Functional equations for Mahler measures of genus-one curves

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Mahler measure for one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

Lehmer's question

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$= \log(1.176280818\ldots) = 0.162357612\ldots$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

$$\text{Does there exist} \quad C > 0, \quad \text{for all} \quad P(x) \in \mathbb{Z}[x]$$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the above polynomial the best possible?

- Dobrowolski (1979): P monic, irreducible, noncyclotomic, of degree d

$$M(P) > 1 + c\left(\frac{\log\log d}{\log d}\right)^3.$$

- Breusch (1951): P monic, irreducible, nonreciprocal,

$$M(P) > 1.324717... = \text{real root of} \quad x^3 - x - 1$$

  (rediscovered by Smyth (1971))
- Bombieri & Vaaler (1983): $M(P) < 2$, then P divides a $Q \in \mathbb{Z}[x]$ whose coefficients belong to $\{-1, 0, 1\}$.

Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

where h is the logarithmic Weil height.

Boyd & Lawton Theorem

$P \in \mathbb{C}[x_1, \ldots, x_n]$

$$\lim_{k_2\to\infty} \ldots \lim_{k_n\to\infty} m(P(x, x^{k_2}, \ldots, x^{k_n})) = m(P(x_1, x_2, \ldots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1)$$
- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

Examples in three variables

- Condon (2003):

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005):

$$\pi^2 m(x^2+1+(x+1)y+(x-1)z) = \pi \mathrm{L}(\chi_{-4},2) + \frac{21}{8}\zeta(3)$$

Examples with more than three variables

L.(2003):

- $$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=24\mathrm{L}(\chi_{-4},4)$$
- $$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$
- $$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)=62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

Known fromulas for n.

The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd (1998)

$$m\,(k) \stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

E_k determined by $x + \frac{1}{x} + y + \frac{1}{y} + k = 0$.

Rodriguez-Villegas (1997)
$k = 4\sqrt{2}$ (CM case)

$$m\left(4\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\sqrt{2}\right) = \mathrm{L}'(E_{4\sqrt{2}}, 0)$$

(By Bloch)

$k = 3\sqrt{2}$ (modular curve $X_0(24)$)

$$m\left(3\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2}\right) = q\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

(By Beilinson)

L. & Rogers (2006)
For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

Kurokawa & Ochiai (2005)
For $h \in \mathbb{R}^*$,

$$m(4h^2)+m\left(\frac{4}{h^2}\right)=2m\left(2\left(h+\frac{1}{h}\right)\right).$$

$h = \frac{1}{\sqrt{2}}$ in both equations, and using K-theory,

Corollary

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2}) = 4\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

The elliptic regulator

F field. Matsumoto Theorem:

$$K_2(F) = \langle \{a, b\}, a, b \in F \rangle \,/\, \langle \text{bilinear}, \{a, 1-a\} \rangle$$

$K_2(E) \otimes \mathbb{Q}$ subgroup of $K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$ determined by kernels of tame symbols.
$x, y \in \mathbb{Q}(E)$, assume trivial tame symbols.
The regulator map (Beilinson, Bloch):

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R}(1))$$

$\omega \in H^0(E, \Omega^1)$,

$$< r(\{x, y\}), \omega >= \frac{1}{2\pi \mathrm{i}} \int_{E(\mathbb{C})} \eta(x, y) \wedge \omega$$

$$\eta(x, y) := \log |x| \mathrm{d} \mathrm{i} \arg y - \log |y| \mathrm{d} \mathrm{i} \arg x$$

$$\eta(x,y) := \log|x| \mathrm{d}i \arg y - \log|y| \mathrm{d}i \arg x$$

$\eta(x, 1-x) = \mathrm{d}D(x)$,

$$D(x) = \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Bloch-Wigner dilogarithm.

Need integrality conditions.

Computing the regulator

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$$

z mod $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$.
Kronecker-Eisenstein series

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right) = \frac{y_\tau^2}{\pi} \sum_{m,n\in\mathbb{Z}}{}' \frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}$$

y_τ is the imaginary part of τ.
Elliptic dilogarithm

$$D_\tau(z) := \sum_{n\in\mathbb{Z}} D(zq^n)$$

Regulator function given by

$$R_\tau = D_\tau - \mathrm{i}J_\tau$$

$\mathbb{Z}[E(\mathbb{C})]^- = \mathbb{Z}[E(\mathbb{C})]/\sim \quad [-P] \sim -[P].$
R_τ is an odd function,

$$\mathbb{Z}[E(\mathbb{C})]^- \to \mathbb{C}.$$

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

$$\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$$

$$(x)^- * (y) = \sum m_i n_j (a_i - b_j).$$

Theorem

(Bloch, Beilinson) $E/\mathbb{R}$ *elliptic curve,* x, y *non-constant functions in* $\mathbb{C}(E)$*,* $\omega \in \Omega^1$

$$\int_{E(\mathbb{C})} \eta(x,y) \wedge \omega = \Omega_0 R_\tau((x)^- * (y))$$

Regulators and Mahler measures

Deninger (1997)

L-functions $\leftarrow$ Bloch-Beilinson's conjectures

In the example,

$$yP_k(x,y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} (\log^+ |y_{(1)}(x)| + \log^+ |y_{(2)}(x)|) \frac{\mathrm{d}x}{x}.$$

By Jensen's formula respect to y.

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_{\mathbb{T}^1} \eta(x,y),$$

Proposition

$E/\mathbb{R}$ elliptic curve, x, y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, $\omega \in \Omega^1$

$$-\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x)^- * (y)\right)\right)$$

where Ω_0 is the real period and $\Omega = \int_\gamma \omega$.

Use results of Beilinson, Bloch, Deninger

Idea of Proof

Modular elliptic surface associated to $\Gamma_0(4)$

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

Weierstrass form:

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

$P = \left(1, \frac{k}{2}\right)$, torsion point of order 4.

$$(x)^- * (y) = 4(P) - 4(-P) = 8(P).$$

$$P \equiv -\frac{1}{4} \mod \mathbb{Z} + \tau\mathbb{Z} \qquad k \in \mathbb{R}$$

$$\tau = \mathrm{i} y_\tau \qquad k \in \mathbb{R}, |k| > 4,$$

$$\tau = \frac{1}{2} + \mathrm{i} y_\tau \qquad k \in \mathbb{R}, |k| < 4$$

Understand cycle $[|x| = 1] \in H_1(E, \mathbb{Z})$

$$\Omega = \tau \Omega_0 \quad k \in \mathbb{R}$$

$$-\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x)^- * (y)\right)\right)$$

$$m(k) = \frac{4}{\pi}\,\mathrm{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right), \quad k \in \mathbb{R}$$

Theorem

(Rodriguez-Villegas)

$$m(k) = \mathrm{Re}\left(\frac{16y_\mu}{\pi^2}\sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})}\right)$$

$$= \mathrm{Re}\left(-\pi \mathrm{i}\mu + 2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2\frac{q^n}{n}\right)$$

where $j(E_k) = j\left(-\frac{1}{4\mu}\right)$

$$q = \mathrm{e}^{2\pi\mathrm{i}\mu} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

and y_μ *is the imaginary part of* μ.

Functional equations

- Functional equations of the regulator

$$J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right) = 2J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) + 2J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}(\mu+1)}{2}}\right)$$

$$\frac{1}{y_{4\mu}}J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right) = \frac{1}{y_{2\mu}}J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) + \frac{1}{y_{2\mu}}J_{2\mu}\left(-\mathrm{e}^{\pi\mathrm{i}\mu}\right)$$

- Hecke operators approach

$$m(k) = \mathrm{Re}\left(-\pi\mathrm{i}\mu + 2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2\frac{q^n}{n}\right)$$

$$= \mathrm{Re}\left(-\pi\mathrm{i}\mu - \pi\mathrm{i}\int_{\mathrm{i}\infty}^{\mu}(e(z)-1)\mathrm{d}z\right)$$

$$e(\mu) = 1 - 4\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2q^n$$

$$q = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)}\right)$$

Second degree modular equation, $|h| < 1$, $h \in \mathbb{R}$,

$$q^2\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(h^4\right).$$

$h \to \mathrm{i}h$

$$-q\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right).$$

Then the equation with J becomes

$$m\left(q\left(\left(\frac{2h}{1+h^2}\right)^2\right)\right)+m\left(q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right)\right)=m\left(q\left(h^4\right)\right).$$

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

Direct approach

Also some equations can be proved directly using isogenies:

$$\phi_1 : E_{2\left(h+\frac{1}{h}\right)} \to E_{4h^2}, \qquad \phi_2 : E_{2\left(h+\frac{1}{h}\right)} \to E_{\frac{4}{h^2}}.$$

$$\phi_1 : (X, Y) \to \left(\frac{X(h^2X+1)}{X+h^2}, -\frac{h^3Y\left(X^2+2h^2X+1\right)}{(X+h^2)^2} \right)$$

$$m\left(4h^2\right) = r_1\left(\{x_1, y_1\}\right) = \frac{1}{2\pi}\int_{|X_1|=1} \eta(x_1, y_1)$$

$$= \frac{1}{4\pi}\int_{|X|=1} \eta(x_1 \circ \phi_1, y_1 \circ \phi_1) = \frac{1}{2} r\left(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}\right)$$

The identity with $h = \frac{1}{\sqrt{2}}$

$$m(2) + m(8) = 2m\left(3\sqrt{2}\right)$$

$$m\left(3\sqrt{2}\right) + m\left(\mathrm{i}\sqrt{2}\right) = m(8)$$

$f = \frac{\sqrt{2}Y - X}{2}$ in $\mathbb{C}(E_{3\sqrt{2}})$.

$$(f)^- * (1-f) = 6(P) - 10(P+Q) \Rightarrow 6(P) \sim 10(P+Q).$$

$Q = \left(-\frac{1}{h^2}, 0\right)$ has order 2.

$$\phi : E_{3\sqrt{2}} \to E_{\mathrm{i}\sqrt{2}} \qquad (X,Y) \to (-X, \mathrm{i}Y)$$

$$r_{\mathrm{i}\sqrt{2}}(\{x,y\}) = r_{3\sqrt{2}}(\{x \circ \phi, y \circ \phi\})$$

But

$$(x \circ \phi)^- * (y \circ \phi) = 8(P + Q)$$

$$(x)^- * (y) = 8(P)$$

$$6r_{3\sqrt{2}}(\{x, y\}) = 10r_{\mathrm{i}\sqrt{2}}(\{x, y\})$$

and

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}).$$

Consequently,

$$m(8) = \frac{8}{5}m(3\sqrt{2})$$

$$m(2) = \frac{2}{5}m(3\sqrt{2})$$

Other families

- Hesse family

$$h(a^3) = m\left(x^3 + y^3 + 1 - \frac{3xy}{a}\right)$$

(studied by Rodriguez-Villegas 1997)

$$h(u^3) = \sum_{j=0}^{2} h\left(1 - \left(\frac{1-\xi_3^j u}{1+2\xi_3^j u}\right)^3\right) \qquad |u| \, \mathrm{small}$$

- More complicated equations for examples studied by Stienstra 2005:

$$m\left((x+1)(y+1)(x+y) - \frac{xy}{t}\right)$$

and Bertin 2004, Zagier $<$ 2005, and Stienstra 2005:

$$m\left((x+y+1)(x+1)(y+1) - \frac{xy}{t}\right)$$