On Mahler measures of several-variable polynomials and polylogarithms

Zeta Functions Seminar – University of California at Berkeley

May 17th, 2004

Matilde N. Lalín – University of Texas at Austin [1]

1. Mahler measure

Definition 1 *For* $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, *the (logarithmic)* Mahler measure *is defined by*

$$m(P) \quad = \quad \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{1}$$

This integral is not singular and $m(P)$ always exists.
Because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha| \tag{2}$$

[2]we have a simple expression for the Mahler measure of one-variable polynomials:

$$m(P) = \log |a_d| + \sum_{n=1}^{d} \log^+ |\alpha_n| \quad \text{for} \quad P(x) = a_d \prod_{n=1}^{d} (x - \alpha_n)$$

2. Properties

Here are some general properties (see [8])

- For $P, Q \in \mathbb{C}[x_1, \dots, x_n]$, we have $m(P \cdot Q) = m(P) + m(Q)$. Because of that, it makes sense to talk about the Mahler measure of rational functions.
- Let $P \in \mathbb{C}[x_1, \dots, x_n]$ such that $a_{m_1,\dots,m_n}$ is the coefficient of $x_1^{m_1} \dots x_n^{m_n}$ and P has degree d_i in x_i. Then

$$|\, a_{m_1,\dots,m_n}| \leq \binom{d_1}{m_1} \dots \binom{d_n}{m_n} M(P) \tag{3}$$

$$M(P) \leq L(P) \leq 2^{d_1 + \dots + d_n} M(P) \tag{4}$$

  where $L(P)$ is the length of the polynomial, the sum of the absolute values of the coefficients, and

$$M(P) := \mathrm{e}^{m(P)}$$

- $m(P) \geq 0$ if P has integral coefficients.

[1] mlalin@math.utexas.edu – http://www.ma.utexas.edu/users/mlalin

[2] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

- By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $m(P) = 0$ if and only if P is the product of powers of x and cyclotomic polynomials.
- Lehmer [13] studied this example in 1933:

  $$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = \log(1.176280818\ldots) = 0.162357612\ldots$$

  This example is special because its Mahler measure is very small, but still greater than zero.

  The following questions are still open: Is there a lower bound for positive Mahler measure of polynomials in one variable with integral coefficients? Does this degree 10 polynomial reach the lowest bound?
- For $P \in \mathbb{C}[x_1, \ldots, x_n]$

  $$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots x^{k_n})) = m(P(x_1, \ldots x_n)) \tag{5}$$

  (this result is due to Boyd and Lawton see [1], [12]).

  In particular Lehmer's problem in several variables reduces to the one variable case.

We have seen a simple expression for the Mahler measure in the one-variable case. A natural question is: What happens with several variables?

3. Examples of Mahler measures in several variables

For two and three variables, several examples are known. The first and simplest examples in two and three variables were given by Smyth [16] and also [1]:

$$m(1 + x + y) = \frac{1}{\pi} D(\zeta_6) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1) \tag{6}$$

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \tag{7}$$

Vandervelde [17] in 2002, studied the example of $axy + bx + cy + d$. He developed a general formula for this case. Some particular cases are:

$$m(1 + x + y + \mathrm{i}xy) = \frac{\sqrt{2}}{\pi} \mathrm{L}(\chi_{-8}, 2) = \frac{1}{4} \mathrm{L}'(\chi_{-8}, -1) \tag{8}$$

$$m(1 + x + y + \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} xy) = \frac{4\sqrt{2}}{15\pi} \mathrm{L}(\chi_{-8}, 2) = \frac{1}{15} \mathrm{L}'(\chi_{-8}, -1) \tag{9}$$

Condon, [5] in 2003,

$$m(1 + x + (1 - x)(y + z)) = \frac{28}{5\pi^2} \zeta(3) \tag{10}$$

4. Polylogarithms

The examples mentioned above have been computed by elementary integrals involving polylogarithms.

Definition 2 *The* kth polylogarithm *is the function defined by the power series*

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1 \tag{11}$$

This function can be continued analytically to $\mathbb{C} \setminus [1, \infty)$ via the integral

$$-\int_{0 \le s_1 \le \ldots s_k \le 1} \frac{\mathrm{d}s_1}{s_1 - \frac{1}{x}} \frac{\mathrm{d}s_2}{s_2} \ldots \frac{\mathrm{d}s_k}{s_k}$$

In order to avoid discontinuities, and to extend polylogarithms to the whole complex plane, several modifications have been proposed. Zagier [18] considers the following version:

$$P_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k} \frac{2^j B_j}{j!} (\log|x|)^j \mathrm{Li}_{k-j}(x) \right) \tag{12}$$

where B_j is the jth Bernoulli number, $\mathrm{Li}_0(x) \equiv -\frac{1}{2}$ and Re_k denotes Re or Im depending on whether k is odd or even.

This function is one-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and continuous in $\mathbb{P}^1(\mathbb{C})$. Moreover, P_k satisfy very clean functional equations. The simplest ones are

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1} P_k(x) \qquad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

there are also lots of functional equations which depend on the index k. For instance, for $k = 2$, we have the Bloch–Wigner dilogarithm,

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1 - x) \log|x|$$

which satisfies the well-known five-term relation

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1 - y}{1 - xy}\right) + D\left(\frac{1 - x}{1 - xy}\right) = 0 \tag{13}$$

5. More examples of Mahler measures in several variables

A generalization of Smyth's first result was due to Cassaigne and Maillot [14]: for $a, b, c \in \mathbb{C}^*$,

$$\pi m(a + bx + cy) = \begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases} \tag{14}$$

where $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively. The term with the dilogarithm can be interpreted as the volume of the ideal hyperbolic tetrahedron which has the triangle as basis and the fourth vertex is infinity. See figure 1.

We would also like to add that Boyd [2] has computed numerically several examples involving L-series of elliptic curves, some of them were proved by Rodriguez-Villegas [15]. For instance

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) \stackrel{?}{=} \mathrm{L}'(E, 0) \tag{15}$$

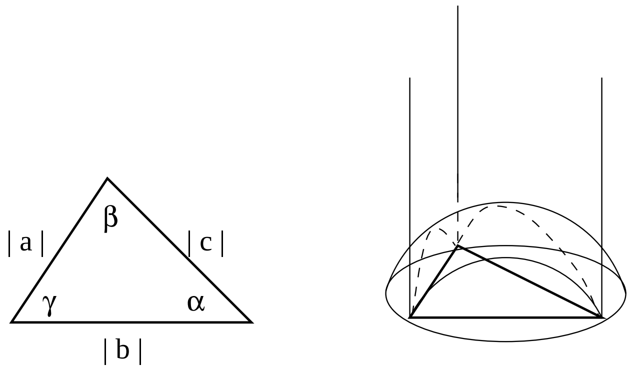


Figure 1: The main term in Cassaigne – Maillot formula is the volume of the ideal hyperbolic tetrahedron over the triangle.

where E is the elliptic curve of conductor 15 which is the projective closure of the curve $x + \frac{1}{x} + y + \frac{1}{y} + 1 = 0$, and $\mathrm{L}(E, s)$ is the L-function of E.

However, for more than three variables, very little is known.

Theorem 3 *For $n \geq 1$ we have:*

$$\pi^{2n} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right)$$

$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} \pi^{2n-2h} (2h)! \frac{2^{2h+1}-1}{2} \zeta(2h+1) \tag{16}$$

For $n \geq 0$:

$$\pi^{2n+1} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right) z\right)$$

$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!} 2^{2h+1} \pi^{2n-2h} (2h+1)! \mathrm{L}(\chi_{-4}, 2h+2) \tag{17}$$

For $n \geq 1$:

$$\pi^{2n+2} m\left(1 + x + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_{2n}}{1+x_{2n}}\right)(1+y)z\right) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h+2} \pi^{2n-2h}$$

$$\cdot (2h-1)! \sum_{k=0}^{h-1} \binom{2h-2k+2}{2} \frac{2^{2h-2k+3}-1}{2^{2h}} \frac{(-1)^k B_{2k} (2\pi)^{2k}}{2(2k)!} \zeta(2h-2k+3) \tag{18}$$

For $n \geq 0$:

$$\pi^{2n+3} m\left(1 + x + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_{2n+1}}{1+x_{2n+1}}\right)(1+y)z\right)$$

$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!} 2^{2h+3} \pi^{2n-2h} \left(\mathrm{i}(2h)! \mathcal{L}_{3,2h+1}(\mathrm{i}, \mathrm{i}) + (2h+1)! \pi^2 \mathrm{L}(\chi_{-4}, 2h+2)\right) \tag{19}$$

Where $\mathcal{L}_{r,s}(\alpha,\alpha)$ are certain linear combinations of polylogarithms and

$$s_l(a_1,\dots,a_k)=\begin{cases}1 & \text{if} \quad l=0\\ \sum_{i_1<\dots<i_l} a_{i_1}\dots a_{i_l} & \text{if} \quad 0<l\le k\\ 0 & \text{if} \quad k<l\end{cases} \tag{20}$$

are the elementary symmetric polynomials, i. e.,

$$\prod_{i=1}^{k}(x+a_i)=\sum_{l=0}^{k} s_l(a_1,\dots,a_k)x^{k-l} \tag{21}$$

For example,

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right) = 24\mathrm{L}(\chi_{-4},4)+6\zeta(2)\mathrm{L}(\chi_{-4},2) \tag{22}$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right) = 62\zeta(5)+28\zeta(2)\zeta(3) \tag{23}$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = 93\zeta(5) \tag{24}$$

$$\tag{25}$$

The idea behind the prove of Theorem 3 is the following. Let $P_\alpha\in\mathbb{C}[x_1,\dots,x_n]$ whose coefficients depend polynomially on a parameter $\alpha\in\mathbb{C}$. We replace α by $\alpha\frac{1-y}{1+y}$ and obtain a polynomial $\tilde{P}_\alpha\in\mathbb{C}[x_1,\dots,x_n,y]$ (multiplying by $1+y$). The Mahler measure of the second polynomial is a certain integral of the Mahler measure of the first polynomial.

$$m(\tilde{P}_\alpha)=\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}^1} m\left(P_{\alpha\frac{1-y}{1+y}}\right)\frac{\mathrm{d}y}{y}$$

Now it is easy to see that if the Mahler measure of the original polynomial depends on polylogarithms, so does the Mahler measure of the new polynomial.

6. Examples coming from the world of resultants

Let us mention some examples of Mahler measure of resultants (this will be part of a joint work with D'Andrea [6]).

Theorem 4

$$m(\mathrm{Res}_{\{0,m,n\}})=m(\mathrm{Res}_t(x+yt^m+t^n,z+wt^m+t^n))$$

$$=\frac{2}{\pi^2}\left(-mP_3(\varphi^n)-nP_3(-\varphi^m)+mP_3\left(\phi^n\right)+nP_3\left(\phi^m\right)\right)$$

where φ is the real root of $x^n+x^{n-m}-1=0$ such that $0\le\varphi\le 1$, and ϕ is the real root of $x^n-x^{n-m}-1=0$ such that $1\le\phi$.

Theorem 5

$$m(\mathrm{Res}_{\{(0,0),(1,0),(0,1)\}})=m\left(\begin{vmatrix} x & y & z\\ u & v & w\\ r & s & t\end{vmatrix}\right)$$

$$=m((1-x)(1-y)-(1-z)(1-w))=\frac{9\zeta(3)}{2\pi^2}$$

7. An algebraic integration for Mahler measure

Here we will follow Deninger [7]. Given a variety X over $K = \mathbb{R}$ or $\mathbb{C}$ there is a transformation

$$r_{\mathcal{D}} : H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \longrightarrow H^i_{\mathcal{D}}(X/K, \mathbb{R}(j))$$

called *Beilinson regulator*.

Here

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) = Gr^j_{\gamma} K_{2j-i}(X) \otimes \mathbb{Q}$$

for X a regular, quasi-projective variety.

There is a natural pairing,

$$\langle , \rangle : H^n(X/K, \mathbb{R}(n)) \times H_n(X/K, \mathbb{R}(-n)) \longrightarrow \mathbb{R}$$

Observe:

$$H^i_{\mathcal{D}}(X, \mathbb{R}(i)) = \{\varphi \in \mathcal{A}^{i-1}(X, \mathbb{R}(i-1)) \,|\, \mathrm{d}\varphi = \pi_{i-1}(\omega), \omega \in F^i(X)\} \,/\, \mathrm{d}\mathcal{A}^{i-2}(X, \mathbb{R}(i-1))$$

Here $\mathcal{A}^i(X, \mathbb{R}(j))$ denotes the space of smooth i-forms with values in $(2\pi \mathrm{i})^j\mathbb{R}$, and $F^i(X)$ denotes the space of holomorphic i-forms on X with at most logarithmic singularities at infinity. $\pi_n : \mathbb{C} \to \mathbb{R}(n)$ is the projection $\pi_n(z) = \frac{z+(-1)^n\bar{z}}{2}$.

For $P \in \mathbb{Z}[x_1, \dots, x_n]$, let $Z(P) = \{P = 0\} \cap \mathbb{G}^n_{m,K}$. Denote $X_P = \mathbb{G}^n_{m,K} \setminus Z(P)$. Since $F^{n+1}(X_P) = 0$,

$$H^{n+1}_{\mathcal{D}}(X_P/K, \mathbb{R}(n+1)) = H^n(X_P/K, \mathbb{R}(n))$$

Deninger observes the following:

$$m(P) = \left\langle r_{\mathcal{D}}\{P, x_1, \dots, x_n\}, [\mathbb{T}^n] \otimes (2\pi \mathrm{i})^{-n} \right\rangle$$

Under certain assumptions, and by means of Jensen's formula,

$$m(P^*) - m(P) = \left\langle r_{\mathcal{D}}\{x_1, \dots, x_n\}, [A] \otimes (2\pi \mathrm{i})^{1-n} \right\rangle$$

where $\{x_1, \dots, x_n\} \in H^n_{\mathcal{M}}(Z^{reg}, \mathbb{Q}(n))$ and $[A] \in H_{n-1}(Z^{reg}, \mathbb{Z}))$, where A is the union of connected components of dimension $n-1$ in $\{P = 0\} \cap \{|x_1| = \ldots = |x_{n-1}| = 1, \, |x_n| \geq 1\}$

8. The two-variable case

Rodriguez-Villegas [15] has worked out the details for two variables. This was further developed by Boyd and Rodriguez-Villegas [3], [4].

Given a smooth projective curve C and x, y rational functions ($x, y \in \mathbb{C}(C)^*$), define

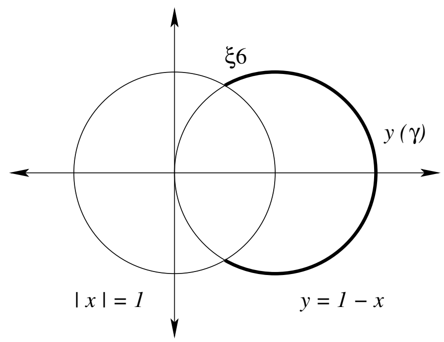
$$\eta(x, y) = \log |x| \mathrm{d} \arg y - \log |y| \mathrm{d} \arg x \tag{26}$$

Here

$$\mathrm{d} \arg x = \mathrm{Im} \left(\frac{\mathrm{d}x}{x} \right) \tag{27}$$

is well defined in $\mathbb{C}$ in spite of the fact that arg is not. η is a 1-form in $C \setminus S$, where S is the set of zeros and poles of x and y. It is also closed, because of

$$\mathrm{d}\eta(x, y) = \mathrm{Im} \left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} \right) = 0$$



Let $P \in \mathbb{C}[x, y]$. Write

$$P(x, y) = a_d(x)y^d + \ldots + a_0(x) = P^*(x) \prod_{n=1}^{d} (y - \alpha_n(x))$$

Then by Jensen's formula,

$$m(P) = m(P^*) + \frac{1}{2\pi \mathrm{i}} \sum_{n=1}^{d} \int_{\mathbb{T}^1} \log^+ |\alpha_n(x)| \frac{\mathrm{d}x}{x} = m(P^*) - \frac{1}{2\pi} \int_\gamma \eta(x, y) \tag{28}$$

Here

$$\gamma = \{P(x, y) = 0\} \cap \{|x| = 1, |y| \geq 1\}$$

is a union of paths in $C = \{P(x, y) = 0\}$. Also note that $\partial\gamma = \{(x, y) \in \mathbb{C}^2 \,|\, |x| = |y| = 1, P(x, y) = 0\}$

In our examples, we will get that η is exact, and $\partial\gamma \neq 0$ and then we can integrate using Stokes' Theorem.

In Smyth's case, we compute the Mahler measure of $P(x, y) = y + x - 1$. We get:

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log |y + x - 1| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_\gamma \eta(x, y)$$

Now write $x = \mathrm{e}^{2\pi \mathrm{i}\theta}$,

$$\gamma(\theta) = (\mathrm{e}^{2\pi \mathrm{i}\theta}, 1 - \mathrm{e}^{2\pi \mathrm{i}\theta}), \quad \theta \in [1/6\,; 5/6]$$

$$x(\partial\gamma) = [\xi_6] - [\bar{\xi}_6]$$

The good point is that $\eta(x, y)$ is exact in this case

Theorem 6

$$\eta(x, 1 - x) = \mathrm{d}D(x) \tag{29}$$

Then

$$2\pi m(x + y + 1) = D(\xi_6) - D(\bar{\xi}_6) = 2D(\xi_6)$$

In general, we associate $\eta(x, y)$ with an element in $H^1(C \setminus S, \mathbb{R})$ in the following way. Given $[\gamma] \in H_1(C \setminus S, \mathbb{Z})$,

$$[\gamma] \to \int_\gamma \eta(x, y) \tag{30}$$

(we identify $H^1(C \setminus S, \mathbb{R})$ with $H_1(C \setminus S, \mathbb{Z})'$). Under certain conditions (tempered polynomials, trivial tame symbols, see [15]) $\eta(x, y)$ can be thought as an element in $H^1(C, \mathbb{R})$.

Note the following

Theorem 7 *η satisfies the following properties*

1. $\eta(x,y) = -\eta(y,x)$
2. $\eta(x_1x_2,y) = \eta(x_1,y) + \eta(x_2,y)$
3. $\eta(x,1-x) = 0$ *in* $H^1(C,\mathbb{R})$

As a consequence, η is a symbol, and can be factored through $K_2(\mathbb{C}(C))$ (by Matsumoto's Theorem). Then we can guarantee that $\eta(x,y)$ is exact by having $\{x,y\}$ is trivial in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$. (Tensoring with $\mathbb{Q}$ kills roots of unity, which is fine, since η is trivial on them).

In general, if

$$x \wedge y = \sum_j r_j \ \ z_j \wedge (1 - z_j)$$

in $\bigwedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$, then

$$\eta(x,y) = \mathrm{d}\left(\sum_j r_j D(z_j)\right) = \mathrm{d}D\left(\sum_j r_j[z_j]\right)$$

We have $\gamma \subset C$ such that

$$\partial\gamma = \sum_k \epsilon_k[w_k] \quad \epsilon_k = \pm 1$$

where $w_k \in C(\mathbb{C})$, $|x(w_k)| = |y(w_k)| = 1$. Then

$$2\pi m(P) = D(\xi) \quad \text{for } \xi = \sum_k \sum_j r_j[z_j(w_k)]$$

We could summarize the whole picture as follows:

$$\ldots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C,\partial\gamma) \to K_2(C) \to \ldots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

There are two "nice" situations:

- $\eta(x,y)$ is exact, then $\{x,y\} \in K_3(\partial\gamma)$. In this case we have $\partial\gamma \neq \emptyset$, we use Stokes' Theorem and we finish with an element $K_3(\partial\gamma) \subset K_3(\bar{\mathbb{Q}})$, leading to polylogarithms and zeta functions (of number fields), due to theorems by Borel, Bloch, Suslim and others.
- $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(C)$. In this case, we have $\eta(x,y)$ is not exact and we get essentially the L-series of a curve, leading to examples of Beilinson's conjectures.

In general, we may get combinations of both situations.

9. The three-variable case

We are going to extend this situation to three variables. We will take

$$\eta(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y|\,\mathrm{d}\log|z| - \mathrm{d}\arg y\,\mathrm{d}\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z|\,\mathrm{d}\log|x| - \mathrm{d}\arg z\,\mathrm{d}\arg x\right) + \log|z|\left(\frac{1}{3}\mathrm{d}\log|x|\,\mathrm{d}\log|y| - \mathrm{d}\arg x\,\mathrm{d}\arg y\right)$$

Then η verifies

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}\wedge\frac{\mathrm{d}z}{z}\right)$$

We can express the Mahler measure of P

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z)$$

Where

$$\Gamma = \{P(x,y,z) = 0\} \cap \{|x| = |y| = 1, |z| \geq 1\}$$

We are integrating on a subset of $S = \{P(x,y,z) = 0\}$. The differential form is defined in this surface.

As in the two-variable case, we would like to apply Stokes' Theorem.

Let us take a look at Smyth's case, we can express the polynomial as $P(x,y,z) = (1-x) + (1-y)z$. We get:

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2}\log^+\left|\frac{1-x}{1-y}\right|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = -\frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z)$$

In general, we have

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y + \frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

Suppose we have

$$x\wedge y\wedge z = \sum r_i\ \ x_i\wedge(1-x_i)\wedge y_i$$

in $\bigwedge^3(\mathbb{C}(S)^*)\otimes\mathbb{Q}$.

Then

$$\int_\Gamma \eta(x,y,z) = \sum r_i\int_\Gamma \eta(x_i, 1-x_i, y_i) = \sum r_i\int_{\partial\Gamma}\omega(x_i,y_i)$$

In Smyth's case, this corresponds to

$$x\wedge y\wedge z = -\ \ x\wedge(1-x)\wedge y -\ \ y\wedge(1-y)\wedge x$$

in other words,

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

Back to the general picture, $\partial\Gamma = \{P(x,y,z) = 0\}\cap\{|x| = |y| = |z| = 1\}$. When $P\in\mathbb{Q}[x,y,z]$, Γ can be thought as

$$\gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}\cap\{|x| = |y| = 1\}$$

Note that we are integrating now on a path inside the curve $C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$. The differential form ω is defined in this new curve (this way of thinking the integral over a new curve has been proposed by Maillot). Now it makes sense to try to apply Stokes' Theorem again. We have

$$\omega(x,x) = \mathrm{d}P_3(x)$$

Suppose we have

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.

Then, as before:

$$\int_\gamma \omega(x,y) = \sum r_i \; P_3(x_i)|_{\partial\gamma}$$

Back to Smyth's case, $C = \{x = y\} \cup \{xy = 1\}$ in this example, and

$$-[x]_2 \otimes y - [y]_2 \otimes x = \pm 2[x]_2 \otimes x$$

Then

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x) = \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

10. The K-theory conditions

We follow Goncharov, [9], [10]. Given a field F, we define subgroups $R_i(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ as

$$\begin{aligned}
R_1(F) &:= [x] + [y] - [xy] \\
R_2(F) &:= [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right] \\
R_3(F) &:= \text{ certain functional equation of the trilogarithm}
\end{aligned}$$

Define

$$B_i(F) := \mathbb{Z}[\mathbb{P}^1_F]/R_i(F) \tag{31}$$

The idea is that $B_i(F)$ is the place where P_i naturally acts. We have the following complexes:

$$\begin{aligned}
B_F(3) &: \; B_3(F) \xrightarrow{\delta_1^3} B_2(F) \otimes F^* \xrightarrow{\delta_2^3} \wedge^3 F^* \\
B_F(2) &: \; B_2(F) \xrightarrow{\delta_1^2} \wedge^2 F^* \\
B_F(1) &: \; F^*
\end{aligned}$$

($B_i(F)$ is placed in degree 1).

$$\delta_1^3([x]_3) = [x]_2 \otimes x \qquad \delta_2^3([x]_2 \otimes y) = x \wedge (1-x) \wedge y \qquad \delta_1^2([x]_2) = x \wedge (1-x)$$

Proposition 8

$$\begin{aligned} H^1(B_F(1)) &\cong K_1(F) && (32)\\ H^2(B_F(2)) &\cong K_2(F) && (33)\\ H^3(B_F(3)) &\cong K_3^M(F) && (34) \end{aligned}$$

Goncharov [9] conjectures:

$$H^i(B_F(3)\otimes\mathbb{Q}) \cong K_{6-i}^{[3-i]}(F)_{\mathbb{Q}}$$

Where $K_n^{[i]}(F)_{\mathbb{Q}}$ is a certain quotient in a filtration of $K_n(F)_{\mathbb{Q}}$.

Note that our first condition is that

$$x\wedge y\wedge z = 0 \quad \text{in} \quad H^3(B_{\mathbb{Q}(S)}(3)\otimes\mathbb{Q}) \cong K_3^{[0]}(\mathbb{Q}(S))_{\mathbb{Q}} \cong K_3^M(\mathbb{Q}(S))\otimes\mathbb{Q}$$

and the second condition is

$$[x_i]_2\otimes y_i = 0 \quad \text{in} \quad H^2(B_{\mathbb{Q}(S)}(3)\otimes\mathbb{Q}) \overset{?}{\cong} K_4^{[1]}(\mathbb{Q}(C))_{\mathbb{Q}}$$

Hence, the conditions can be translated as certain elements in different K-theories must be zero, which is analogous to the two-variable case.

We could summarize this picture as follows. We first integrate in this picture

$$\ldots \to K_4(\partial\Gamma) \to K_3(S,\partial\Gamma) \to K_3(S) \to \ldots$$

$$\partial\Gamma = S\cap\mathbb{T}^3$$

As before, we have two situations. All the examples we have talked about fit into the situation when $\eta(x,y,z)$ is exact and $\partial\Gamma\neq\emptyset$. Then we finish with an element in $K_4(\partial\Gamma)$.

Then we go to

$$\ldots \to (K_5(\bar{\mathbb{Q}})\supset)K_5(\partial\gamma) \to K_4(C,\partial\gamma) \to K_4(C) \to \ldots$$

$$\partial\gamma = C\cap\mathbb{T}^2$$

Again we have two possibilities, but in our context, $\omega(x,y)$ is exact and we finish with an element in $K_5(\partial\gamma)\subset K_5(\bar{\mathbb{Q}})$ leading to trilogarithms and zeta functions, due to Zagier's conjecture and Borel's theorem.

References

[1] D. W. Boyd, Speculations concerning the range of Mahler's measure, *Canad. Math. Bull.* **24** (1981), 453 - 469.

[2] D. W. Boyd, Mahler's measure and special values of L-functions, *Experiment. Math.* **7** (1998), 37 – 82.

[3] D. W. Boyd, F. Rodriguez Villegas, Mahler's measure and the dilogarithm (I), *Canad. J. Math.* **54** (2002), 468 - 492.

[4] D. W. Boyd, F. Rodriguez Villegas, with an appendix by N. M. Dunfield, Mahler's measure and the dilogarithm (II), (preprint, July 2003)

[5] J. Condon, Calculation of the Mahler measure of a three variable polynomial, (preprint, October 2003)

[6] C. D'Andrea, M. N. Lalín, On The Mahler measure of resultants in small dimensions. (in preparation).

[7] C. Deninger, Deligne periods of mixed motives, K-theory and the entropy of certain Z^n-actions, *J. Amer. Math. Soc.* **10** (1997), no. 2, 259–281.

[8] G. Everest, T. Ward, Heights of Polynomials and Entropy in Algebraic Dynamics, *Universitex (UTX), Springer-Verlag* London (1999).

[9] A. B. Goncharov, Geometry of Configurations, Polylogarithms, and Motivic Cohomology, *Adv. Math.* **114** (1995), no. 2, 197–318.

[10] A. B. Goncharov, The Classical Polylogarithms, Algebraic K-theory and $\zeta_F(n)$, *The Gelfand Mathematical Seminars 1990-1992, Birkhauser*, Boston (1993), 113 - 135.

[11] M. N. Lalín, Some examples of Mahler measures as multiple polylogarithms, *J. Number Theory* **103** (2003), no. 1, 85–108.

[12] W. M. Lawton, A problem of Boyd concerning geometric means of polynomials, *J. Number Theory* **16** (1983), no. 3, 356 – 362.

[13] D. H. Lehmer, Factorization of certain cyclotomic functions. *Ann. of Math. (2)* **34** (1933), no. 3, 461–479.

[14] V. Maillot, Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables. *Mém. Soc. Math. Fr. (N.S.)* **80** (2000), 129pp.

[15] F. Rodriguez Villegas, Modular Mahler measures I, *Topics in number theory* (University Park, PA 1997), 17-48, Math. Appl., 467, Kluwer Acad. Publ. Dordrecht, 1999.

[16] C. J. Smyth, On measures of polynomials in several variables, *Bull. Austral. Math. Soc. Ser. A* **23** (1981), 49 - 63. Corrigendum (with G. Myerson): *Bull. Austral. Math. Soc.* **26** (1982), 317 - 319.

[17] S. Vandervelde, A formula for the Mahler measure of $axy + bx + cy + d$, *J. Number Theory* **100** (2003) 184–202.

[18] D. Zagier, Polylogarithms, Dedekind Zeta functions, and the Algebraic K-theory of Fields, *Arithmetic algebraic geometry* (Texel, 1989), 391–430, Progr. Math., 89, Birkhuser Boston, Boston, MA, 1991.