Mahler measure and evaluation of regulators

Matilde N. Lalín

Institut des Hautes Etudes Scientifiques and Clay Mathematics Institute
mlalin@ihes.fr
http://www.math.ubc.ca/~mlalin

June 2nd, 2006

Mahler measure for one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\lim_{n\to\infty}\frac{|\alpha^{n+1}-1|}{|\alpha^n-1|}=\begin{cases}|\alpha| & \text{if } |\alpha|>1\\ 1 & \text{if } |\alpha|<1\end{cases}$$

For

$$P(x)=a\prod_i (x-\alpha_i)$$

$$M(P)=|a|\prod_i \max\{1,|\alpha_i|\}$$

$$m(P)=\log M(P)=\log|a|+\sum_i \log^+|\alpha_i|$$

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

where Φ_{n_i} are cyclotomic polynomials

Lehmer's question

Lehmer (1933)

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$$

$= \log(1.176280818\ldots) = 0.162357612\ldots$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

Does there exist $C>0$, for all $P(x)\in\mathbb{Z}[x]$

$$m(P)=0 \quad \text{or} \quad m(P)>C??$$

Is the above polynomial the best possible?

Lehmer's question

Lehmer (1933)

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$= \log(1.176280818\dots) = 0.162357612\dots$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

$$\text{Does there exist} \quad C > 0, \quad \text{for all} \quad P(x) \in \mathbb{Z}[x]$$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the above polynomial the best possible?

Lehmer's question

Lehmer (1933)

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$$

$= \log(1.176280818\dots) = 0.162357612\dots$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

Does there exist $C > 0$, for all $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is the above polynomial the best possible?

Mahler measure of polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Given

$$P(x) = a_d \prod_{n=1}^{d} (x - \alpha_n) \in \mathbb{C}[x]$$

$$m(P) = \log |a_d| + \sum_{n=1}^{d} \log^+ |\alpha_n|$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

where h is the logarithmic Weil height.

Boyd & Lawton

$P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathrm{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathrm{L}'(A,0)$$

$$A: Y^2 = X^3-44X+112$$

Examples in three variables

- Condon (2003):

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005):

$$\pi^2 m(x^2+1+(x+1)y+(x-1)z) = \pi \mathrm{L}(\chi_{-4},2) + \frac{21}{8}\zeta(3)$$

Examples with more than three variables

L.(2003):

- $$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 24\mathrm{L}(\chi_{-4},4)$$

- $$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right) = 62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

- $$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = 93\zeta(5)$$

Known formulas for n.

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$\mathcal{L}_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.
Zagier:

$$\mathcal{L}_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number
$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.
One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

$\mathcal{L}_k$ satisfies lots of functional equations

$$\mathcal{L}_k\left(\frac{1}{x}\right) = (-1)^{k-1}\mathcal{L}_k(x) \qquad \mathcal{L}_k(\bar{x}) = (-1)^{k-1}\mathcal{L}_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

Philosophy of Beilinson's conjectures

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \text{ rank } 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997): General framework

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x,y) = \log|x|\,\mathrm{d}\arg y - \log|y|\,\mathrm{d}\arg x$$

$$\eta(x,1-x) = \mathrm{d}D(x) \qquad \mathrm{d}\eta(x,y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}\right)$$

An algebraic integration for Mahler measure

Deninger (1997): General framework

Rodriguez-Villegas (1997) : $P(x, y) \in \mathbb{C}[x, y]$

$$m(P) = m(P^*) - \frac{1}{2\pi} \int_\gamma \eta(x, y)$$

$$\eta(x, y) = \log |x| \mathrm{d} \arg y - \log |y| \mathrm{d} \arg x$$

$$\eta(x, 1 - x) = \mathrm{d}D(x) \qquad \mathrm{d}\eta(x, y) = \mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right)$$

The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x,y,z)$$

The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \log |z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_{\Gamma} \eta(x,y,z)$$

The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_\Gamma \log |z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_\Gamma \eta(x,y,z)$$

The three-variable case

$$P(x,y,z) = (1-x) - (1-y)z \quad X = \{P(x,y,z) = 0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3} \int_{\mathbb{T}^3} \log \left| z - \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} \frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log^+ \left| \frac{1-x}{1-y} \right| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2} \int_\Gamma \log|z| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$

$$= -\frac{1}{(2\pi)^2} \int_\Gamma \eta(x,y,z)$$

$$\eta(x,y,z) = \log|x| \left(\frac{1}{3}\mathrm{d}\log|y| \wedge \mathrm{d}\log|z| - \mathrm{d}\arg y \wedge \mathrm{d}\arg z\right)$$
$$+ \log|y| \left(\frac{1}{3}\mathrm{d}\log|z| \wedge \mathrm{d}\log|x| - \mathrm{d}\arg z \wedge \mathrm{d}\arg x\right)$$
$$+ \log|z| \left(\frac{1}{3}\mathrm{d}\log|x| \wedge \mathrm{d}\log|y| - \mathrm{d}\arg x \wedge \mathrm{d}\arg y\right)$$

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} \wedge \frac{\mathrm{d}z}{z}\right)$$

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x, y)$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$
$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$z = \frac{1-x}{1-y}$$

$$\eta(x,y,z) = -\eta(x, 1-x, y) - \eta(y, 1-y, x)$$

$$m(P) = \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,1-x,y)+\eta(y,1-y,x) = \frac{1}{(2\pi)^2}\int_{\partial\Gamma}\omega(x,y)+\omega(y,x)$$

$$\eta(x,1-x,y) = \mathrm{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$z = \frac{1-x}{1-y}$$

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

$$m(P) = \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,1-x,y)+\eta(y,1-y,x) = \frac{1}{(2\pi)^2}\int_{\partial\Gamma}\omega(x,y)+\omega(y,x)$$

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$\Gamma = X \cap \{|x| = |y| = 1, |z| \geq 1\}$$

Maillot: if $P \in \mathbb{R}[x,y,z]$,

$$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$$

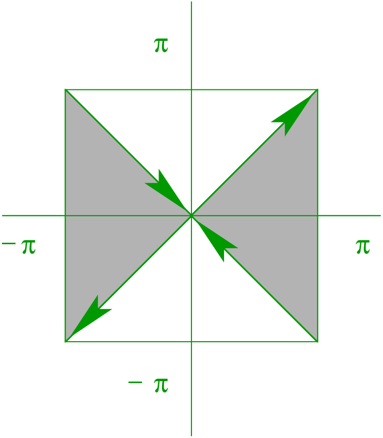
ω defined in

$$C = \{P(x,y,z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\}$$

Want to apply Stokes' Theorem again.

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m((1-x)-(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

$$\omega(x,x) = \mathrm{d}\mathcal{L}_3(x)$$

$$= \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2} \int_\Gamma \eta(x, y, z)$$

Need

$$x \wedge y \wedge z = \sum r_i \;\; x_i \wedge (1 - x_i) \wedge y_i$$

in $\bigwedge^3(\mathbb{C}(X)^*) \otimes \mathbb{Q}$,

($\{x, y, z\} = 0$ in $K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$) then

$$\int_\Gamma \eta(x, y, z) = \sum r_i \int_\Gamma \eta(x_i, 1 - x_i, y_i)$$

$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

$$\omega(x,y) = -D(x)\,\mathrm{d}\arg y$$
$$+\frac{1}{3}\log|y|(\log|1-x|\,\mathrm{d}\log|x| - \log|x|\,\mathrm{d}\log|1-x|)$$

Let

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right]$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.
F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\left\langle [0], [\infty], R_2(x,y)\right\rangle$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.
Then

$$\int_\gamma \omega(x,y) = \sum r_i\, \mathcal{L}_3(x_i)|_{\partial\gamma}$$

$$\omega(x,y) = -D(x)\,\mathrm{d}\arg y$$
$$+\frac{1}{3}\log|y|(\log|1-x|\,\mathrm{d}\log|x| - \log|x|\,\mathrm{d}\log|1-x|)$$

Let

$$R_2(x,y) = [x] + [y] + [1-xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right]$$

in $\mathbb{Z}\left[\mathbb{P}^1_{\mathbb{C}(C)}\right]$.
F field,

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\left\langle [0], [\infty], R_2(x,y)\right\rangle$$

Need

$$[x]_2 \otimes y = \sum r_i [x_i]_2 \otimes x_i$$

in $(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$.
Then

$$\int_\gamma \omega(x,y) = \sum r_i\ \mathcal{L}_3(x_i)|_{\partial\gamma}$$

Big picture in three variables

$$\cdots \to K_4(\partial\Gamma) \to K_3(X,\partial\Gamma) \to K_3(X) \to \dots$$

$$\partial\Gamma = X \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C,\partial\gamma) \to K_4(C) \to \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

Deninger(1997)

$$m(P) = m(P^*) + \frac{1}{(-2\mathrm{i}\pi)^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$$\pi^{2n} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right)$$

$$= \sum_{h=1}^{n} c_{n,h} \pi^{2n-2h} \zeta(2h+1)$$

Deninger(1997)

$$m(P) = m(P^*) + \frac{1}{(-2\mathrm{i}\pi)^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \cdots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$$\pi^{2n} m\left(1 + \left(\frac{1-x_1}{1+x_1}\right) \dots \left(\frac{1-x_{2n}}{1+x_{2n}}\right) z\right)$$
$$= \sum_{h=1}^{n} c_{n,h} \pi^{2n-2h} \zeta(2h+1)$$

Identities

Boyd (1997), Rodriguez-Villegas (2000)

$$7m(y^2+2xy+y-x^3-2x^2-x) = 5m(y^2+4xy+y-x^3+x^2)$$

Rogers (2005)

$$m(4n^2) + m\left(\frac{4}{n^2}\right) = 2m\left(2n+\frac{2}{n}\right)$$

where

$$m(k) := m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right)$$

Identities

Boyd (1997), Rodriguez-Villegas (2000)

$$7m(y^2+2xy+y-x^3-2x^2-x) = 5m(y^2+4xy+y-x^3+x^2)$$

Rogers (2005)

$$m(4n^2)+m\left(\frac{4}{n^2}\right) = 2m\left(2n+\frac{2}{n}\right)$$

where

$$m(k) := m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right)$$

Idea in the Elliptic Curve case

- For $\{x, y\} \in K_2(E)$:

$$r(\{x, y\}) = -\frac{1}{2\pi} \int_\gamma \eta(x, y)$$

  γ generates $H_1(E, \mathbb{Z})^-$

- $$r(\{x, y\}) = D^E((x) \diamond (y))$$

  if $(x), (y)$ supported on $E_{tors}(\bar{\mathbb{Q}})$.

- $$\pi D^E \sim L(E, 2)$$

  is HARD (requires complex multiplication or modular case)

Idea in the Elliptic Curve case

- For $\{x, y\} \in K_2(E)$:

$$r(\{x,y\}) = -\frac{1}{2\pi}\int_\gamma \eta(x,y)$$

  γ generates $H_1(E,\mathbb{Z})^-$

- $$r(\{x,y\}) = D^E((x)\diamond(y))$$

  if $(x),(y)$ supported on $E_{tors}(\bar{\mathbb{Q}})$.

- $$\pi D^E \sim L(E,2)$$

  is HARD (requires complex multiplication or modular case)

Idea in the Elliptic Curve case

- For $\{x, y\} \in K_2(E)$:

$$r(\{x, y\}) = -\frac{1}{2\pi} \int_\gamma \eta(x, y)$$

γ generates $H_1(E, \mathbb{Z})^-$

-

$$r(\{x, y\}) = D^E((x) \diamond (y))$$

if $(x), (y)$ supported on $E_{tors}(\bar{\mathbb{Q}})$.

-

$$\pi D^E \sim L(E, 2)$$

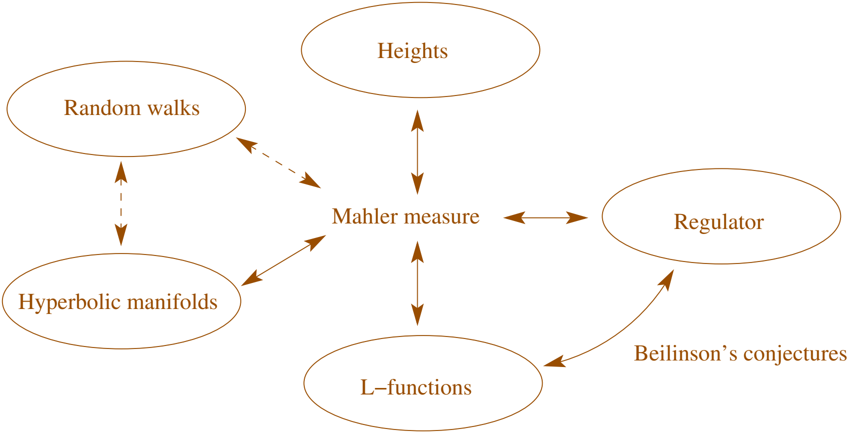
is HARD (requires complex multiplication or modular case)

$$m\left(\frac{4}{n^2}\right) = m\left(2n + \frac{2}{n}\right) + m\left(2\mathrm{i}n + \frac{2}{\mathrm{i}n}\right)$$

Rogers & L. (in progress)

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

curve is $X_0(24)$


Heights
Random walks
Mahler measure
Regulator
Hyperbolic manifolds
L–functions
Beilinson's conjectures