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right) &\stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{B_k} \qquad k \in \mathbb{N} \\ m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\right) &= 2\mathrm{L}'(\chi_{-4}, -1) \\ m\left(x + \frac{1}{x} + y + \frac{1}{y} - 4\sqrt{2}\right) &= \mathrm{L}'(A, 0) \end{aligned}$$

  Where B_k is a rational number, and E_k is the elliptic curve with corresponds to the zero set of the polynomial. When $k = 4$ the curve has genus zero. When $k = 4\sqrt{2}$ the elliptic curve is

$$A : y^2 = x^3 - 44x + 112,$$

  which has complex multiplication.

5. More examples in several variables

Theorem 6 *For* $n \geq 1$ *we have:*

$$\pi^{2n} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right)$$

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \dots, (2n-2)^2)}{(2n-1)!} \pi^{2n-2h} (2h)! \frac{2^{2h+1}-1}{2} \zeta(2h+1). \tag{10}$$

For $n \geq 0$:

$$\pi^{2n+1} m \left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right) z\right)$$

$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \dots, (2n-1)^2)}{(2n)!} 2^{2h+1} \pi^{2n-2h} (2h+1)! \mathrm{L}(\chi_{-4}, 2h+2). \tag{11}$$

For $n \geq 1$:

$$\pi^{2n+2} m \left(1 + x + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) (1+y) z\right) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \dots, (2n-2)^2)}{(2n-1)!} 2^{2h+2} \pi^{2n-2h}$$

$$\cdot (2h-1)! \sum_{k=0}^{h-1} \binom{2h-2k+2}{2} \frac{2^{2h-2k+3}-1}{2^{2h}} \frac{(-1)^k B_{2k} (2\pi)^{2k}}{2(2k)!} \zeta(2h-2k+3). \tag{12}$$

For $n \geq 0$:

$$\pi^{2n+3} m \left(1 + x + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right) (1+y) z\right)$$

$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \dots, (2n-1)^2)}{(2n)!} 2^{2h+3} \pi^{2n-2h} \left(\mathrm{i}(2h)! \mathcal{L}_{3,2h+1}(\mathrm{i}, \mathrm{i}) + (2h+1)! \pi^2 \mathrm{L}(\chi_{-4}, 2h+2)\right). \tag{13}$$

There is a similar result for the family

$$\pi^n m \left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_n}{1+x_n}\right) x + \left(1 - \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_n}{1+x_n}\right)\right) y\right)$$

$=$ *combination of $\zeta(odd)$.*

Where $\mathcal{L}_{r,s}(\alpha, \alpha)$ are certain linear combinations of polylogarithms and

$$s_l(a_1, \dots, a_k) = \begin{cases} 1 & \text{if} \quad l = 0 \\ \sum_{i_1 < \dots < i_l} a_{i_1} \dots a_{i_l} & \text{if} \quad 0 < l \leq k \\ 0 & \text{if} \quad k < l \end{cases} \tag{14}$$

are the elementary symmetric polynomials, i. e.,

$$\prod_{i=1}^{k} (x + a_i) = \sum_{l=0}^{k} s_l(a_1, \dots, a_k) x^{k-l}. \tag{15}$$

For example,

$$\pi^3 m \left(1 + \left(\frac{1-x_1}{1+x_1}\right) \left(\frac{1-x_2}{1+x_2}\right) \left(\frac{1-x_3}{1+x_3}\right) z\right) = 24\mathrm{L}(\chi_{-4}, 4) + \pi^2 \mathrm{L}(\chi_{-4}, 2) \tag{16}$$

$$\pi^4 m \left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_4}{1+x_4}\right) z\right) = 62\zeta(5) + \frac{14}{3}\pi^2 \zeta(3) \tag{17}$$

$$\pi^4 m \left(1 + x + \left(\frac{1-x_1}{1+x_1}\right) \left(\frac{1-x_2}{1+x_2}\right) (1+y) z\right) = 93\zeta(5) \tag{18}$$

(19)

6. Beilinson's conjectures

One of the main problems in Number Theory is finding rational (or integral) solutions of polynomial equations with rational coefficients (global solutions). Any global solution can be seen as a local solution by reducing modulo a prime number p. Hence, if a equation does not have local solutions, then we know that it does not have global solutions. The converse problem would be: Does having solutions modulo p for every prime p guarantee that the equation has global solutions? This question is known as the local-global principle and its answer is negative in general.

There are several theorems and conjectures which predict that one may obtain global information from local information and that that relation is made through values of L-functions. These statements include the Dirichlet class number formula, the Birch–Swinnerton-Dyer conjecture, and more generally, Bloch's and Beilinson's conjectures.

Typically, there are four elements involved in this setting: an arithmetic-geometric object X (typically, an algebraic variety), its L-function (which codify local information), a finitely generated abelian group K, and a regulator map $K \to \mathbb{R}$. When K has rank 1, Beilinson's conjectures predict that the $\mathrm{L}'_X(0)$ is, up to a rational number, equal to a value of the regulator.

For instance, for a number field F, Dirichlet class number formula states that

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}h_F \mathrm{reg}_F}{\omega_F\sqrt{|D_F|}}.$$

Here, $X = \mathcal{O}_F$ (the ring of integers), $\mathrm{L}_X = \zeta_F$, and the group is $\mathcal{O}_F^*$. Hence, when F is a real quadratic field, Dirichlet class number formula may be written as $\zeta'_F(0)$ is equal to, up to a rational number, $\log|\epsilon|$, for some $\epsilon \in \mathcal{O}_F^*$.

7. An algebraic integration for Mahler measure

The appearance of the L-functions in Mahler measures formulas is a common phenomenon. Deninger [7] interpreted the Mahler measure as a Deligne period of a mixed motive. More specifically, in two variables, and under certain conditions, he proved that

$$m(P) = \mathrm{reg}(\xi_i),$$

where reg is the determinant of the regulator matrix, which we are evaluating in some class in an appropriate group in K-theory.

Rodriguez-Villegas [17] made explicit the relationship between Mahler measure and regulators by computing the regulator for the two-variable case, and using this machinery to explain the formulas for two variables.

For example, let us start with Smyth's example, $P(x,y) = y + x - 1$. Then its Mahler measure is

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log|y+x-1|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}.$$

By Jensen's equality,

$$m(P) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1} \log^+|1-x|\frac{\mathrm{d}x}{x} = \frac{1}{2\pi \mathrm{i}}\int_{\gamma} \log|y|\frac{\mathrm{d}x}{x} = -\frac{1}{2\pi}\int_{\gamma}\eta(x,y)$$

where $\gamma = \{P(x,y)=0\} \cap \{|x|=1, |y|\geq 1\}$ and

$$\eta(x,y) = \log|x|\,\mathrm{d}\arg y - \log|y|\,\mathrm{d}\arg x.$$

This is a closed differential form defined in $C = \{P(x, y) = 0\}$ minus the sets of zeros and poles of x and y. It satisfies the following properties:

- $\eta(x, y) = -\eta(y, x)$
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$

We would like to apply Stokes Theorem. The question is, when is $\eta(x, y)$ exact? Fortunately, there is

Theorem 7

$$\eta(x, 1-x) = \mathrm{d}D(x).$$

Where $D(x)$ is the Bloch–Wigner dilogarithm,

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|.$$

Here

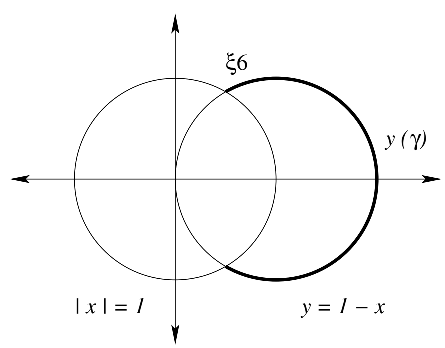
$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad |x| < 1.$$

If we use Stokes Theorem, we get

$$m(P) = -\frac{1}{2\pi} D(\partial\gamma).$$

Now we parametrize

$$\gamma : x = \mathrm{e}^{2\pi \mathrm{i}\theta} \quad y(\gamma(\theta)) = 1 - \mathrm{e}^{2\pi \mathrm{i}\theta}, \quad \theta \in [1/6\,; 5/6] \qquad \partial\gamma = [\bar{\xi}_6] - [\xi_6]$$



Then we obtain

$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi}_6) = 2D(\xi_6) = \frac{3\sqrt{3}}{2}\mathrm{L}(\chi_{-3}, 2).$$

In general, given $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x, y).$$

For $\eta(x, y)$ to be exact we need $\{x, y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$, or equivalently,

$$x \wedge y = \sum_j r_j \;\; z_j \wedge (1 - z_j) \quad \text{in} \quad \bigwedge^2 (\mathbb{C}(C)^*) \otimes \mathbb{Q}.$$

Then we get

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}.$$

We could summarize the whole situation as follows:

$$\ldots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C,\partial\gamma) \to K_2(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x,y)$ is exact, then $\{x,y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq 0$ and we use Stokes Theorem. We obtain an element in $K_3(\partial\gamma) \subset K_3(\bar{Q})$. This leads to dilogarithms and then to zeta functions of number fields by Theorems of Borel, Zagier, and others.
- $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(C)$. We have $\eta(x,y)$ is not exact. We get L-series of a curve and examples of Beilinson's conjectures.

In general we may get combinations of both situations.

For the three variable case, let us start with Smyth example, $P(x,y,z) = (1-x)+(1-y)z$. Then,

$$m(P) = m(1-y) + \frac{1}{(2\pi\mathrm{i})^3}\int_{\mathbb{T}^3} \log\left|z - \frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathbb{T}^2} \log^+\left|\frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = -\frac{1}{(2\pi)^2}\int_\Gamma \log|z|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z)$$

where $\Gamma = \{P(x,y,z)=0\} \cap \{|x|=|y|=1, |z| \geq 1\}$, and

$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\,\mathrm{d}\log|y|\ \mathrm{d}\log|z| -\ \mathrm{d}\arg y\ \mathrm{d}\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\,\mathrm{d}\log|z|\ \mathrm{d}\log|x| -\ \mathrm{d}\arg z\ \mathrm{d}\arg x\right) + \log|z|\left(\frac{1}{3}\,\mathrm{d}\log|x|\ \mathrm{d}\log|y| -\ \mathrm{d}\arg x\ \mathrm{d}\arg y\right).$$

This is a closed differential form defined in $S = \{P(x,y,z)=0\}$ minus the set of zeros and poles of x, y and z. Now note that

$$\eta(x, 1-x, y) = \mathrm{d}w(x,y) \tag{20}$$

where

$$\omega(x,y) = -D(x)\,\mathrm{d}\arg y + \frac{1}{3}\log|y|(\log|1-x|\,\mathrm{d}\log|x| - \log|x|\,\mathrm{d}\log|1-x|).$$

In our case,

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x).$$

The computation of $\gamma = \partial\Gamma$ can be made in an efficient way (for polynomials with real coefficients) by applying certain ideas of Maillot. If $P \in \mathbb{Q}[x,y,z]$, we can think of

$$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x|=|y|=1\}.$$

Now ω is defined in $C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$.

So,

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

leads to $C = \{x = y\} \cup \{xy = 1\}$ and

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y) + \omega(y,x).$$

The exactness of $\omega(x,y)$ is guaranteed by the condition

$$\omega(x,x) = \mathrm{d}P_3(x) \tag{21}$$

where $P_3(x)$ is Zagier's modified version of the trilogarithm:

$$P_3(x) := \mathrm{Re}\left(\mathrm{Li}_3(x) - \log|x|\mathrm{Li}_2(x) + \frac{1}{3}\log^2|x|\mathrm{Li}_1(x)\right).$$

This leads to

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2}\zeta(3).$$

In general, let $P(x,y,z) \in \mathbb{C}[x,y,z]$, then we can write

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z) \tag{22}$$

where P^* is a two-variable polynomial which is the principal coefficient of $P \in \mathbb{C}[x,y][z]$.

In order to use equation (20), we need that $\{x,y,z\} = 0$ as an element of the Milnor K-theory group $K_3^M(\mathbb{C}(S))$ or

$$x \wedge y \wedge z = \sum r_i\ x_i \wedge (1-x_i) \wedge y_i \quad \text{in} \quad \bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}.$$

We obtain

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_{\partial\Gamma} \omega(x_i,y_i).$$

The problem is that $\omega(x,y)$ is only multiplicative in the second variable. For the first variable, its behavior is ruled by the five term relation:

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] = 0$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.

In general, for a field F, define

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\{[0],[\infty],R_2(x,y)\}.$$

In order to achieve that $\omega(x,y)$ is exact, we need the condition

$$[x]_2 \otimes y = \sum r_i\ [x_i]_2 \otimes x_i$$

in $B_2(\mathbb{Q}(C)) \otimes \mathbb{Q}(C)^*$. By a conjecture of Goncharov [8], this translates into an element in $K_4^{[1]}(\mathbb{Q}(C))$ which has to be zero.

If the condition is satisfied, we get

$$\int_\gamma \omega(x,y) = \sum r_i \; P_3(x_i)|_{\partial\gamma}\,.$$

We could summarize this picture as follows. We first integrate in this picture

$$\ldots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \ldots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

As before, we have two situations. All the examples we have talked about fit into the situation when $\eta(x,y,z)$ is exact and $\partial\Gamma \neq \emptyset$. Then we finish with an element in $K_4(\partial\Gamma)$.

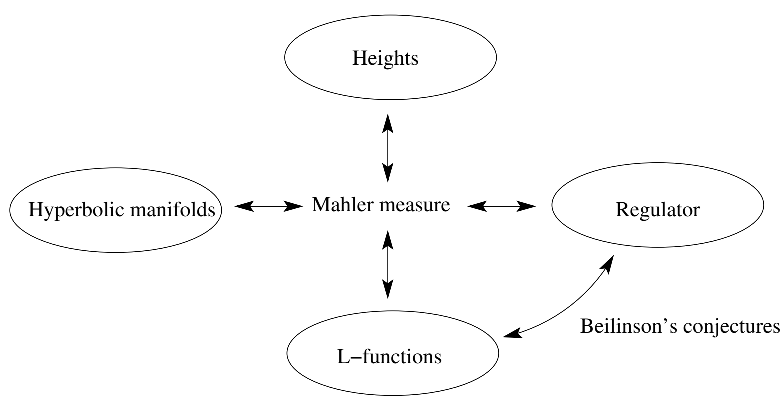
Then we go to

$$\ldots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Again we have two possibilities, but in our context, $\omega(x,y)$ is exact and we finish with an element in $K_5(\partial\gamma) \subset K_5(\bar{\mathbb{Q}})$ leading to trilogarithms and zeta functions, due to Zagier's conjecture and Borel's theorem.

The next picture shows how Mahler measure interacts with several elements (some have been discussed here and some have not). We can see the key role of Mahler measure in the relation among special values of L-functions and regulators (which are related via Beilinson's conjectures), heights, and hyperbolic manifolds (that are related by Beilinson's conjectures as well). It is our general goal to bring more light to the nature of these relationships.



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