Mahler measure and regulators

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Mahler measure of one-variable polynomials

Pierce (1918) $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933)

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha|>1 \\ 1 & \text{if } |\alpha|<1 \end{cases}$$

For

$$P(x) = a\prod_i (x-\alpha_i)$$

$$M(P) = |a|\prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+ |\alpha_i|$$

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Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^n \prod \phi_i(x)$$

Lehmer's Question

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$$

$$=0.162357612\ldots$$

Lehmer(1933) Does there exist $C>0$ such that $P(x)\in\mathbb{Z}[x]$

$$m(P)=0 \quad \text{or} \quad m(P)>C??$$

$$\sqrt{\Delta_{379}}=1,794,327,140,357$$

Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi\mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi\mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

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Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

where h is the logarithmic Weil height.

Boyd & Lawton Theorem

$P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

More examples in several variables

- Condon (2003)

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2007)

$$\pi^2 m\left(\mathrm{Res}_t(x + yt + t^2, z + wt + t^2)\right)$$

$$= \pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = 4\sqrt{5}\,\mathrm{L}(\chi_5, 3)$$

- Boyd & L. (2005)

$$\pi^2 m(x^2 + 1 + (x+1)y + (x-1)z) = \pi \mathrm{L}(\chi_{-4}, 2) + \frac{21}{8}\zeta(3)$$

- L. (2006)

$$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 24\mathrm{L}(\chi_{-4},4)$$

- $$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

- Known formulas for

$$\pi^{n+2} m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

Why do we get nice numbers?

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$\mathcal{L}_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

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$\mathcal{L}_k$ satisfies lots of functional equations

$$\mathcal{L}_k\left(\frac{1}{x}\right) = (-1)^{k-1}\mathcal{L}_k(x) \qquad \mathcal{L}_k(\bar{x}) = (-1)^{k-1}\mathcal{L}_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \,\mathrm{rank}\, 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997): General framework

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x,y) = \log|x|\,\mathrm{d}\arg y - \log|y|\,\mathrm{d}\arg x$$

$$\eta(x,1-x) = \mathrm{d}D(x) \qquad \mathrm{d}\eta(x,y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}\right)$$

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The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi i)^3}\int_{\mathbb{T}^3} \log\left|z - \frac{1-x}{1-y}\right| \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}$$

$$= \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \log^+\left|\frac{1-x}{1-y}\right| \frac{dx}{x}\frac{dy}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \log|z| \frac{dx}{x}\frac{dy}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \eta(x,y,z)$$

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$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y|\wedge\mathrm{d}\log|z| - \mathrm{d}\arg y\wedge\mathrm{d}\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z|\wedge\mathrm{d}\log|x| - \mathrm{d}\arg z\wedge\mathrm{d}\arg x\right)$$

$$+\log|z|\left(\frac{1}{3}\mathrm{d}\log|x|\wedge\mathrm{d}\log|y| - \mathrm{d}\arg x\wedge\mathrm{d}\arg y\right)$$

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}\wedge\frac{\mathrm{d}z}{z}\right)$$

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x, y)$$

where

$$\omega(x, y) = -D(x)\mathrm{d} \arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$z = \frac{1-x}{1-y}$$

$$\eta(x, y, z) = -\eta(x, 1-x, y) - \eta(y, 1-y, x)$$

$$m(P) = \frac{1}{(2\pi)^2}\int_\Gamma \eta(x, 1-x, y) + \eta(y, 1-y, x) = \frac{1}{(2\pi)^2}\int_{\partial\Gamma} \omega(x, y) + \omega(y, x)$$

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$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

Maillot: if $P \in \mathbb{R}[x,y,z]$,

$$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

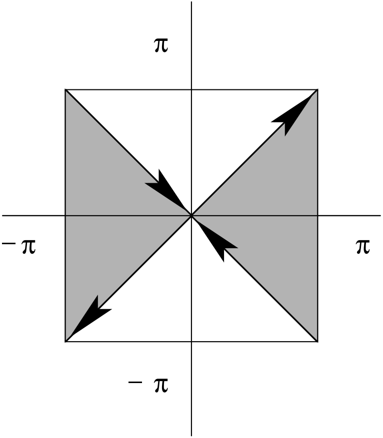
ω defined in

$$C = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)-(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$= \frac{1}{4\pi^2} 8(\mathcal{L}_3(1)-\mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

The three-variable case

Theorem

L. (2005)
$P(x,y,z) \in \mathbb{Q}[x,y,z]$ *irreducible, nonreciprocal,*

$$X = \{P(x,y,z)=0\}, \qquad C = \{\mathrm{Res}_z(P(x,y,z), P(x^{-1},y^{-1},z^{-1}))=0\}$$

$$x \wedge y \wedge z = \sum_i r_i x_i \wedge (1-x_i) \wedge y_i \qquad in \qquad \bigwedge^3 (\mathbb{C}(X)^*) \otimes \mathbb{Q},$$

$$\{x_i\}_2 \otimes y_i = \sum_j r_{i,j} \{x_{i,j}\}_2 \otimes x_{i,j} \qquad in \qquad (\mathcal{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$$

Then

$$4\pi^2 (m(P^*) - m(P)) = \mathcal{L}_3(\xi) \qquad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}$$

F field. Bloch group:

$$\mathcal{B}_2(F) := \mathbb{Z}[\mathbb{P}^1_F] / \langle \{0\}, \{\infty\}, R_2(x, y) \rangle$$

$$R_2(x, y) = \{x\}_2 + \{y\}_2 + \{1 - xy\}_2 + \left\{ \frac{1 - x}{1 - xy} \right\}_2 + \left\{ \frac{1 - y}{1 - xy} \right\}_2$$

is the five-term relation for D.

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$$\{x,y,z\} = 0 \quad \textit{in} \quad K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$$

$$\{x_i\}_2 \otimes y_i \quad \textit{trivial in} \quad gr_3^{\gamma} K_4(\mathbb{C}(C)) \otimes \mathbb{Q}\,(?)$$

Then

$$4\pi^2(m(P^*) - m(P)) = \mathcal{L}_3(\xi) \qquad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}$$

- Explains all the known cases involving $\zeta(3)$ (by Borel's Theorem).
- It is constructive (no need of "happy idea" integrals).
- Integration sets hard to describe.
- Conjecture for n-variables using Goncharov's regulator currents. Provides motivation for Goncharov's construction.

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The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd (1998)

$$m(k) \stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

E_k determined by $x + \frac{1}{x} + y + \frac{1}{y} + k = 0$.

Rogers & L (2007)
For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

Kurokawa & Ochiai (2005)
For $h \in \mathbb{R}^*$,

$$m(4h^2)+m\left(\frac{4}{h^2}\right)=2m\left(2\left(h+\frac{1}{h}\right)\right).$$

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$h = \frac{1}{\sqrt{2}}$ in both equations, and some K-theory,

Corollary

$$m(8) = 4m(2) = \frac{8}{5} m\left(3\sqrt{2}\right)$$

Rodriguez-Villegas (1997)

$k = 3\sqrt{2}$ (modular curve $X_0(24)$)

$$m\left(3\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2}\right) = q\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

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$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

For $|k| > 4$, $x + \frac{1}{x} + y + \frac{1}{y} + k$ does not intersect $\mathbb{T}^2$.

$$m(k) = -\frac{1}{2\pi \mathrm{i}} \int_\gamma \eta(x, y)$$

where

$$\gamma = X \cap \{|x| = 1\}$$

$$\eta(x, y) = \log |x| \mathrm{d}\mathrm{i} \arg y - \log |y| \mathrm{d}\mathrm{i} \arg x$$

We are evaluating the regulator in $\{x, y\} \in K_2(E)_{\mathbb{Q}}$.

Computing the regulator

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$$

z mod $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$.
Bloch regulator function

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right) = \frac{y_\tau^2}{\pi} \sum_{m,n\in\mathbb{Z}}{}' \frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}$$

y_τ is the imaginary part of τ.

Theorem

after results of Beilinson, Bloch, idea of Deninger

$E/\mathbb{R}$ elliptic curve, x, y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, $\omega \in \Omega^1$

$$-\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x)\diamond(y)\right)\right)$$

where Ω_0 is the real period and $\Omega = \int_\gamma \omega$.

In our case,

$$\mathbb{Z}[E(\mathbb{C})]^- \ni (x) \diamond (y) = 8(P), \qquad P \quad \text{4-torsion.}$$

Isogenies $\rightsquigarrow$ Functional eq for the regulator.

Functional eq for the regulator $\rightsquigarrow$ Functional eq for the Mahler measure

Big picture

$$\cdots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(X, \partial\gamma) \to K_2(X) \to \dots$$

$$\partial\gamma = X \cap \mathbb{T}^2$$

- $\eta(x,y)$ is exact, then $\{x,y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes's Theorem.
  $\rightsquigarrow D$, $1 + x + y$

- $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(C)$. We have $\eta(x,y)$ is not exact.
  $\rightsquigarrow L$-function, $1 + x + \frac{1}{x} + y + \frac{1}{y}$

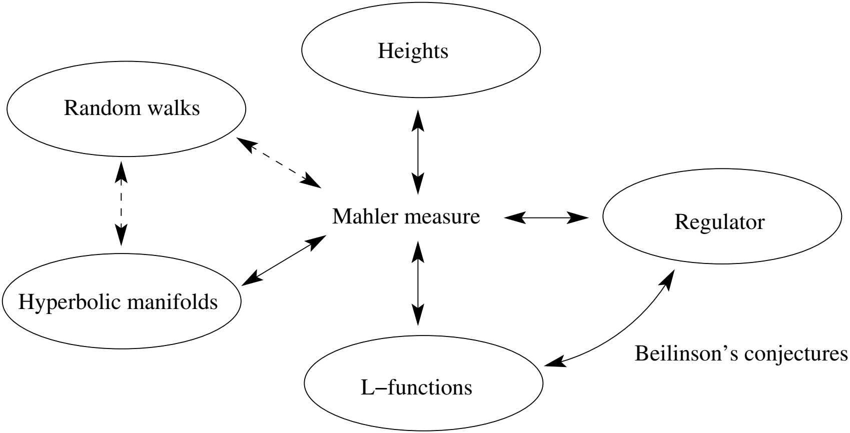
Big picture in three variables

$$\cdots \to K_4(\partial\Gamma) \to K_3(X, \partial\Gamma) \to K_3(X) \to \dots$$

$$\partial\Gamma = X \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$


Heights
Random walks
Mahler measure
Regulator
Hyperbolic manifolds
L–functions
Beilinson's conjectures