On Mahler measures of several-variable polynomials and polylogarithms

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1. Mahler measure

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

$$m(P) = \log |a_d| + \sum_{n=1}^{d} \log^+ |\alpha_n|$$

for

$$P(x) = a_d \prod_{n=1}^{d} (x - \alpha_n) \in \mathbb{C}[x]$$

because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

2. Properties

- $m(P \cdot Q) = m(P) + m(Q)$

- $m(P) \geq 0$ if P has integral coefficients.

- Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

- Boyd – Lawton : $P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots x^{k_n}))$$

$$= m(P(x_1, \dots x_n))$$

- Lehmer (1933)

  $m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$

  $= \log(1.176280818\ldots) = 0.162357612\ldots$

  $\forall\epsilon > 0, \exists P(x) \in \mathbb{Z}[x] \quad \text{with} \quad 0 < m(P) < \epsilon??$

  Is the polynomial above the best possible?

Jensen's formula ⟶ simple expression in one-variable case.

Several-variable case?

3. Examples in several variables

Smyth (1981)

$$\begin{aligned} m(1+x+y) &= \frac{3\sqrt{3}}{4\pi}\mathsf{L}(\chi_{-3},2) = \mathsf{L}'(\chi_{-3},-1) \\ m(1+x+y+z) &= \frac{7}{2\pi^2}\zeta(3) \end{aligned}$$

Vandervelde (2002)

$$m(1+x+y+\mathsf{i}xy) = \frac{\sqrt{2}}{\pi}\mathsf{L}(\chi_{-8},2) = \frac{1}{4}\mathsf{L}'(\chi_{-8},-1)$$

Condon (2003)

$$m(1+x+(1-x)(y+z)) = \frac{28}{5\pi^2}\zeta(3)$$

4. Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

$$\mathrm{Li}_k(x) = -\int_{0 \le s_1 \le \ldots s_k \le 1} \frac{\mathrm{d}s_1}{s_1 - \frac{1}{x}} \frac{\mathrm{d}s_2}{s_2} \cdots \frac{\mathrm{d}s_k}{s_k}$$

An analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$P_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number, $\mathrm{Li}_0(x) \equiv -\frac{1}{2}$,

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

P_k satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1} P_k(x) \qquad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

Bloch–Wigner dilogarithm ($k=2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0$$

5. More examples in several variables

Theorem 1 *(2003)*

$$\pi^n m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)z\right)$$

$=$ *combination of* ζ(*odd*) / L(χ_{-4}, *even*)

$$\pi^n m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

$=$ *combination of* ζ(*odd*) / L(χ_{-4}, *even*), *polylogarithms*

Examples

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$=24\mathsf{L}(\chi_{-4},4)+6\zeta(2)\mathsf{L}(\chi_{-4},2)$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)$$

$$=62\zeta(5)+28\zeta(2)\zeta(3)$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

Proof of Theorem (Idea).

- $P_\alpha \in \mathbb{C}[x_1, \ldots, x_n]$, coefficients depend polynomially on a parameter $\alpha \in \mathbb{C}$.

  For example, $P_\alpha(x) = 1 + \alpha x$.

  $m(P_\alpha) = \log^+ |\alpha|$.

- $\alpha \to \alpha\frac{1-y}{1+y}$, get $\tilde{P}_\alpha \in \mathbb{C}[x_1, \ldots, x_n, y]$

  In the example,
  $\tilde{P}_\alpha(x, y) = 1 + y + \alpha(1 - y)x$.

- $$m(\tilde{P}_\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} m\left(P_{\alpha\frac{1-y}{1+y}}\right) \frac{\mathrm{d}y}{y}$$

6. Examples from the world of resultants

(joint with D'Andrea 2003).

- $m(\mathrm{Res}_{\{0,m,n\}})$

  $$= m(\mathrm{Res}_t(x + yt^m + t^n, z + wt^m + t^n)) =$$

  $$\frac{2}{\pi^2}(-mP_3(\varphi^n) - nP_3(-\varphi^m) + mP_3(\phi^n) + nP_3(\phi^m))$$

  $$0 \le \varphi \le 1 \quad \text{root of} \quad x^n + x^{n-m} - 1 = 0$$

  $$1 \le \phi \quad \text{root of} \quad x^n - x^{n-m} - 1 = 0$$

- $m(\mathrm{Res}_{\{(0,0),(1,0),(0,1)\}}) = m\left(\begin{vmatrix} x & y & z \\ u & v & w \\ r & s & t \end{vmatrix}\right)$

  $$= m((1-x)(1-y) - (1-z)(1-w)) = \frac{9\zeta(3)}{2\pi^2}$$

7. An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$\eta(x,y) = \log|x| \,\mathrm{d}\arg y - \log|y| \,\mathrm{d}\arg x$$

$$\mathrm{d}\arg x = \mathrm{Im}\left(\frac{\mathrm{d}x}{x}\right)$$

$$\mathrm{d}\eta(x,y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$$

and

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$\gamma = \{P(x,y) = 0\} \cap \{|x| = 1, |y| \geq 1\}$

Smyth's case:

$$P(x,y) = y + x - 1$$

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log|y+x-1|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

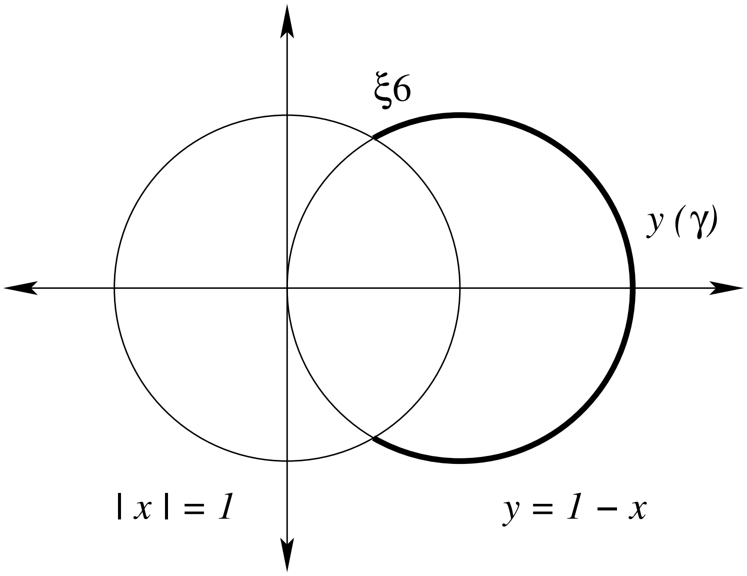
$$= \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{T}^1} \log^+|1-x|\frac{\mathrm{d}x}{x}$$

$$= -\frac{1}{2\pi}\int_{\gamma} \eta(x, 1-x)$$

write $x = \mathrm{e}^{2\pi \mathrm{i}\theta}$,

$$y(\gamma(\theta)) = 1 - \mathrm{e}^{2\pi \mathrm{i}\theta}, \quad \theta \in [1/6\,;5/6]$$

$$x(\partial\gamma) = [\xi_6] - [\bar{\xi}_6]$$



$\eta(x, y)$ is exact:

Theorem 2

$$\eta(x, 1 - x) = \mathsf{d}D(x)$$

$$2\pi m(x + y + 1) = D(\xi_6) - D(\bar{\xi_6})$$

$$= 2D(\xi_6) = \frac{3\sqrt{3}}{2}\mathsf{L}(\chi_{-3}, 2)$$

In general,

- $\eta(x,y) = -\eta(y,x)$
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$
- $\eta(x, 1-x) = 0$ in $H^1(C, \mathbb{R})$

$\Rightarrow \eta$ is a symbol, can be factored through $K_2(\mathbb{C}(C))$ (by Matsumoto)

Need $\{x, y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$.

$$x \wedge y = \sum_j r_j \;\; z_j \wedge (1 - z_j)$$

in $\bigwedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$, then

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}$$

Big picture

$$\ldots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C, \partial\gamma) \to K_2(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x,y)$ is exact, then $\{x,y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

  $\rightsquigarrow$ dilogarithms, zeta function

- $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(C)$. We have $\eta(x,y)$ is not exact.

  $\rightsquigarrow$ L-series of a curve

We may get combinations of both situations.

9. The three-variable case

$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y|\,\mathrm{d}\log|z| - \mathrm{d}\arg y\,\mathrm{d}\arg z\right)$$
$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z|\,\mathrm{d}\log|x| - \mathrm{d}\arg z\,\mathrm{d}\arg x\right)$$
$$+\log|z|\left(\frac{1}{3}\mathrm{d}\log|x|\,\mathrm{d}\log|y| - \mathrm{d}\arg x\,\mathrm{d}\arg y\right)$$

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}\wedge\frac{\mathrm{d}z}{z}\right)$$

and

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z)$$

$$\Gamma = \{P(x,y,z)=0\}\cap\{|x|=|y|=1, |z|\geq 1\}$$

Smyth's case:

$$P(x,y,z) = (1-x) + (1-y)z$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log^+ \left|\frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \eta(x,y,z)$$

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

In general,

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x, y)$$

where

$$\omega(x, y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

Need

$$x \wedge y \wedge z = \sum r_i \;\; x_i \wedge (1 - x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$, then

$$\int_\Gamma \eta(x, y, z) = \sum r_i \int_\Gamma \eta(x_i, 1 - x_i, y_i)$$

$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

in Smyth's case:

$$x \wedge y \wedge z = - \;\; x \wedge (1 - x) \wedge y - \;\; y \wedge (1 - y) \wedge x$$

Maillot: if $P \in \mathbb{Q}[x, y, z]$,

$\partial\Gamma = \gamma = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$

ω defined in

$$C = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\omega(x, x) = \mathrm{d}P_3(x)$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$, then

$$\int_\gamma \omega(x, y) = \sum r_i \, P_3(x_i)|_{\partial\gamma}$$

in Smyth's case

$$C = \{x = y\} \cup \{xy = 1\}$$

$$-[x]_2 \otimes y - [y]_2 \otimes x = \pm 2[x]_2 \otimes x$$

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

$$= \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

10. A little bit of K-theory

F field, define subgroups $R_i(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ as

$$\begin{aligned} R_1(F) &:= [x]+[y]-[xy] \\ R_2(F) &:= [x]+[y]+[1-xy]+\left[\frac{1-x}{1-xy}\right]+\left[\frac{1-y}{1-xy}\right] \\ R_3(F) &:= \text{functional equation of the trilogarithm} \end{aligned}$$

$$B_i(F) := \mathbb{Z}[\mathbb{P}^1_F]/R_i(F)$$

$$\begin{aligned} B_F(3) &: B_3(F) \xrightarrow{\delta_1^3} B_2(F)\otimes F^* \xrightarrow{\delta_2^3} \wedge^3 F^* \\ B_F(2) &: B_2(F) \xrightarrow{\delta_1^2} \wedge^2 F^* \\ B_F(1) &: F^* \end{aligned}$$

($B_i(F)$ is placed in degree 1).

$\delta_1^3([x]_3) = [x]_2 \otimes x \qquad \delta_2^3([x]_2\otimes y) = x\wedge(1-x)\wedge y$

$$\delta_1^2([x]_2) = x\wedge(1-x)$$

Goncharov conjectures:

$$H^i(B_F(3) \otimes \mathbb{Q}) \overset{?}{\cong} K_{6-i}^{[3-i]}(F)_{\mathbb{Q}}$$

Our first condition is $x \wedge y \wedge z$ is 0 in

$$H^3(B_{\mathbb{Q}(S)}(3) \otimes \mathbb{Q}) \cong K_3^M(\mathbb{Q}(S)) \otimes \mathbb{Q}$$

Our second condition is $[x_i]_2 \otimes y_i$ is 0 in

$$H^2(B_{\mathbb{Q}(S)}(3) \otimes \mathbb{Q}) \overset{?}{\cong} K_4^{[1]}(\mathbb{Q}(C))_{\mathbb{Q}}$$

Big picture II

$$\ldots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \ldots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\ldots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.