ON CEVA POINTS OF (ALMOST) EQUILATERAL TRIANGLES

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Abstract. A Ceva point of a rational-sided triangle is any internal or external point such that the lengths of the three cevians through this point are rational. Buchholz [Buc89] studied Ceva points and showed a method to construct new Ceva points from a known one. We prove that almost-equilateral and equilateral rational triangles have infinitely many Ceva points by establishing a correspondence to points in certain elliptic surfaces of positive rank.

1. Introduction

In 1801 Euler [Eul01] published an article presenting a parametrization of the triangles with the property that the distance from a vertex to the center of gravity is rational. Because this distance is two-thirds of the length of the median, this amounts to parametrizing the triangles whose medians have rational length. A Heron triangle is a triangle with rational sides and rational area. In 1981 Guy ([Guy04], Problem D21) posed the question of finding perfect triangles, namely, Heron triangles whose three medians are also rational. This particular problem remains open, with the main contribution being the parametrization of Heron triangles with *two* rational medians by Buchholz and Rathbun [BR97, BR98]. Further contributions to parametrizations of triangles with rational medians were made by Buchholz and various collaborators [Buc02, BBRS03, BS19], and by Ismail [Ism20]. Elliptic curves arising from such problems were studied by Dujella and Peral [DP13, DP14]. Heron triangles have been extensively studied by various authors, see for example [Sas99, KL00, GM06, Bre06, vL07, ILS07, SSSG+13, BS15, HH20].

In his PhD thesis [Buc89] Buchholz considered the more general situation of three cevians. He defined a Ceva point of a rational-sided triangle to be any internal or external point such that the lengths of the three cevians through this point are rational. Buchholz showed a method for constructing a new Ceva point from a known one. As Buchholz remarked, the orthocenter of a Heron triangle is necessarily a Ceva point, since the heights must be rational for the triangle to have a rational area. He used this to conclude that Heron triangles have infinitely many Ceva points.

An integral-sided $\triangle ABC$ is said to be almost-equilateral if the three sides have lengths that are consecutive integers. We may extend this notion by scaling. In that case, an almost-equilateral triangle has three rational sides in arithmetic progression.

We prove the following result.

Theorem 1. *Let $\triangle ABC$ be an almost-equilateral rational triangle. For almost all rational cevians, there are infinitely many Ceva points in that cevian. In particular, $\triangle ABC$ has infinitely many Ceva points. The same applies if $\triangle ABC$ is an equilateral rational triangle.*

We remark that the above result *does not require* that $\triangle ABC$ be a Heron triangle. The result for the equilateral triangle was also considered by Buchholz [Buc89].

This paper is organized as follows. In Section 2 we recall the results of Buchholz and describe how to combine them for the general set-up of our problem. In Section 3 we consider the almost equilateral case. The Ceva points under consideration are parametrized by an elliptic $K3$ surface. We study the arithmetic of a rational elliptic surface in detail, showing that it has rank 2, and use this to prove that the elliptic $K3$ surface has positive rank. We further analyze the $K3$ surface in a particular example in Subsection 3.1. We

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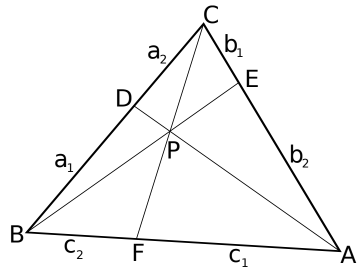

FIGURE 1. The $\overset{\triangle}{ABC}$ with three cevians.

consider the elliptic $K3$ surface of the equilateral triangle in Section 4 and we prove that in this case the rank is exactly 2. We conclude the proof of Theorem 1 and add some further discussion in the last Section.

2. SET-UP OF THE PROBLEM

Consider a triangle $\triangle ABC$ with sides of lengths $|BC| = a, |CA| = b$, and $|AC| = c$, consider the cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ of lengths p, q, and r. The points D, E, F divide the sides of the triangle into two segments each, of lengths $a_1, a_2, b_1, b_2, c_1, c_2$ (see Figure 1). Denote the area of $\triangle ABC$ by $\mathcal{A}$. The parameters p, q, r are non negative, since they represent the lengths of cevians. Two of the a_i, b_i, c_i may be negative when the bases of two of the cevians lie outside the segment of the corresponding side.

Ceva's theorem implies that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent if and only if

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1.$$

Stewart's theorem implies that

$$a(p^2 + a_1 a_2) = b^2 a_1 + c^2 a_2,$$

and analogously for the other cevians. From this, Buchholz proves the following result.

Theorem 2 (Theorem 18, [Buc89]). *For any integer-sided triangle $\overset{\triangle}{ABC}$ with the notation described above, any rational cevian from the vertex A is given by*

$$\begin{aligned} p =& \frac{\alpha u^2 + \beta v^2}{4auv} \\ a_1 =& \frac{\alpha u^2 - \beta v^2 + 2uv(a^2 - b^2 + c^2)}{4auv} \\ a_2 =& a - a_1, \end{aligned} \tag{1}$$

where α, β, u, v are all integers and $\alpha\beta = 16\mathcal{A}^2$.

Indeed, Heron's formula for the area of the triangle gives

$$\mathcal{A}^2 = s(s-a)(s-b)(s-c), \quad \text{for } s = \frac{a+b+c}{2},$$

and therefore $16\mathcal{A}^2$ is an integral number.

Now assume that both p and q are rational. Combining the theorem above with Ceva's and Stewart's theorems, Buchholz arrives to the following equation (Eq. (5.2) [Buc89]):

$$r^2(a_1 b_1 + a_2 b_2)^2 = a_1 b_1 a_2 b_2 (a^2 + b^2 - c^2) + a^2 a_2^2 b_2^2 + b^2 a_1^2 b_1^2, \tag{2}$$

and we are interested in the case in which r is also rational.

In Theorem 2, set $U = \frac{u}{v}$ and $\alpha = 1$, $\beta = 16\mathcal{A}^2$, and proceed similarly introducing a variable V for the equations corresponding to the side b. Then

$$
\begin{cases}
a_1 = \dfrac{1}{4a}\left(U - \dfrac{16\mathcal{A}^2}{U}\right) + \dfrac{a^2 - b^2 + c^2}{2a}, \qquad a_2 = a - a_1, \\[2ex]
b_1 = \dfrac{1}{4b}\left(V - \dfrac{16\mathcal{A}^2}{V}\right) + \dfrac{a^2 + b^2 - c^2}{2b}, \qquad b_2 = b - b_1.
\end{cases} \tag{3}
$$

Replacing the formulas above in equation (2), and setting $y = UVr(a_1b_1 + a_2b_2)$ yields an equation of the form $y^2 = P(U, V)$, where P is a polynomial of degree 4 in each of the variables U, V. If we think of P as a polynomial of degree 4 in $\mathbb{Q}[V][U]$, the curve has a solution at $U = b^2 - (a - c)^2$. This corresponds to $a_1 = 0$ and we can take $y = a^2b_2UV$ while V is a free parameter. Using this rational solution one can turn the equation into a Weierstrass form. The coefficients are polynomials in V satisfying $\deg a_j \leq 2j$, thus giving an elliptic $K3$ surface as long as the resulting equation is not singular. We can then proceed to study the arithmetic structure of the surface.

Keeping track of the geometric problem, we have the additional condition that

$$
U, V \geq 0 \tag{4}
$$

so that p, given by equation (1) and representing the length of a cevian, is a non negative number (and similarly for the cevian from the vertex B). The goal is to prove that there are infinitely many points in the surface that satisfy condition (4). In some cases, it may be easier to first work with the rational elliptic surface resulting from considering the variable $v = V - \frac{16\mathcal{A}^2}{V}$ instead.

Once we succeed in proving that the elliptic $K3$ surface has positive rank, Silverman's Specialization Theorem ([Sil94], Theorem 11.4) tells us that for all but finitely many values of $V = V_0$ the rank of $E(\mathbb{Q}(V_0))$ is greater than or equal to the rank of $E(\mathbb{Q}(V))$.

Finally we must ensure that there are infinitely many points that satisfy condition (4) in order to prove Theorem 1. This can be deduced from the following result.

Theorem 3 (Poincaré and Hurwitz ([Sko50]p.78))**.** *Let E be an elliptic curve over $\mathbb{Q}$ with positive rank. If $E(\mathbb{R})$ is connected, then $E(\mathbb{Q})$ is dense in $E(\mathbb{R})$. If $E(\mathbb{R})$ has two connected components, then $E(\mathbb{Q})$ is dense in the connected components of $E(\mathbb{R})$ containing points of $E(\mathbb{Q})$ of infinite order.*

3. Almost-equilateral triangles

We consider in this section the case of an almost-equilateral triangle. More specifically, we construct an elliptic surface associated to the Ceva points of a rational cevian in an almost-equilateral triangle and study its arithmetic. Our main reference for studying the arithmetic of elliptic surface is the book of Schütt and Shioda [SS19].

We start by setting $a = b - 1$ and $c = b + 1$. Technically, we will fix the cevian coming from the side of length b. The reasoning for the other three cevians is similar.

By applying Heron's formula for the area of the triangle, we obtain

$$
\mathcal{A} = \frac{b\sqrt{3(b^2 - 4)}}{4}.
$$

Thus, (3) becomes

$$
\begin{cases}
a_1 = \dfrac{1}{4(b-1)}\left(U - \dfrac{3b^2(b^2-4)}{U}\right) + \dfrac{b^2+2}{2(b-1)}, \qquad a_2 = -\dfrac{1}{4(b-1)}\left(U - \dfrac{3b^2(b^2-4)}{U}\right) - \dfrac{b(b-4)}{2(b-1)}, \\[2ex]
b_1 = \dfrac{1}{4b}\left(V - \dfrac{3b^2(b^2-4)}{V}\right) + \dfrac{b-4}{2}, \qquad b_2 = -\dfrac{1}{4b}\left(V - \dfrac{3b^2(b^2-4)}{V}\right) + \dfrac{b+4}{2}.
\end{cases} \tag{5}
$$

We consider the change of variables

$$
v = V - \frac{3b^2(b^2-4)}{V},
$$

and let $y = 16U(b-1)br(a_1b_1 + a_2b_2)$. Replacing in (2) we obtain

$$\begin{aligned}
y^2 =&\Big((3b^2-6b+1)v^2 + 4b(-10b^2+23b-4)v + 4b^2(b^4+2b^3+33b^2-88b+16)\Big)U^4 \\
&+ 8b^2\Big((4b-7)v^2 + 2(b-4)b(2b^2+2b-13)v - 24(b-4)(b-2)b^2(b+2)\Big)U^3 \\
&- 2b^2\Big((7b^4-14b^3-71b^2+120b+12)v^2 - 12(b-2)b(b+2)(14b^2-37b-4)v \\
&- 12(b-2)b^2(b+2)(b^4-14b^3-31b^2+136b+16)\Big)U^2 \\
&- 24(b-2)b^4(b+2)\Big((4b-7)v^2 + 2(b-4)b(2b^2+2b-13)v - 24(b-4)(b-2)b^2(b+2)\Big)U \\
&+ 9(b-2)^2b^4(b+2)^2\Big((3b^2-6b+1)v^2 + 4b(-10b^2+23b-4)v + 4b^2(b^4+2b^3+33b^2-88b+16)\Big).
\end{aligned}$$

As noted earlier, a rational point is given by $U = b^2-4$, $y = 4(b-1)^3(b^2-4)(v-2b^2-8b)$. Following a standard algorithm (see for example, Cassels [Cas91], chapter 8) the quartic above is birational to the Weierstrass form

$$\begin{aligned}
E_b : Y^2 =&X^3 + 4\Big[(b^4-38b^3+13b^2+120b+12)v^2 - 12(b-2)b(b+2)(3b^3+2b^2-37b-4)v \\
&- 12(b-2)b^2(b+2)(b^4-14b^3-31b^2+136b+16)\Big]X^2 \\
&- 384(b-2)b(b+2)(v+2b^2-8b)^2\Big[(b^5-3b^4-14b^3+9b^2+49b+12)v^2 \\
&- 3(b-2)b(b+2)(b^4+8b^3-10b^2-56b-15)v \\
&+ 12(b-2)b^2(b+2)(b^4+12b^3-5b^2-48b-14)\Big]X \\
&+ 2304(b-2)^2b^2(b+1)^2(b+2)^2(v+2b(b-4))^4\Big((b^2-7)v-12b(b^2-4)\Big)^2. \qquad (6)
\end{aligned}$$

By examining the degrees on v of the coefficients, we conclude that E_b is a rational elliptic surface. Indeed, following the standard notation for the Weierstrass form, the coefficients satisfy $\deg_v a_j \le j$.

We will now study the arithmetic of E_b.

Proposition 4. *Let E_b be the rational elliptic surface given by equation* (6) *and let*

$$\begin{aligned}
P_b(v) =&\Big[0, 48(b-2)b(b+1)(b+2)\Big(v+2b(b-4)\Big)^2\Big((b^2-7)v-12b(b^2-4)\Big)\Big], \\
R_b(v) =&\Big[8(b-2)(b+2)\Big(v+2b(b-4)\Big)\Big((b^2+4b+1)v+4b(b+2)(b^2-3b-1)\Big), \\
&32(b-2)(b-1)^3(b+2)\Big(v-2b(b+4)\Big)\Big(v+2b(b-4)\Big)\Big(v+2b(b^2-4)\Big)\Big], \\
Q_b(v) =&\Big[12(b-2)(b+1)^2(b+2)\Big(v+2b(b-4)\Big)^2, 0\Big].
\end{aligned}$$

Then

$$E_b(\mathbb{C}(v)) \cong \mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z},$$

P_b and R_b generate the free part, while Q_b generates the torsion part.

Proof. The discriminant of the Weierstrass form (6) is given by

$$\begin{aligned}
\Delta_b =&339738624(b-2)^4(b-1)^{12}b^2(b+2)^4\Big(v-2b(b+4)\Big)^4\Big(v+2b(b-4)\Big)^4\Big(v^2+12b^2(b^2-4)\Big) \\
&\times\Big((b^2-3)v^2-12b(b^2-4)v+48b^2(b^2-4)\Big).
\end{aligned}$$

Observe that the singularities at $v = 2b(b+4)$ and $v = -2b(b-4)$ correspond to $b_2 = 0$ and $b_1 = 0$ respectively in (5). We remark that the values $b = 0, 1, \pm 2$ lead to singular surfaces, but these cases are already excluded from the geometric problem, since they do not yield triangles.

By applying Tate's algorithm ([Sil94], IV.9) the singularities at $v = 2b(b+4)$ and $v = -2b(b-4)$ are of type I_4 in the Kodaira classification. By Proposition 5.16 in [CD89] (Theorem 5.47 in [SS19]), we have

$$e(E_b) = 12 = \sum_{\nu} e(F_\nu),$$

where ν goes over the singularities and $e(F_\nu) = m_\nu$, the number of components of the singular fiber F_ν, if it is multiplicative and $m_\nu + 1$ if F_ν is additive. We have $m_\nu = n$ if ν if of type is I_n. Thus, we have

$$4 = \sum_{\substack{\nu \text{ root of} \\ (v^2+12b^2(b^2-4))((b^2-3)v^2-12b(b^2-4)v+48b^2(b^2-4))}} e(F_\nu)$$

Thus, if the roots are all different, then the four singularities have $m_\nu = 1$ and are of type I_1. By checking the discriminant of $\Big(v-2b(b+4)\Big)\Big(v+2b(b-4)\Big)\Big(v^2+12b^2(b^2-4)\Big)\Big((b^2-3)v^2-12b(b^2-4)v+48b^2(b^2-4)\Big)$ one can see that there are extra multiplicities for the roots only if $b = 0, \pm1, \pm2$, and those values are excluded from the geometric problem.

By the Shioda–Tate formula ([Shi72], Corollary 1.5 or [SS19], Corollary 6.7), we have

$$\rho(E_b) = \operatorname{rk} E_b(\mathbb{C}(v)) + 2 + \sum_{\nu}(m_\nu - 1). \tag{7}$$

In our case, the Picard number of a rational elliptic surface is 10, and therefore,

$$10 = \rho(E_b) = \operatorname{rk} E_b(\mathbb{C}(v)) + 2 + 2\cdot(4-1) + 4(1-1),$$

and we conclude that $\operatorname{rk} E_b(\mathbb{C}(v)) = 2$.

By inspection we find the points P_b, R_b, Q_b. We see that Q_b is a point of order 2. It is also possible to see (by inspecting equation (6)) that there are no other points of order 2 over $\mathbb{C}(v)$. By Table (4.5) in [MP89] we conclude that $E_b(\mathbb{C}(v))_{\text{tor}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (because in our case the rank of the Mordell–Weil group is $R = 2$ and the Euler characteristic for a rational elliptic surface is $\chi = 1$).

By the Determinant formula (Corollary 6.39 in [SS19]), we have

$$|\operatorname{disc} \operatorname{NS}(E_b)| = \frac{|\operatorname{disc} \operatorname{Triv}(E_b)\cdot \operatorname{disc} \operatorname{MWL}(E_b)|}{|E_b(\mathbb{C}(v))_{\text{tor}}|^2}, \tag{8}$$

where $\operatorname{MWL}(E_b)$ is the Mordell–Weil lattice and $\operatorname{Triv}(E_b)$ is the trivial lattice.

By Definition 7.3 in [Shi90],

$$\operatorname{disc} \operatorname{Triv}(E_b) = \prod_{\nu} m_\nu^{(1)}, \tag{9}$$

where $m_\nu^{(1)}$ is the number of simple components of the corresponding singular fiber. We have $m_\nu^{(1)} = n$ if ν is of type I_n. We thus get

$$\operatorname{disc} \operatorname{Triv}(E_b) = 16.$$

Since $\operatorname{disc} \operatorname{NS}(E_b) = -1$ (as the Néron–Severi lattice of a rational elliptic surface is unimodular) and $|E(\mathbb{C}(v))_{\text{tor}}| = 2$, equation (8) becomes

$$|\operatorname{disc} \operatorname{MWL}(E_b)| = \frac{1}{4}. \tag{10}$$

Now we proceed to compute the determinant of the Gram matrix of P_b and R_b. In order to do this we need to find the height pairing of both points. By formulas (6.14) and (6.15) in [SS19],

$$\langle P, R\rangle = \chi + (P.O) + (R.O) - (P.R) - \sum_{\nu} \operatorname{contr}_\nu(P, R), \tag{11}$$

$$h(P) := \langle P, P\rangle = 2\chi + 2(P.O) - \sum_{\nu} \operatorname{contr}_\nu(P), \tag{12}$$

and similarly for R. In the above formulas, $(P.R)$ represents the intersection multiplicity of P and R and $\operatorname{contr}_\nu(P, R)$ represent certain correction terms given by the local contribution from the fiber at ν (see [SS19], Definition 6.23).

Notice that P_b and R_b do not intersect O since the formulas for their coordinates involve polynomials (the denominators are trivial), and therefore $(P_b.O) = (R_b.O) = 0$. Also notice that P_b and R_b intersect at $[0,0]$ when $v = -2b(b-4)$. We will see later that upon desingularization, P_b and R_b intersect different fibers and therefore $(P_b \cdot R_b) = 0$.

We need the following result in order to compute the correction terms.

Theorem 5 (Néron [Nér64]). *Let E_s be an elliptic curve defined over $\mathbb{C}[s]$ given by a Weierstrass model, and denote by v_s the s-adic valuation. Suppose that E_0 has a double point with distinct tangents and $v_s(j(E_s)) = -m < 0$ (this happens if and only if E_0 is singular of type I_m). Then, for every integer $l > m/2$, there exists a Weierstrass model $\mathcal{E}_s$ deduced from E_s by a transformation of the form*

$$\begin{aligned} X &= x + qz, \\ Y &= y + ux + rz, \\ Z &= z, \end{aligned}$$

with $q, r, u \in \mathbb{C}[s]$. The Weierstrass model $\mathcal{E}_s$ is given by

$$Y^2Z + \lambda XYZ + \mu YZ^2 = X^3 + \alpha X^2 Z + \beta XZ^2 + \gamma Z^3, \tag{13}$$

with coefficients satisfying

$$v_s(\lambda^2 + 4\alpha) = 0, \quad v_s(\mu) \geq l, \quad v_s(\beta) \geq l, \quad v_s(\gamma) = m, \text{ and } \quad v_s(j(\mathcal{E}_s)) = -m. \tag{14}$$

We follow the exposition of [Ber10, LM19]. A singular fiber of type I_m over $s = 0$ is composed by nonsingular rational curves $\Theta_{0,0}, \Theta_{0,1}, \ldots, \Theta_{0,m-1}$. When $m = 2h$, the configuration of these curves can be found in $(\mathbb{P}^2)^h$ with a point $[X : Y : Z] \in E_0$ over $s = 0$ corresponding to the point

$$[X : Y : Z^{(1)}] \times [X : Y : Z^{(2)}] \times \cdots \times [X : Y : Z^{(h)}] \in (\mathbb{P}^2)^h,$$

where $[X : Y : Z^{(i+1)}] = [X : Y : sZ^{(i)}]$.

If $[X : Y : Z]$ is singular over s, it satisfies equation (13), then $[X : Y : Z^{(1)}]$ satisfies equation

$$Y^2Z^{(1)} + \lambda XYZ^{(1)} + (\mu/s)Y(Z^{(1)})^2 = sX^3 + \alpha X^2 Z^{(1)} + (\beta/s)X(Z^{(1)})^2 + (\gamma/s^2)(Z^{(1)})^3.$$

Under conditions (14) together with $m = 2h \geq 4$, the equation above simplifies upon evaluation at $s = 0$ to

$$Z^{(1)}(Y^2 + \lambda_0 XY - \alpha_0 X^2) = 0,$$

where the subscript 0 indicates evaluation at $s = 0$.

We remark that in our case $\lambda = 0$ and therefore the equation above becomes

$$Z^{(1)}(Y - \mu X)(Y + \mu X) = 0,$$

where $\alpha_0 = \mu^2$.

When $s = 0$ the point $[X : Y : Z^{(1)}]$ becomes either $[0 : 0 : 1], [x_1 : \mu x_1 : 1]$, or $[x_1 : -\mu x_1 : 1]$. In the last two cases, it has been desingularized either on $\Theta_{0,1}$ or $\Theta_{0,m-1}$. We tend to think that $\mu > 0$ corresponds to $\Theta_{0,1}$ and $-\mu < 0$ corresponds to $\Theta_{0,m-1}$, but this is just a convention, and it does not make a difference which component is identified with μ and which component is identified with $-\mu$.

The components that will be relevant to us are given by

$$\begin{aligned} \Theta_{0,0} =& [X : Y : 0] \times \cdots \times [X : Y : 0], \\ \Theta_{0,1} =& [X : \mu X : Z] \times [1 : \mu : 0] \times \cdots \times [1 : \mu : 0], \\ \Theta_{0,h} =& [0 : 0 : 1] \times \cdots \times [0 : 0 : 1] \times [X_0 : Y_0 : Z_0], \\ \Theta_{0,m-1} =& [X : -\mu X : Z] \times [1 : -\mu : 0] \times \cdots \times [1 : -\mu : 0]. \end{aligned}$$

Once we have identified the image of $[X : Y : Z]$ in $(\mathbb{P}^2)^h$, the correction terms can be computed following table 6.1 in [SS19]. In this work we only consider singularities of type I_n. In this case, if P intersects $\Theta_{\nu,i}$, we have $\mathrm{contr}_\nu(P) = \frac{i(n-i)}{n}$. If, in addition, R intersects $\Theta_{\nu,j}$, and $i < j$, we have $\langle P, R \rangle = \frac{i(n-j)}{n}$. If $j < i$, we simply reverse the formula.

Notice that for I_1, all the correction terms are necessarily 0. Therefore, we only need to compute the correction terms for the two singularities of type I_4.

First consider the singularity at $v = -2b(b-4)$. By writing $v = w - 2b(b-4)$ we shift the singularity at $w = 0$. Then we perform the change of variables $X = X_1 + \frac{3(b-2)(b+2)(5b^2+16b+20)}{2} w^2$. The Weierstrass form (6) becomes (in homogenized form)

$$\begin{aligned} Y^2 Z =& X_1^3 + \frac{1}{2}\Big[(53b^4 - 160b^3 + 104b^2 + 384b - 624)w^2 - 64(b-1)^2 b^2 (5b^2 - 8b - 12)w + 512(b-1)^4 b^4\Big] X_1^2 Z \\ & - \frac{9}{4}(b-2)^2(b+2)^2\Big[(69b^4 - 64b^3 - 392b^2 + 1024b - 880)w - 128(b-4)(b-1)^2 b^2 (b+4)\Big] w^3 X_1 Z^2 \\ & + \frac{81}{8}(b-2)^4(b+2)^4\Big[(7b^4 + 96b^3 - 424b^2 + 640b - 400)w^2 + 64(b-1)^2 b^2 (3b^2 - 8b + 20)w - 512(b-1)^4 b^4\Big] w^4 Z^3, \end{aligned}$$

and the points become

$$\begin{aligned} P_b(w) =& \Big[-\frac{3(b-2)(b+2)(5b^2+16b+20)}{2} w^2, 48(b-2)b(b+1)(b+2)\Big((b^2-7)w - 2(b-1)^2 b(b+4)\Big)w^2 \Big] \\ R_b(w) =& \Big[\frac{(b-2)(b+2)\Big((b^2+16b-44)w + 32(b-1)^2 b^2\Big)w}{2}, 32(b-2)(b-1)^3(b+2)(w - 4b^2)\Big(w + 2b^2(b-1)\Big)w \Big]. \end{aligned}$$

We see that R_b intersects $\Theta_{-2b(b-4),3}$. Since w^2 divides both coordinates of P_b, we conclude that P_b intersects $\Theta_{-2b(b-4),2}$.

Thus, we obtain that

$$\mathrm{contr}_{-2b(b-4)}(P_b) = 1, \quad \mathrm{contr}_{-2b(b-4)}(P_b, R_b) = \frac{1}{2}, \quad \mathrm{contr}_{-2b(b-4)}(R_b) = \frac{3}{4}.$$

Now consider the singularity at $v = 2b(b+4)$. By writing $v = w + 2b(b+4)$ we shift the singularity at $w = 0$. Then we perform the change of variables

$$X = X_1 + \frac{3(b-2)(b+2)}{2(b+1)^2}\Big((5b^4 + 38b^3 + 57b^2 + 8b + 20)w^2 + 64b^2(b+1)^4 w + 128b^4(b+1)^4\Big).$$

The Weierstrass form (6) becomes (in homogenized form)

$$\begin{aligned} Y^2 Z =& X_1^3 + \frac{(b-1)^2}{2(b+1)^2}\Big[(53b^4 + 160b^3 + 104b^2 - 384b - 624)w^2 + 64b^2(b+1)^2(5b^2 + 8b - 12)w + 512b^4(b+1)^4\Big] X_1^2 Z \\ & - \frac{9(b-2)^2(b-1)^4(b+2)^2}{4(b+1)^4}\Big[(69b^4 + 64b^3 - 392b^2 - 1024b - 880)w + 128(b-4)b^2(b+1)^2(b+4)\Big] w^3 X_1 Z^2 \\ & + \frac{81(b-2)^4(b-1)^6(b+2)^4}{8(b+1)^6}\Big[(7b^4 - 96b^3 - 424b^2 - 640b - 400)w^2 - 64b^2(b+1)^2(3b^2 + 8b + 20)w \\ & - 512b^4(b+1)^4\Big] w^4 Z^3, \end{aligned}$$

and the points become

$$\begin{aligned} P_b(w) =& \Big[-\frac{3(b-2)(b+2)}{2(b+1)^2}\Big((5b^4 + 38b^3 + 57b^2 + 8b + 20)w^2 + 64b^2(b+1)^4 w + 128b^4(b+1)^4\Big), \\ & 48(b-2)b(b+1)(b+2)(w + 4b^2)^2\Big((b^2-7)w + 2(b-4)b(b+1)^2\Big) \Big] \\ R_b(w) =& \Big[\frac{(b-2)(b-1)^2(b+2)}{2(b+1)^2}\Big((b^2 - 16b - 44)w - 32b^2(b+1)^2\Big)w, 32(b-2)(b-1)^3(b+2)(w + 4b^2)\Big(w + 2b^2(b+1)\Big)w \Big]. \end{aligned}$$

We see that P_b intersects $\Theta_{2b(b+4),0}$ and that R_b intersects $\Theta_{2b(b+4),3}$.

Thus, we obtain that

$$\mathrm{contr}_{2b(b+4)}(P_b) = 0, \quad \mathrm{contr}_{2b(b+4)}(P_b, R_b) = 0, \quad \mathrm{contr}_{2b(b+4)}(R_b) = \frac{3}{4}.$$

Finally, combining equations (11) and (12), we obtain

$$\begin{aligned}
h(P_b) =& 2\chi + 2(P_b.O) - \sum_\nu \mathrm{contr}_\nu(P_b) = 2 - (1+0) = 1,\\
\langle P_b, R_b\rangle =& \chi + (P_b.O) + (R_b.O) - (P_b.R_b) - \sum_\nu \mathrm{contr}_\nu(P_b, R_b) = 1 - \left(\frac{1}{2} + 0\right) = \frac{1}{2},\\
h(R_b) =& 2\chi + 2(R_b.O) - \sum_\nu \mathrm{contr}_\nu(P_b) = 2 - \left(\frac{3}{4} + \frac{3}{4}\right) = \frac{1}{2}.
\end{aligned}$$

Thus, the Gram determinant of P_b and R_b equals

$$\begin{vmatrix} h(P_b) & \langle P_b, R_b\rangle \\ \langle R_b, P_b\rangle & h(R_b) \end{vmatrix} = \frac{1}{4}.$$

Comparing with equation (10), and using the fact that P_b and R_b are elements in the Mordell–Weil group, we conclude that they are generators of $E_b(\mathbb{C}(v))$.

This concludes the proof of Proposition 4. $\square$

To make a full analysis of the original geometric problem we need to replace v by $V - \frac{3b^2(b^2-4)}{V}$ and complete all the computations for the resulting $K3$ surface F_b. We will not write the formula for F_b here.

Notice that we have a morphism

$$\begin{aligned}
\varphi : \mathbb{P}^1 &\to \mathbb{P}^1\\
[V:1] &\to [V^2 - 3b^2(b^2-4) : V]
\end{aligned}$$

yielding a base change

$$\begin{aligned}
F_b &\to E_b\\
[X:Y:Z] &\to \left[VX : Y : V^3 Z\right].
\end{aligned}$$

Let $P_b', R_b' \in F_b(\mathbb{C}(V))$ be induced by P_b, R_b.

By Theorem 6.53 in [SS19], since $\deg\varphi = 2$,

$$h(P_b') = 2h(P_b) = 2 \text{ and } h(R_b') = 2h(R_b) = 1.$$

In particular, since their heights are non-zero, we conclude that both P_b' and R_b' are non-torsion elements in $F_b(\mathbb{C}(V))$.

It would be interesting to explore whether P_b' and R_b' are generators of $F_b(\mathbb{C}(V))$. We can prove that they are generators *if we assume that* $\mathrm{rk}F_b(\mathbb{C}(V)) = 2$. For the sake of simplicity we do this proof in the particular case of the triangle $(2,3,4)$ in the next subsection.

3.1. **The (2,3,4) triangle.** Here we consider the triangle $(2,3,4)$, corresponding to the rational elliptic surface E_3 and the elliptic $K3$ surface F_3 from the previous section. Equations (5) become

$$\left\{\begin{aligned}
a_1 &= \frac{1}{8}\left(U - \frac{135}{U}\right) + \frac{11}{4}, \qquad a_2 = -\frac{1}{8}\left(U - \frac{135}{U}\right) - \frac{3}{4},\\
b_1 &= \frac{1}{12}\left(V - \frac{135}{V}\right) - \frac{1}{2}, \qquad b_2 = -\frac{1}{12}\left(V - \frac{135}{V}\right) + \frac{7}{2}.
\end{aligned}\right. \tag{15}$$

As before, we take

$$v = V - \frac{135}{V}$$

and $y = 96Ur(a_1b_1 + a_2b_2)$ and replace in (2) in order to obtain

$$\begin{aligned}
y^2 =& 2(5v^2 - 150v + 3312)U^4 + 72(5v^2 - 66v + 1080)U^3 + 108(13v^2 + 330v - 13680)U^2\\
&- 9720(5v^2 - 66v + 1080)U + 36450(5v^2 - 150v + 3312).
\end{aligned}$$

A rational point is given by $U=5$, $y=160(v-42)$. By considering the change of variables

$$X = \frac{5\Big[4(v-42)y + (103v^2-768v-9468)U^2 - 18(59v^2-768v+5940)U + 135(25v^2-768v+14076)\Big]}{9(U-5)^2}$$

$$\begin{aligned} Y =& \frac{40(v-42)}{27(U-5)^3}\Big[\Big((v-150)U - 45(v-54)\Big)y - (35v^2-672v+13140)U^3 \\ & -27(19v^2-4140)U^2 + 135(61v^2-1056v+21420)U - 6075(5v^2-192v+4428)\Big] \end{aligned}$$

we obtain the Weierstrass form

$$E_3 : Y^2 = X^3 - 6(19v^2-120v-3420)X^2 + 135(23v^2+180v-18180)(v-6)^2X + 8100(v-6)^4(v-90)^2. \tag{16}$$

We remark that the above equation is slightly different than the one obtained by replacing $b=3$ in equation (6), as we have chosen the change of variables to absorb some extra powers of 2.

Proposition 6. *Let E_3 be the rational elliptic surface given by equation (16) and*

$$\begin{aligned} P_3(v) =& \Big[0, 90(v-6)^2(v-90)\Big], \\ R_3(v) =& \Big[5(v-6)(11v-30), 20(v-42)(v-6)(v+30)\Big], \\ Q_3(v) =& \Big[60(v-6)^2, 0\Big]. \end{aligned}$$

Then

$$E_3(\mathbb{C}(v)) \cong \mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z},$$

P_3 and R_3 generate the free part, while Q_3 generates the torsion part.

Proof. The proof of this statement follows the same lines as the proof of Proposition 4. The discriminant of (16) is given by

$$\Delta_3 = 2799360000(v-42)^4(v-6)^4(v^2-30v+360)(v^2+540).$$

As before, we conclude that $v=42$ and $v=6$ correspond to singularities of type I_4, while the roots of $(v^2-30v+360)(v^2+540)$ yield singularities of type I_1. By the Shioda–Tate formula (7) we have $\operatorname{rk} E_3(\mathbb{C}(v)) = 2$.

Following the same reasoning as in the previous section, P_3 and R_3 generate the free part of $E_3(\mathbb{C}(v))$, while Q_3 is a point of order 2 that generates the torsion. $\square$

Now we proceed to replace v by $V - \frac{135}{V}$ and replace X by $\frac{X}{V^2}$ and Y by $\frac{Y}{V^3}$. This gives the Weierstrass form

$$\begin{aligned} F_3 : Y^2 =& X^3 - 6(19V^4 - 120V^3 - 8550V^2 + 16200V + 346275)X^2 \\ & + 135(23V^4 + 180V^3 - 24390V^2 - 24300V + 419175)(V+9)^2(V-15)^2X \\ & + 8100(V^2-90V-135)^2(V+9)^4(V-15)^4. \end{aligned} \tag{17}$$

Since $\deg_V a_j \leq 2j$ (with at least one equal) we have an elliptic $K3$ surface.

Proposition 7. *Let F_3 be the elliptic $K3$ surface given by equation (17) and*

$$\begin{aligned} P'_3(V) =& \Big[0, 90(V^2-90V-135)(V+9)^2(V-15)^2\Big] \\ R'_3(V) =& \Big[5(V-15)(V+9)(11V^2-30V-1485), 20(V-45)(V-15)(V+3)(V+9)(V^2+30V-135)\Big] \\ Q'_3(V) =& \Big[45(V-15)(V-9)(V+9)(V+15), 90(V-45)(V-15)(V+3)(V+9)(V^2+135)\Big]. \end{aligned}$$

Then

$$\operatorname{rk} F_3(\mathbb{C}(V)) \geq 2, \qquad F_3(\mathbb{C}(V))_{\mathrm{tor}} \cong \mathbb{Z}/4\mathbb{Z}.$$

Moreover, if $\operatorname{rk} F_3(\mathbb{C}(V)) = 2$, then P'_3 and R'_3 generate the free part. In addition, Q'_3 generates the torsion part.

Proof. The discriminant of the Weierstrass form (17) is given by

$$\Delta = 2799360000(V-45)^4(V-15)^4(V+3)^4(V+9)^4(V^2+135)^2(V^4-30V^3+90V^2+4050V+18225).$$

Observe that the singularities at $V = 15, -9$ and at $V = 45, -3$ correspond to the cases $b_1 = 0$ and $b_2 = 0$ in (15). By applying Tate's algorithm, the singularities at $V = 45, 15, -3, -9$ are of type I_4, those at $V = \pm 3\sqrt{15}i$ are of type I_2, and the roots of $(V^4 - 30V^3 + 90V^2 + 4050V + 18225)$ are of type I_1.

By the Shioda–Tate formula (7),

$$\rho(F_3) = \mathrm{rk} F_3(\mathbb{C}(V)) + 2 + 4\cdot(4-1) + 2\cdot(2-1),$$

and since F_3 is a $K3$ surface, the Picard number satisfies $\rho(F_3) \leq 20$. From this we conclude that

$$\mathrm{rk} F_3(\mathbb{C}(V)) \leq 4.$$

By inspection we find the points in the statement. We see that Q_3' is a point of order 4 and that the only point of order 2 is $2Q_3'$, and we will later see that P_3' and R_3' are independent non-torsion points. By Table (4.5) in [MP89] we conclude that $F_3(\mathbb{C}(V))_{\mathrm{tor}}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ (because in our case the rank R of the Mordell–Weil group is at least 2 and the Euler characteristic is $\chi = 2$).

Applying the Determinant formula (8), we obtain

$$|\mathrm{disc}\,\mathrm{NS}(F_3)| = 2^6 |\mathrm{disc}\,\mathrm{MWL}(F_3)|.$$

By the discussion before this section, we have that $h(P_3') = 2$, $\langle P_3', R_3'\rangle = h(R_3') = 1$. Therefore, the Gram matrix of P_3' and R_3' has determinant 1. From this we confirm that P_3', R_3' are independent non-torsion points. In addition, we obtain

$$|\mathrm{disc}\,\mathrm{NS}(F_3)| \text{ divides } 2^6.$$

Now assume that $\mathrm{rk} F_3(\mathbb{C}(V)) = 2$.

In that case, the lattice generated by P_3' and R_3' has index a power of 2 in $\mathrm{MWL}(F_3)$. If this index is greater than 1, then $\mathrm{MWL}(F_3)$ must contain a point S such that $2S$ is one of the following points: $P_3', R_3', P_3' + R_3', P_3' + Q_3', R_3' + Q_3'$, or $P_3' + R_3' + Q_3'$.

We will prove that this is not possible. To do this, we proceed to study the intersection of P_3', R_3', Q_3' with the singular fibers, by applying Theorem 5 in each singularity.

First consider $V = 45$. By making $V = 45 + w$, and $X = X_1 + 270(2273w^2 + 60480w + 583200)$, we obtain a Weierstrass form that can be written as

$$Y^2 Z = X_1^3 + (\alpha_1 w + 167961600)X_1^2 Z + \beta_1 w^3 X_1 Z^2 + \gamma_1 w^4 Z^3.$$

We have omitted most coefficients for the sake of simplicity. The points become

$$\begin{aligned}
P_3'(w) =& \Big[-270(2273w^2 - 60480w - 583200), 90(w+30)^2(w+54)^2(w^2-2160)\Big],\\
R_3'(w) =& \Big[5(11w^3 + 1884w^2 - 4842w - 77760)w, 20(w+30)(w+48)(w+54)(w^2+120w+3240)w\Big],\\
Q_3'(w) =& \Big[45(w^3 + 180w^2 - 1794w - 25920)w, 90(w+30)(w+48)(w+54)(w^2+90w+2160)w\Big].
\end{aligned}$$

By inspecting the above formulas, we conclude that P_3' intersects $\Theta_{45,0}$, while R_3' and Q_3' intersect $\Theta_{45,3}$. Therefore, we obtain,

$$\mathrm{contr}_{45}(P_3') = 0, \quad \mathrm{contr}_{45}(P_3', R_3') = 0, \quad \mathrm{contr}_{45}(R_3') = \frac{3}{4}.$$

For $V = 15$, we write $V = 15 + w$ and $X = X_1 + 30510w^2$. This leads to the Weierstrass equation

$$Y^2 Z = X_1^3 + (\alpha_1 w + 4665600)X_1^2 Z + \beta_1 w^3 X_1 Z^2 + \gamma_1 w^4 Z^3.$$

The points become

$$\begin{aligned}
P_3'(w) =& \Big[-30510w^2, 90(w+24)^2(w^2-60w-1260)w^2\Big],\\
R_3'(w) =& \Big[5(11w^3 + 564w^2 + 1638w + 12960)w, 20(w-30)(w+18)(w+24)(w^2+60w+540)w\Big],\\
Q_3'(w) =& \Big[45(w^3 + 60w^2 + 366w + 4320)w, 90(w-30)(w+18)(w+24)(w^2+30w+360)w\Big].
\end{aligned}$$

By inspecting the above formulas, we conclude that P_3' intersects $\Theta_{15,2}$, while R_3' and Q_3' intersect $\Theta_{15,3}$. Therefore, we obtain,

$$\mathrm{contr}_{15}(P_3') = 1, \quad \mathrm{contr}_{15}(P_3', R_3') = \frac{1}{2}, \quad \mathrm{contr}_{15}(R_3') = \frac{3}{4}.$$

For $V = -3$, we write $V = -3 + w$ and $X = X_1 - 270(31w^2 - 576w - 2592)$. This leads to the Weierstrass equation

$$Y^2Z = X_1^3 + (\alpha_1 w + 746496)X_1^2 Z + \beta_1 w^3 X_1 Z^2 + \gamma_1 w^4 Z^3.$$

The points become

$$\begin{aligned} P_3'(w) =& \Big[270(31w^2 - 576w - 2592), 90(w-18)^2(w+6)^2(w^2 - 96w + 144)\Big], \\ R_3'(w) =& \Big[5(11w^3 - 228w^2 + 342w - 5184)w, 20(w-48)(w-18)(w+6)(w^2 + 24w - 216)w\Big], \\ Q_3'(w) =& \Big[45(w^3 - 12w^2 - 66w - 1728)w, 90(w-48)(w-18)(w+6)(w^2 - 6w + 144)w\Big]. \end{aligned}$$

Thus P_3' intersects $\Theta_{-3,0}$, R_3' intersects $\Theta_{-3,1}$, and Q_3' intersects $\Theta_{-3,3}$, and we obtain

$$\mathrm{contr}_{-3}(P_3') = 0, \quad \mathrm{contr}_{-3}(P_3', R_3') = 0, \quad \mathrm{contr}_{-3}(R_3') = \frac{3}{4}.$$

For $V = -9$, we write $V = -9 + w$ and $X = X_1 + 30510w^2$. This leads to the Weierstrass equation

$$Y^2Z = X_1^3 + (\alpha_1 w + 1679616)X_1^2 Z + \beta_1 w^3 X_1 Z^2 + \gamma_1 w^4 Z^3.$$

The points become

$$\begin{aligned} P_3'(w) =& \Big[-30510w^2, 90(w-24)^2(w^2 - 108w + 756)w^2\Big], \\ R_3'(w) =& \Big[5(11w^3 - 492w^2 - 954w + 7776)w, 20(w-54)(w-24)(w-6)(w^2 + 12w - 324)w\Big], \\ Q_3'(w) =& \Big[45(w^3 - 36w^2 - 498w + 2592)w, 90(w-54)(w-24)(w-6)(w^2 - 18w + 216)w\Big]. \end{aligned}$$

Thus P_3' intersects $\Theta_{-9,2}$, R_3' intersects $\Theta_{-9,1}$, and Q_3' intersects $\Theta_{-9,3}$, and we obtain

$$\mathrm{contr}_{-9}(P_3') = 1, \quad \mathrm{contr}_{-9}(P_3', R_3') = \frac{1}{2}, \quad \mathrm{contr}_{-9}(R_3') = \frac{3}{4}.$$

For $V = 3\sqrt{15}i$, we write $V = 3\sqrt{15}i + w$ and $X = X_1 + 77760(-2\sqrt{15}iw + 45)$. This leads to the Weierstrass equation

$$Y^2Z = X_1^3 + (\alpha_1 w - 583200(1 + \sqrt{15}i))X_1^2 Z + \beta_1 w^2 X_1 Z^2 + \gamma_1 w^2 Z^3.$$

The points become

$$\begin{aligned} P_3'(w) =& \Big[p_1 w - 3499200, p_2 w + 1889568000(1 - \sqrt{15}i)\Big], \\ R_3'(w) =& \Big[r_1 w + 388800(1 + \sqrt{15}i), r_2 w - 139968000(9 + \sqrt{15}i)\Big], \\ Q_3'(w) =& \Big[q_1 w^2, q_2 w\Big]. \end{aligned}$$

We see that P_3' and R_3' intersect $\Theta_{3\sqrt{15}i,0}$, while Q_3' intersects $\Theta_{3\sqrt{15}i,1}$. Thus,

$$\mathrm{contr}_{3\sqrt{15}i,0}(P_3') = 0, \quad \mathrm{contr}_{3\sqrt{15}i,0}(P_3', R_3') = 0, \quad \mathrm{contr}_{3\sqrt{15}i,0}(R_3') = 0.$$

The case of $V = -3\sqrt{15}i$ is similar, as it is just the conjugate of the previous case.

From the above computation we recover the result that $h(P_3') = 2, h(R_3') = \langle P_3', R_3' \rangle = 1$.

As we remarked before, if P_3', R_3' are not generators, then $\mathrm{MWL}(F_3)$ must contain a point S such that $2S$ is one of the following points: $P_3', R_3', P_3' + R_3', P_3' + Q_3', R_3' + Q_3', P_3' + R_3' + Q_3'$. By looking at the intersections with the components of $V = 3\sqrt{15}i$, we see that $P_3' + Q_3', R_3' + Q_3'$, and $P_3' + R_3' + Q_3'$ intersect $\Theta_{3\sqrt{15}i,1}$. However, if S exists, $2S$ must intersect $\Theta_{3\sqrt{15}i,0}$. Therefore those points can not be $2S$. Similarly, $V = 45$ allows us to eliminate the possibility that R_3' and $P_3' + R_3'$ be $2S$. Unfortunately the intersections with the fibers do not allow us to eliminate the possibility of $P_3' = 2S$. In order to eliminate this last possibility, we can find directly with a computer a general formula for the X coordinate of $2S$ if $S = [x, y]$, and verify that the resulting rational function has no roots in $\mathbb{C}(V)$ and therefore it can never equal 0 (the X coordinate of P_3').

Therefore, *if the rank is 2*, we conclude that P_3' and R_3' are generators.

This concludes the proof of Proposition 4. $\square$

4. Equilateral triangle

Here we consider the equilateral triangle of sides $(1, 1, 1)$. We consider directly the elliptic $K3$ surface, as we are able to describe its Mordell–Weil group completely.

In this case we have $\mathcal{A} = \frac{\sqrt{3}}{4}$. As in the almost equilateral case, we take $\alpha = 1, \beta = 16\mathcal{A}^2 = 3$ in Theorem 2 and equation (3) becomes

$$\begin{cases} a_1 = \dfrac{1}{4}\left(U - \dfrac{3}{U}\right) + \dfrac{1}{2}, \qquad a_2 = 1 - a_1, \\[2ex] b_1 = \dfrac{1}{4}\left(V - \dfrac{3}{V}\right) + \dfrac{1}{2}, \qquad b_2 = 1 - b_1. \end{cases}$$

Replacing in (2), we obtain

$$y^2 = (3V^4 - 14V^2 + 27)U^4 + 32(V^2 - 3)VU^3 - 2(7V^4 - 54V^2 + 63)U^2 - 96(V^2 - 3)VU + 9(3V^4 - 14V^2 + 27),$$

where we have taken $y = 16UVr(a_1b_1 + a_2b_2)$. A rational solution is given by $U = 1$, $y = 4(V - 3)(V + 1)$.

The change of variables

$$\begin{aligned} X =& \frac{2}{(U-1)^2}\Big[(V-3)(V+1)y + 2(5V^2+2V-15)VU^2 - 2(V^2-3)(V^2+10V-3)U \\ &+ 6(V^4 - V^3 - 4V^2 + 3V + 9)\Big] \\ Y =& \frac{2(V-3)(V+1)}{(U-1)^3}\Big[\Big((V-1)(V+3)U - (5V^2-6V-15)\Big)y - (V-1)(V+3)(3V^2+2V-9)U^3 \\ &+ (7V^4 - 24V^3 - 54V^2 + 72V + 63)U^2 + (7V^4 + 72V^3 - 54V^2 - 216V + 63)U \\ &- 3(9V^4 - 8V^3 - 42V^2 + 24V + 81)\Big] \end{aligned}$$

leads to the Weierstrass form

$$E : Y^2 = X^3 + (V^4 - 36V^3 - 18V^2 + 108V + 9)X^2 - 24(V^2 - 3V - 3)(V^2 - 3)(V+3)^2(V-1)^2X + 36(V^2-3)^2(V+3)^4(V-1)^4. \tag{18}$$

This is an elliptic $K3$ surface.

Proposition 8. *Let E be the rational elliptic surface given by equation* (18) *and*

$$\begin{aligned} P(V) =& \Big[0, 6(V-1)^2(V+3)^2(V^2-3)\Big], \\ R(V) =& \Big[2(V^2-3)(V^2+6V-3), 8\sqrt{3}V(V^2-3)(V^2+3)\Big], \\ Q(V) =& [6(V-1)(V+3)(V^2-3), 12(V-3)(V-1)(V+1)(V+3)(V^2-3)\Big], \\ S(V) =& \Big[2(V+3)^2(V^2-3), 0\Big]. \end{aligned}$$

Then

$$E(\mathbb{C}(V)) \cong \mathbb{Z}^2 \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

P and R generate the free part, while Q and S generate the torsion part (Q has order 4 and S has order 2).

Proof. The discriminant of E is given by

$$\Delta = 82944(V-3)^4(V-1)^4(V+1)^4(V+3)^4(V^2-3)^2(V^2+3)^2.$$

Notice that the singularities at $V = 1, -3$ and $V = -1, 3$ correspond to $b_1 = 0$ and $b_2 = 0$ respectively. The singularities at $V = \pm\sqrt{3}$ correspond to $b_1 = b_2 = \frac{1}{2}$, and the corresponding cevian is the median/altitude/bisector. By Tate's algorithm, the singularities at $V = \pm 1, \pm 3$ are of type I_4 and those at $V = \pm\sqrt{3}, \pm\sqrt{3}i$ are of type I_2.

The Shioda–Tate formula (7) then implies

$$\rho(E) = \mathrm{rk}E(\mathbb{C}(v)) + 2 + 4 \cdot (4-1) + 4 \cdot (2-1)$$

and, since we are working with an elliptic $K3$ surface, $\rho(E) \leq 20$ and therefore $\mathrm{rk}E(\mathbb{C}(v)) \leq 2$.

By inspection we find the points P, Q, S. Since the rank is likely 2, we search for another point of infinite order supported over $\mathbb{Q}(\sqrt{3})$ (since $\sqrt{3}$ is associated to $\mathcal{A}$). To do this, we consider the quadratic twist

$$3Y^2 = X^3+(V^4-36V^3-18V^2+108V+9)X^2-24(V^2-3V-3)(V^2-3)(V+3)^2(V-1)^2X+36(V^2-3)^2(V+3)^4(V-1)^4,$$

and, upon setting $Y_1 = 9Y$, $X_1 = 3X$, search among the integral points of

$$Y_1^2 = X_1^3+3(V^4-36V^3-18V^2+108V+9)X_1^2-216(V^2-3V-3)(V^2-3)(V+3)^2(V-1)^2X_1+972(V^2-3)^2(V+3)^4(V-1)^4$$

in order to find R.

We will later see that P and R are non-torsion, and therefore the rank is at least 1. We also see that Q is a point of order 4 and S is of order 2 and independent of Q. By Table (4.5) in [MP89] we conclude that $E(\mathbb{C}(V))_{\mathrm{tor}}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (because in our case the rank of the Mordell–Weil group is $R \geq 1$ and the Euler characteristic is $\chi = 2$).

Applying the Determinant formula (8), we obtain

$$|\det \mathrm{NS}(E)| = 2^6 |\det \mathrm{MWL}(E)|. \tag{19}$$

We proceed to compute the intersections of the points with the singular fibers by applying Theorem 5.

First consider the case of $V = 3$. After setting $V = 3 + w$ and $X = X_1 + 6(41w^2 + 96w + 72)$, we obtain

$$\begin{aligned} Y^2Z =& X_1^3 + (w^4 - 24w^3 + 450w^2 + 864w + 576)X_1^2Z - 12(2w^5 + 50w^4 + 465w^3 + 3528w^2 + 6315w + 4032)w^3X_1Z^2 \\ &+ 36(w^2 + 16w + 6)(w^2 + 18w - 9)(w^4 + 10w^3 + 75w^2 + 144w + 96)w^4Z^3 \end{aligned}$$

and the points become

$$\begin{aligned} P(w) =& \Big[-6(41w^2 + 96w + 72), 6(w+2)^2(w+6)^2(w^2+6w+6)\Big], \\ R(w) =& \Big[2(w^4 + 18w^3 - 21w^2 - 72w - 72), 8\sqrt{3}(w+3)(w^2+6w+6)(w^2+6w+12)\Big], \\ Q(w) =& \Big[6(w^3 + 14w^2 + 25w + 24)w, 12(w+2)(w+4)(w+6)(w^2+6w+6)w\Big], \\ S(w) =& \Big[2(w^2 + 18w - 9)w^2, 0\Big]. \end{aligned}$$

We conclude from the above that P and R intersect $\Theta_{3,0}$, Q intersects $\Theta_{3,1}$, while S intersects $\Theta_{3,2}$. This implies that

$$\mathrm{contr}_3(P) = 0, \quad \mathrm{contr}_3(P,R) = 0, \quad \mathrm{contr}_3(R) = 0.$$

Now consider $V = 1$. After making the change $V = 1 + w$, and $X = X_1 + 30w^2$, we obtain

$$\begin{aligned} Y^2Z =& X_1^3 + (w^4 - 32w^3 - 30w^2 - 32w + 64)X_1^2Z - 12(2w^5 + 18w^4 + 25w^3 + 32w^2 - 21w + 64)w^3X_1Z^2 \\ &+ 36(w^2 + 2w + 3)(w^2 + 8w + 6)(w^4 + 10w^3 + 15w^2 + 16w - 32)w^4Z^3 \end{aligned}$$

and the points become

$$\begin{aligned} P(w) =& \Big[-30w^2, 6(w+4)^2(w^2+2w-2)w^2\Big], \\ R(w) =& \Big[2(w^4 + 10w^3 + 3w^2 - 8w - 8), \sqrt{3}(w+1)(w^2+2w-2)(w^2+2w+4)\Big], \\ Q(w) =& \Big[6(w-1)(w^2+7w+8)w, 12(w-2)(w+2)(w+4)(w^2+2w-2)w\Big], \\ S(w) =& \Big[2(w^4 + 10w^3 + 15w^2 + 16w - 32), 0\Big]. \end{aligned}$$

We have that P intersects $\Theta_{1,2}$, R and S intersect $\Theta_{1,0}$, and Q intersects $\Theta_{1,3}$. Therefore,

$$\mathrm{contr}_1(P) = 1, \quad \mathrm{contr}_1(P,R) = 0, \quad \mathrm{contr}_1(R) = 0.$$

For $V = -1$, we consider the change $V = -1 + w$, and $X = X_1 + 6(8 - 7w^2)$. We obtain

$$\begin{aligned} Y^2Z =& X_1^3 + (w^4 - 40w^3 + 30w^2 + 32w + 64)X_1^2Z - 12(2w^5 - 14w^4 + 9w^3 - 152w^2 + 107w + 192)w^3X_1Z^2 \\ &+ 36(w^2 + 6)(w^2 - 6w + 3)(w^4 + 2w^3 + 15w^2 - 16w - 32)w^4Z^3 \end{aligned}$$

and the points become

$$\begin{aligned} P(w) =& \Big[6(7w^2 - 8), 6(w - 2)^2(w + 2)^2(w^2 - 2w - 2)\Big], \\ R(w) =& \Big[2(w^4 + 2w^3 + 3w^2 + 8w - 8), 8\sqrt{3}(w - 1)(w^2 - 2w - 2)(w^2 - 2w + 4)\Big], \\ Q(w) =& \Big[6(w^3 - 2w^2 + w + 8)w, 12(w - 4)(w - 2)(w + 2)(w^2 - 2w - 2)w\Big], \\ S(w) =& \Big[2(w^4 + 2w^3 + 15w^2 - 16w - 32), 0\Big]. \end{aligned}$$

We conclude that P, R, and S intersect $\Theta_{-1,0}$, Q intersects $\Theta_{1,3}$, and

$$\mathrm{contr}_{-1}(P) = 0, \quad \mathrm{contr}_{-1}(P,R) = 0, \quad \mathrm{contr}_{-1}(R) = 0.$$

For $V = -3$, we consider $V = -3 + w$ and $X = X_1 + 30w^2$. This leads to

$$\begin{aligned} Y^2Z =& X_1^3 + (w^4 - 48w^3 + 450w^2 - 864w + 576)X_1^2Z - 12(2w^5 - 46w^4 + 417w^3 - 1728w^2 + 2859w - 1728)w^3X_1Z^2 \\ &+ 36(w^2 - 8w + 6)(w^2 - 6w - 9)(w^4 - 14w^3 + 75w^2 - 144w + 96)w^4Z^3 \end{aligned}$$

and the points become

$$\begin{aligned} P(w) =& \Big[-30w^2, 6(w - 4)^2(w^2 - 6w + 6)w^2\Big], \\ R(w) =& \Big[2(w^4 - 6w^3 - 21w^2 + 72w - 72), 8\sqrt{3}(w - 3)(w^2 - 6w + 6)(w^2 - 6w + 12)\Big], \\ Q(w) =& \Big[6(w^3 - 10w^2 + 25w - 24)w, 12(w - 6)(w - 4)(w - 2)(w^2 - 6w + 6)w\Big], \\ S(w) =& \Big[2(w^2 - 6w - 9)w^2, 0\Big]. \end{aligned}$$

Thus, P and S intersect $\Theta_{-3,2}$, R intersects $\Theta_{-3,0}$, and Q intersects $\Theta_{-3,1}$. Finally,

$$\mathrm{contr}_{-3}(P) = 1, \quad \mathrm{contr}_{-3}(P,R) = 0, \quad \mathrm{contr}_{-3}(R) = 0.$$

For $V = \sqrt{3}$, set $V = \sqrt{3} + w$ and $X = X_1 + 72w$. Then we obtain

$$Y^2Z = X_1^3 + (\alpha_1 w - 36)X_1^2Z + \beta_1 w^2X_1Z^2 + \gamma_1 w^2Z^3$$

and the points become

$$\begin{aligned} P(w) =& \Big[-72w, p_2w\Big], \\ R(w) =& \Big[r_1w^2, r_2w\Big], \\ Q(w) =& \Big[q_1w^2, q_2w\Big], \\ S(w) =& \Big[s_1w, 0\Big]. \end{aligned}$$

We conclude that P, R, Q, and S intersect $\Theta_{\sqrt{3},1}$ and that

$$\mathrm{contr}_{\sqrt{3}}(R) = \frac{1}{2}, \quad \mathrm{contr}_{\sqrt{3}}(P,R) = \frac{1}{2}, \quad \mathrm{contr}_{\sqrt{3}}(R) = \frac{1}{2}.$$

The case $V = -\sqrt{3}$ is analogous to the previous one by Galois conjugation.

If $V=\sqrt{3}i$, we make the change of variables $V=\sqrt{3}i+w$, $X=X_1-72(1+\sqrt{3}i)+48(-3+\sqrt{3}i)w$. We obtain

$$Y^2Z=X_1^3+(\alpha_1 w-144)X_1^2Z+\beta_1 w^2X_1Z^2+\gamma_1 w^2Z^3$$

and the points become

$$\begin{aligned}
P(w)=&\Big[p_1w+72(1+\sqrt{3}i),p_2w+864(-1+\sqrt{3}i)\Big],\\
R(w)=&\Big[r_1w+144,r_2w\Big],\\
Q(w)=&\Big[q_1w+288,q_2w-3456\Big],\\
S(w)=&\Big[s_1w,0\Big].
\end{aligned}$$

Thus, we conclude that P,R,Q intersect $\Theta_{\sqrt{3}i,0}$, while S intersects $\Theta_{\sqrt{3}i,1}$ and

$$\mathrm{contr}_{\sqrt{3}i}(P)=0,\quad \mathrm{contr}_{\sqrt{3}i}(P,R)=0,\quad \mathrm{contr}_{\sqrt{3}i}(R)=0.$$

The case $V=-\sqrt{3}i$ is analogous by Galois conjugation.

Before proceeding to the computation of the heights, we remark that $P(V)$ and $R(V)$ intersect the same component at $[0,0]$ and they also intersect transversally at $[0,1728(45-26\sqrt{3})]$ and therefore $(P\cdot R)=2$. Since the coordinates of P and R are polynomials, they do not intersect O. Consequently, $(P\cdot O)=(R\cdot O)=0$.

By applying formulas (11) and (12), we obtain

$$\begin{aligned}
h(P)=&2\chi+2(P\cdot O)-\sum_\nu \mathrm{contr}_\nu(P)=2\cdot 2-\left(1+1+\frac{1}{2}+\frac{1}{2}\right)=1,\\
\langle P,R\rangle=&\chi+(P\cdot O)+(R\cdot O)-(P\cdot R)-\sum_\nu \mathrm{contr}_\nu(P,R)=2-2-\left(\frac{1}{2}+\frac{1}{2}\right)=-1,\\
h(R)=&2\chi+2(R\cdot O)-\sum_\nu \mathrm{contr}_\nu(R)=2\cdot 2-\left(\frac{1}{2}+\frac{1}{2}\right)=3.
\end{aligned}$$

Thus, the Gram determinant of P and R equals

$$\begin{vmatrix} h(P) & \langle P,R\rangle \\ \langle R,P\rangle & h(R)\end{vmatrix}=2.$$

In particular, P and R are independent non-torsion elements. From equation (19), we deduce that

$$|\det \mathrm{NS}(E)| \text{ divides } 2^7.$$

Therefore, the lattice generated by P and R has index a power of 2 in $\mathrm{MWL}(E)$. If this index is greater than 1, then $\mathrm{MWL}(E)$ must contain a point T such that $2T$ is one of the following points: $P,R,P+R,P+Q,R+Q,P+R+Q,P+S,R+S,P+R+S,P+Q+S,R+Q+S$, or $P+R+Q+S$. Now consider the intersections with $V=\sqrt{3}$. The following sections intersect $\Theta_{\sqrt{3},1}$ and therefore can not be equal to $2T$: $P,R,P+Q+S,R+Q+S,P+R+Q$, and $P+R+S$. By taking $V=\sqrt{3}i$ we further eliminate $P+S$, $R+S$, and $P+R+Q+S$. By considering $V=3$ we eliminate $P+Q$ and $R+Q$. We can not eliminate $P+R$ with this method. It can be verified with a computer that there is no T such that $P+R=2T$. Indeed, the X coordinate of $P+R$ is given by

$$\frac{6(V+3)^2(V-1)^2(V^2+(3+2\sqrt{3})^2)}{(V+3+2\sqrt{3})^2}.$$

We can find directly with a computer a general formula for the X coordinate of $2T$ when $T=[x,y]$ and verify that the resulting rational function can not equal the X coordinate of $P+R$ for any value of x.

We conclude that P and R are generators in this case.

□

5. Proof of Theorem 1 and further discussion

Proof of Theorem 1. Propositions 4 and 7 can be combined with Silverman's Specialization Theorem and Theorem 3 to conclude that for all but finitely many rational values of $V \geq 0$ there are infinitely many values of $U \geq 0$ leading to Ceva points in the cevian determined by V.

The application of Theorem 3 in the case of Proposition 8 is less straightforward. In previous cases where the torsion was cyclic, this was immediate, but in this case, the torsion is non-cyclic. This means that $E(\mathbb{R})$ can be written as the union of two components, one that is infinite, in the sense that it contains O, and the other finite, not containing O (here we write E in place of $E(V)$ for simplicity of notation). In principle we could have a component of $E(\mathbb{R})$ that does not contain a point of $E(\mathbb{Q})$ of infinite order. We notice that the points of order two are

$$\begin{aligned} 2Q(V) =& \Big[3(V-1)^2(V+3)^2, 0\Big], \\ S(V) =& \Big[2(V^2-3)(V+3)^2, 0\Big], \\ 2Q(V) + S(V) =& \Big[-6(V-1)^2(V^2-3), 0\Big]. \end{aligned}$$

We recall that condition (4) implies that we must choose $V \geq 0$. Assume that $V \neq 1, 3$ (the statement allows us to exclude a finite number of cases).

We have that $3(V-1)^2(V+3)^2 > 2(V^2-3)(V+3)^2$ and $3(V-1)^2(V+3)^2 > -6(V-1)^2(V^2-3)$. Therefore, the finite component of $E(\mathbb{R})$ passes through $2Q+S$ and S, while the infinite component passes through $2Q$. Now,

$$2P(V) = \Big[3(V^4+4V^3+10V^2-12V+9), -18V(V^2+1)(V^2+9)\Big]$$

and since $3(V^4+4V^3+10V^2-12V+9) > 3(V^4+4V^3-2V^2-12V+9) = 3(V-1)^2(V+3)^2$, we immediately conclude that $2P$ belongs to the infinite component of $E(\mathbb{R})$. Thus we can apply Theorem 3 in order to conclude the validity of Theorem 1 for the equilateral triangle. $\square$

As a final note, we remark that this method opens the door to study other families of triangles. For example, one could consider close-to-equilateral triangles, namely, those whose sides do not differ by more than one unit. If the lengths of the sides are integral, then the triangle is necessarily isosceles. We have studied the case of the $(5,5,6)$ triangle, and it is very similar to the almost equilateral cases.

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